Algebra $dtsiPBC$: discrete time stochastic Petri box calculus with immediate multiactions

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Abstract: In [MVF01], a continuous time stochastic extension $sPBC$ of finite Petri box calculus PBC [BDH92] was proposed. In [MVCC03], iteration operator was added to $sPBC$.

Algebra $sPBC$ has an interleaving semantics, but PBC has a step one.

We constructed a discrete time stochastic extension $dtsPBC$ of finite PBC [Tar05] and enriched it with iteration [Tar06].

We present the extension $dtsiPBC$ of $dtsPBC$ with immediate multiactions [TMV10,TMV13]. $dtsiPBC$ is a discrete time analog of $sPBC$ with immediate multiactions.

The step operational semantics is defined in terms of labeled probabilistic transition systems.

The denotational semantics is defined in terms of a subclass of labeled DTSPNs with immediate transitions (LDTSIPNs), called discrete time stochastic and immediate Petri boxes (dtsi-boxes).

The corresponding semi-Markov chain and (reduced) discrete time Markov chain are analyzed to evaluate performance.

We propose step stochastic bisimulation equivalence and investigate its interrelations with others.

We explain how to use this equivalence for reduction of transition systems and semi-Markov chains.

We demonstrate how to apply this equivalence to compare stationary behaviour and simplify performance analysis.

The case study of performance evaluation is presented: the shared memory system.

Keywords: stochastic Petri net, stochastic process algebra, Petri box calculus, discrete time, immediate multiaction, transition system, operational semantics, immediate transition, dtsi-box, denotational semantics, Markov chain, performance evaluation, stochastic equivalence, reduction, shared memory system.

Contents

Introduction

Previous work

- Continuous time (subsets of $\mathbb{R}_{\geq 0}$): interleaving semantics
  - *Continuous time stochastic Petri nets (CTSPNs)* [Mol82,FN85]:

    exponential transition firing delays,

    Continuous time Markov chain (CTMC).
  - *Generalized stochastic Petri nets (GSPNs)* [MCB84,CMBC93]:

    exponential and zero transition firing delays,

    Semi-Markov chain (SMC).
  - *Extended generalized stochastic Petri nets (EGSPNs)* [HS89,MBBCCC89]:

    hyper-exponential or Erlang or phase and zero transition firing delays.
  - *Deterministic stochastic Petri nets (DSPNs)* [MC87,MCF90]:

    exponential and deterministic transition firing delays,

    Semi-Markov process (SMP), if no two deterministic transitions are enabled in any marking.
  - *Extended deterministic stochastic Petri nets (EDSPNs)* [GL94]:

    non-exponential and deterministic transition firing delays.
  - *Extended stochastic Petri nets (ESPNs)* [DTGN85]:

    arbitrary transition firing delays.

- Discrete time (subsets of $\mathbb{N}$): interleaving and step semantics
  - *Discrete time stochastic Petri nets (DTSPNs)* [Mol85,ZG94]:

    geometric transition firing delays,

    Discrete time Markov chain (DTMC).
  - *Discrete time deterministic and stochastic Petri nets (DTDSPNs)* [ZFH01]:

    geometric and fixed transition firing delays,

    Semi-Markov chain (SMC).
  - *Discrete deterministic and stochastic Petri nets (DDSPNs)* [ZCH97]:

    phase and fixed transition firing delays,

    Semi-Markov chain (SMC).

Stochastic process algebras

- $MTIPP$ [HR94]
- $GSPA$ [BKLL95]
- $PEPA$ [Hil96]
- $S\pi$ [Pri96]
- $EMPA$ [BGo98]
- $GSMPA$ [BBGo98]
- $sACP$ [AHR00]
- TCP^{dst} [MVi08]

More stochastic process calculi

- $TIPP$ [GHR93]
- $WSCCS$ [Tof94]
- $PM-TIPP$ [Ret95]
- $SPADES$ [AKB98]
- $NMSPA$ [LN00]
- $SM-PEPA$ [Brad05]
- $dsCCP$ [Bort06]
- IPC [CHLS09]
- $iPEPA$ [HBC13]
- $mCCS$ [DH13]
- $PHASE$ [CR14]

Algebra PBC and its extensions

- *Petri box calculus* PBC [BDH92]
- *Time Petri box calculus* $tPBC$ [Kou00]
- *Timed Petri box calculus* $TPBC$ [MF00]
- *Stochastic Petri box calculus* $sPBC$ [MVF01,MVCC03]
- *Ambient Petri box calculus* $APBC$ [FM03]
- *Arc time Petri box calculus* $atPBC$ [Nia05]
- *Generalized stochastic Petri box calculus* $gsPBC$ [MVCR08]
- *Discrete time stochastic Petri box calculus* $dtsPBC$ [Tar05,Tar06]
- *Discrete time stochastic and immediate Petri box calculus* $dtsiPBC$ [TMV10,TMV13]

SPACLS: Classification of stochastic process algebras

Time	Immediate (multi)actions	Interleaving semantics	Non-interleaving semantics
Continuous	No	$MTIPP$ (CTMC), $PEPA$ (CTMP), $sPBC$ (CTMC)	$GSPA$ (GSMP), $S\pi$, $GSMPA$ (GSMP)
	Yes	$EMPA$ (SMC, CTMC), $gsPBC$ (SMC)	—
Discrete	No	$WSCCS$ (DTMC), $dsCCP$ (DTMC)	$dtsPBC$ (DTMC)
	Yes	TCP^{dst} (DTMRC), IPC (DTMC)	$sACP$, $dtsiPBC$ (SMC, DTMC)

The SPNs-based denotational semantics: orange SPA names.

The underlying stochastic process: in parentheses near the SPA names.

Transition labeling

- CTSPNs [Buc95]
- GSPNs [Buc98]
- DTSPNs [BT00]

Stochastic equivalences

- Probabilistic transition systems (PTSs) [BM89,Chr90,LS91,BHe97,KN98]
- SPAs [HR94,Hil94,BGo98]
- Markovian process algebras (MPAs) [Buc94,BKe01]
- CTSPNs [Buc95]
- GSPNs [Buc98]
- Stochastic automata (SAs) [Buc99]
- Stochastic event structures (SESs) [MCW03]

Syntax

The *set of all finite multisets* over X is $\mathbb{N}^X_{fin}$. The *set of all subsets (powerset)* of X is 2^X.

$Act = \{a, b, \ldots\}$ is the set of *elementary actions*.

$\widehat{Act} = \{\hat{a}, \hat{b}, \ldots\}$ is the set of *conjugated actions (conjugates)* s.t. $\hat{a} \neq a$ and $\hat{\hat{a}} = a$.

$\mathcal{A} = Act \cup \widehat{Act}$ is the set of *all actions*.

$\mathcal{L} = \mathbb{N}^{\mathcal{A}}_{fin}$ is the set of *all multiactions*.

The *alphabet* of $\alpha \in \mathcal{L}$ is $\mathcal{A}(\alpha) = \{x \in \mathcal{A} \mid \alpha(x) > 0\}$.

A *stochastic multiaction* is a pair (α, ρ), where
$\alpha \in \mathcal{L}$ and $\rho \in (0; 1)$ is the *probability* of the multiaction α.

$\mathcal{SL}$ is the set of *all stochastic multiactions*.

An *immediate multiaction* is a pair $(\alpha, \natural_l)$, where
$\alpha \in \mathcal{L}$ and $l \in \mathbb{R}_{>0} = (0; +\infty)$ is the *weight* of the multiaction α.

$\mathcal{IL}$ is the set of *all immediate multiactions*.

$\mathcal{SIL} = \mathcal{SL} \cup \mathcal{IL}$ is the set of *all activities*.

The *alphabet* of $(\alpha, \kappa) \in \mathcal{SIL}$ is $\mathcal{A}(\alpha, \kappa) = \mathcal{A}(\alpha)$.

The *alphabet* of $\Upsilon \in \mathbb{N}_{fin}^{\mathcal{SIL}}$ is $\mathcal{A}(\Upsilon) = \cup_{(\alpha,\kappa)\in\Upsilon}\mathcal{A}(\alpha)$.

For $(\alpha, \kappa) \in \mathcal{SIL}$, its *multiaction part* is $\mathcal{L}(\alpha, \kappa) = \alpha$ and

its *probability* or *weight part* is $\Omega(\alpha, \kappa) = \kappa$ if $\kappa \in (0; 1)$; or $\Omega(\alpha, \kappa) = l$ if $\kappa = \natural_l,\ l \in \mathbb{R}_{>0}$.

The *multiaction part* of $\Upsilon \in \mathbb{N}_{fin}^{\mathcal{SIL}}$ is $\mathcal{L}(\Upsilon) = \sum_{(\alpha,\kappa)\in\Upsilon} \alpha$.

The operations: *sequential execution* ;, *choice* $[]$, *parallelism* $||$, *relabeling* $[f]$, *restriction* rs, *synchronization* sy and *iteration* $[**]$.

Sequential execution and choice have the standard interpretation.

Parallelism does not include synchronization unlike that in standard process algebras.

Relabeling functions $f : \mathcal{A} \to \mathcal{A}$ are bijections preserving conjugates: $\forall x \in \mathcal{A}\ f(\hat{x}) = \widehat{f(x)}$.
For $\alpha \in \mathcal{L}$, let $f(\alpha) = \sum_{x \in \alpha} f(x)$. For $\Upsilon \in I\!N_{fin}^{\mathcal{SIL}}$, let $f(\Upsilon) = \sum_{(\alpha,\kappa) \in \Upsilon} (f(\alpha), \kappa)$.

Restriction over $a \in Act$: any process behaviour containing a or its conjugate $\hat{a}$ is not allowed.

Let $\alpha, \beta \in \mathcal{L}$ be two multiactions s.t. for $a \in Act$ we have $a \in \alpha$ and $\hat{a} \in \beta$, or $\hat{a} \in \alpha$ and $a \in \beta$.
Synchronization of α and β by a is $\alpha \oplus_a \beta = \gamma$:

$$\gamma(x) = \begin{cases} \alpha(x) + \beta(x) - 1, & x = a \text{ or } x = \hat{a}; \\ \alpha(x) + \beta(x), & \text{otherwise}. \end{cases}$$

In the iteration, the initialization subprocess is executed first,
then the body one is performed zero or more times, finally, the termination one is executed.

Static expressions specify the structure of processes.

Definition 1 *Let* $(\alpha, \kappa) \in \mathcal{SIL}$ *and* $a \in Act$. *A* static expression *of* $dtsiPBC$ *is*

$$E ::= (\alpha, \kappa) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E*E*E].$$

$StatExpr$ is the set of *all static expressions* of $dtsiPBC$.

Definition 2 *Let* $(\alpha, \kappa) \in \mathcal{SIL}$ *and* $a \in Act$. *A* regular static expression *of* $dtsiPBC$ *is*

$$E ::= (\alpha, \kappa) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E*D*E],$$

where $D ::= (\alpha, \kappa) \mid D;E \mid D[]D \mid D[f] \mid D \text{ rs } a \mid D \text{ sy } a \mid [D*D*E].$

$RegStatExpr$ is the set of *all regular static expressions* of $dtsiPBC$.

Dynamic expressions specify the states of processes.

Dynamic expressions are obtained from static ones annotated with upper or lower bars.

The *underlying static expression* of a dynamic one: removing all upper and lower bars.

Definition 3 *Let $E \in StatExpr$ and $a \in Act$. A dynamic expression of $dtsiPBC$ is*

$$G ::= \overline{E} \mid \underline{E} \mid G;E \mid E;G \mid G[]E \mid E[]G \mid G\|G \mid G[f] \mid G \text{ rs } a \mid G \text{ sy } a \mid$$
$$[G*E*E] \mid [E*G*E] \mid [E*E*G].$$

$DynExpr$ is the set of *all dynamic expressions* of $dtsiPBC$.

Definition 4 *A dynamic expression is regular if its underlying static expression is regular.*

$RegDynExpr$ is the set of *all regular dynamic expressions* of $dtsiPBC$.

We shall consider regular expressions only and omit the word “regular”.

Operational semantics

Inaction rules

Inaction rules: instantaneous structural transformations.

Let $E, F, K \in RegStatExpr$ and $a \in Act$.

IRULES1: Inaction rules for overlined and underlined regular static expressions

$\overline{E;F} \Rightarrow \overline{E};F$	$\underline{E};F \Rightarrow E;\overline{F}$	$E;\underline{F} \Rightarrow \underline{E;F}$
$\overline{E[]F} \Rightarrow \overline{E}[]F$	$\overline{E[]F} \Rightarrow E[]\overline{F}$	$\underline{E}[]F \Rightarrow \underline{E[]F}$
$E[]\underline{F} \Rightarrow \underline{E[]F}$	$\overline{E\|F} \Rightarrow \overline{E}\|\overline{F}$	$\underline{E}\|\underline{F} \Rightarrow \underline{E\|F}$
$\overline{E[f]} \Rightarrow \overline{E}[f]$	$\underline{E}[f] \Rightarrow \underline{E[f]}$	$\overline{E \text{ rs } a} \Rightarrow \overline{E} \text{ rs } a$
$\underline{E} \text{ rs } a \Rightarrow \underline{E \text{ rs } a}$	$\overline{E \text{ sy } a} \Rightarrow \overline{E} \text{ sy } a$	$\underline{E} \text{ sy } a \Rightarrow \underline{E \text{ sy } a}$
$\overline{[E*F*K]} \Rightarrow [\overline{E}*F*K]$	$[\underline{E}*F*K] \Rightarrow [E*\overline{F}*K]$	$[E*\underline{F}*K] \Rightarrow [E*\overline{F}*K]$
$[E*\underline{F}*K] \Rightarrow [E*F*\overline{K}]$	$[E*F*\underline{K}] \Rightarrow \underline{[E*F*K]}$	

Let $E, F \in RegStatExpr$, $G, H, \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

IRULES2: Inaction rules for arbitrary regular dynamic expressions

$$\frac{G \Rightarrow \widetilde{G},\ \circ \in \{;,[]\}}{G \circ E \Rightarrow \widetilde{G} \circ E} \qquad \frac{G \Rightarrow \widetilde{G},\ \circ \in \{;,[]\}}{E \circ G \Rightarrow E \circ \widetilde{G}} \qquad \frac{G \Rightarrow \widetilde{G}}{G \| H \Rightarrow \widetilde{G} \| H} \qquad \frac{H \Rightarrow \widetilde{H}}{G \| H \Rightarrow G \| \widetilde{H}} \qquad \frac{G \Rightarrow \widetilde{G}}{G[f] \Rightarrow \widetilde{G}[f]}$$

$$\frac{G \Rightarrow \widetilde{G},\ \circ \in \{\mathsf{rs},\mathsf{sy}\}}{G \circ a \Rightarrow \widetilde{G} \circ a} \qquad \frac{G \Rightarrow \widetilde{G}}{[G * E * F] \Rightarrow [\widetilde{G} * E * F]} \qquad \frac{G \Rightarrow \widetilde{G}}{[E * G * F] \Rightarrow [E * \widetilde{G} * F]} \qquad \frac{G \Rightarrow \widetilde{G}}{[E * F * G] \Rightarrow [E * F * \widetilde{G}]}$$

Definition 5 *A regular dynamic expression is operative if no inaction rule can be applied to it.*

$OpRegDynExpr$ is the set of *all operative regular dynamic expressions* of $dtsiPBC$.

We shall consider regular expressions only and omit the word "regular".

Definition 6 $\approx\ = (\Rightarrow \cup \Leftarrow)^*$ *is the structural equivalence of dynamic expressions in* $dtsiPBC$. G *and* G' *are structurally equivalent,* $G \approx G'$*, if they can be reached each from other by applying inaction rules in a forward or backward direction.*

Action and empty loop rules

Action rules with stochastic multiactions: execution of non-empty multisets of stochastic multiactions.

Action rules with immediate multiactions: execution of non-empty multisets of immediate multiactions.

Empty loop rule: execution of the empty multiset of activities at a time step.

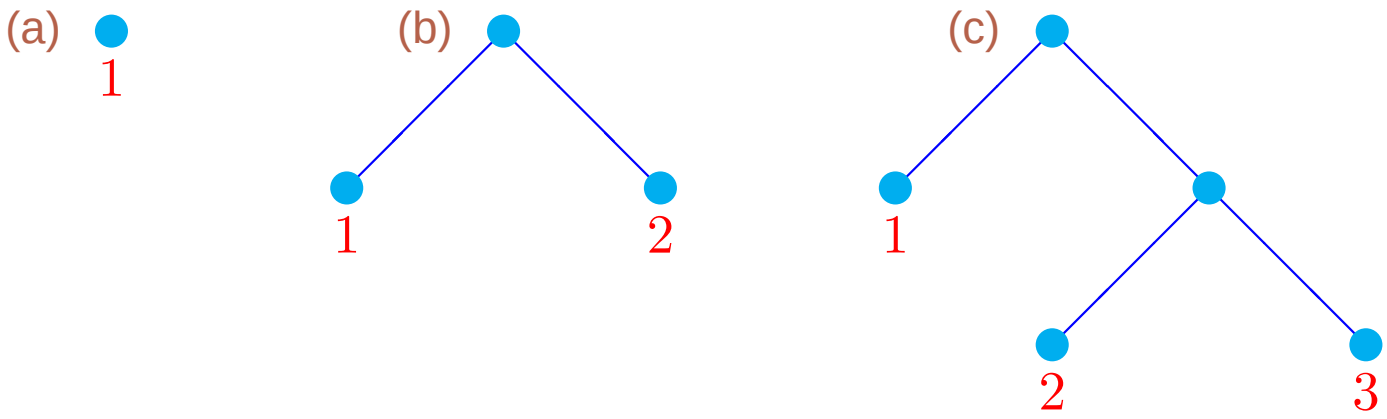
Definition 7 *Let $n \in \mathbb{N}$. The numbering of expressions is*

$$\iota ::= n \mid (\iota)(\iota).$$

Num is the set of *all numberings* of expressions.

The *content* of a numbering $\iota \in Num$ is

$$Cont(\iota) = \begin{cases} \{\iota\}, & \iota \in \mathbb{N}; \\ Cont(\iota_1) \cup Cont(\iota_2), & \iota = (\iota_1)(\iota_2). \end{cases}$$



BTRNUM: The binary trees encoded with the numberings 1, $(1)(2)$ and $(1)((2)(3))$

$[G]_{\approx} = \{H \mid G \approx H\}$ is the equivalence class of $G \in RegDynExpr$ w.r.t. structural equivalence.

G is an *initial* dynamic expression, $init(G)$, if $\exists E \in RegStatExpr\ G \in [\overline{E}]_{\approx}$.

G is a *final* dynamic expression, $final(G)$, if $\exists E \in RegStatExpr\ G \in [\underline{E}]_{\approx}$.

Definition 8 *Let* $G \in OpRegDynExpr$. *The* set of all non-empty multisets of activities which can be potentially executed from G *is* $Can(G)$. *Let* $(\alpha, \kappa) \in \mathcal{SIL}$, $E, F \in RegStatExpr$, $H \in OpRegDynExpr$ *and* $a \in Act$.

1. *If* $final(G)$ *then* $Can(G) = \emptyset$.
2. *If* $G = \overline{(\alpha, \kappa)}$ *then* $Can(G) = \{\{(\alpha, \kappa)\}\}$.
3. *If* $\Upsilon \in Can(G)$ *then*
   $\Upsilon \in Can(G \circ E)$, $\Upsilon \in Can(E \circ G)$ $(\circ \in \{;, [], \|\})$, $f(\Upsilon) \in Can(G[f])$,
   $\Upsilon \in Can(G \text{ rs } a)$ (*when* $a, \hat{a} \notin \mathcal{A}(\Upsilon)$), $\Upsilon \in Can(G \text{ sy } a)$,
   $\Upsilon \in Can([G * E * F])$, $\Upsilon \in Can([E * G * F])$, $\Upsilon \in Can([E * F * G])$.
4. *If* $\Upsilon \in Can(G)$ *and* $\Xi \in Can(H)$ *then* $\Upsilon + \Xi \in Can(G \| H)$.
5. *If* $\Upsilon \in Can(G \text{ sy } a)$ *and* $(\alpha, \kappa), (\beta, \lambda) \in \Upsilon$ *are different activities such that* $a \in \alpha$, $\hat{a} \in \beta$, *then*
   (a) $\Upsilon - \{(\alpha, \kappa), (\beta, \lambda)\} + \{(\alpha \oplus_a \beta, \kappa \cdot \lambda)\} \in Can(G \text{ sy } a)$, *if* $\kappa, \lambda \in (0; 1)$;
   (b) $\Upsilon - \{(\alpha, \kappa), (\beta, \lambda)\} + \{(\alpha \oplus_a \beta, \natural_{l+m})\} \in Can(G \text{ sy } a)$ *if* $\kappa = \natural_l$, $\lambda = \natural_m$, $l, m \in \mathbb{R}_{>0}$.

If $\Upsilon \in Can(G)$ then by definition of $Can(G)$ $\forall \Xi \subseteq \Upsilon,\ \Xi \neq \emptyset$ we have $\Xi \in Can(G)$.

If there are only stochastic (or only immediate) multiactions in the multisets from $Can(G) \neq \emptyset$ then these stochastic (or immediate) multiactions can be executed from G.

Otherwise, besides stochastic ones, there are immediate multiactions in the multisets from $Can(G)$.

By the note above, there are non-empty multisets of immediate multiactions in $Can(G)$ as well:
$\exists \Upsilon \in Can(G)\ \Upsilon \in \mathbb{N}^{\mathcal{IL}}_{fin} \setminus \{\emptyset\}$.

Then no stochastic multiactions can be executed from G,
even if $Can(G)$ contains non-empty multisets of stochastic multiactions:
immediate multiactions have a priority over stochastic ones, and should be executed first.

Definition 9 *Let $G \in OpRegDynExpr$. The set of all non-empty multisets of activities which can be executed from G is*

$$Now(G) = \begin{cases} Can(G), & (Can(G) \subseteq \mathbb{N}^{\mathcal{SL}}_{fin} \setminus \{\emptyset\}) \vee (Can(G) \subseteq \mathbb{N}^{\mathcal{IL}}_{fin} \setminus \{\emptyset\}); \\ Can(G) \cap \mathbb{N}^{\mathcal{IL}}_{fin}, & \textit{otherwise}. \end{cases}$$

G is *tangible*, $tang(G)$, if $Now(G) \subseteq I\!N_{fin}^{\mathcal{SL}} \setminus \{\emptyset\}$. We have $tang(G)$, if $Now(G) = \emptyset$.

G is *vanishing*, $vanish(G)$, if $\emptyset \neq Now(G) \subseteq I\!N_{fin}^{\mathcal{IL}} \setminus \{\emptyset\}$.

Let $G = (\overline{(\{a\}, \natural_1)}[](\{b\}, \natural_2))\|\overline{(\{c\}, \frac{1}{2})}$ and $G' = ((\{a\}, \natural_1)[]\overline{(\{b\}, \natural_2)})\|\overline{(\{c\}, \frac{1}{2})}$.

We have $G \approx G'$, since $G \Leftarrow G'' \Rightarrow G'$ for $G'' = \overline{((\{a\}, \natural_1)[](\{b\}, \natural_2))}\|\overline{(\{c\}, \frac{1}{2})}$, but
$Can(G) = \{\{(\{a\}, \natural_1)\}, \{(\{c\}, \frac{1}{2})\}, \{(\{a\}, \natural_1), (\{c\}, \frac{1}{2})\}\}$,
$Can(G') = \{\{(\{b\}, \natural_2)\}, \{(\{c\}, \frac{1}{2})\}, \{(\{b\}, \natural_2), (\{c\}, \frac{1}{2})\}\}$ and
$Now(G) = \{\{(\{a\}, \natural_1)\}\}$, $Now(G') = \{\{(\{b\}, \natural_2)\}\}$.

Clearly, $vanish(G)$ and $vanish(G')$.

The executions like that of $\{(\{c\}, \frac{1}{2})\}$ (and all multisets including it) from G and G' must be disabled using pre-conditions in the action rules.

Immediate multiactions have a priority over stochastic ones: the former are always executed first.

Let $H = \overline{(\{a\}, \natural_1)}[](\{b\}, \frac{1}{2})$ and $H' = (\{a\}, \natural_1)[]\overline{(\{b\}, \frac{1}{2})}$.

Then $H \approx H'$, since $H \Leftarrow H'' \Rightarrow H'$ for $H'' = \overline{(\{a\}, \natural_1)[](\{b\}, \frac{1}{2})}$, but
$Can(H) = Now(H) = \{\{(\{a\}, \natural_1)\}\}$ and $Can(H') = Now(H') = \{\{(\{b\}, \frac{1}{2})\}\}$.

We have $vanish(H)$, but $tang(H')$.

To make the action rules correct under structural equivalence: the executions like that of $\{(\{b\}, \frac{1}{2})\}$ from H' must be disabled using the pre-conditions.

Immediate multiactions have a priority over stochastic ones:
the choices between them are always resolved in favour of the former.

Let $G \in RegDynExpr$. We write $tang([G]_\approx)$, if $\forall H \in [G]_\approx \cap OpRegDynExpr\ tang(H)$.

Otherwise, we write $vanish([G]_\approx)$, and in this case $\exists H \in [G]_\approx \cap OpRegDynExpr\ vanish(H)$.

Let $(\alpha, \rho), (\beta, \chi) \in \mathcal{SL}$, $(\alpha, \natural_l), (\beta, \natural_m) \in \mathcal{IL}$ and $(\alpha, \kappa) \in \mathcal{SIL}$. Further,
$E, F \in RegStatExpr$, $G, H \in OpRegDynExpr$, $\widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.
Next, $\Gamma, \Delta \in \mathbb{N}^{\mathcal{SL}}_{fin} \setminus \{\emptyset\}$, $\Gamma' \in \mathbb{N}^{\mathcal{SL}}_{fin}$, $I, J \in \mathbb{N}^{\mathcal{IL}}_{fin} \setminus \{\emptyset\}$, $I' \in \mathbb{N}^{\mathcal{IL}}_{fin}$ and $\Upsilon \in \mathbb{N}^{\mathcal{SIL}}_{fin} \setminus \{\emptyset\}$.

The names of the action rules with immediate multiactions have a suffix '$\mathbf{i}$'.

ARULES: Action and empty loop rules

El $\dfrac{tang([G]_{\approx}))}{G \xrightarrow{\emptyset} G}$

B $\dfrac{}{(\alpha,\kappa)} \xrightarrow{\{(\alpha,\kappa)\}} \underline{(\alpha,\kappa)}$

S $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{G;E \xrightarrow{\Upsilon} \widetilde{G};E \quad E;G \xrightarrow{\Upsilon} E;\widetilde{G}}$

C $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ \neg init(G) \vee (init(G) \wedge tang([\overline{E}]_{\approx}))}{G[]E \xrightarrow{\Gamma} \widetilde{G}[]E \quad E[]G \xrightarrow{\Gamma} E[]\widetilde{G}}$

Ci $\dfrac{G \xrightarrow{I} \widetilde{G}}{G[]E \xrightarrow{I} \widetilde{G}[]E \quad E[]G \xrightarrow{I} E[]\widetilde{G}}$

P1 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ tang([H]_{\approx}))}{G\|H \xrightarrow{\Gamma} \widetilde{G}\|H \quad H\|G \xrightarrow{\Gamma} H\|\widetilde{G}}$

P1i $\dfrac{G \xrightarrow{I} \widetilde{G}}{G\|H \xrightarrow{I} \widetilde{G}\|H \quad H\|G \xrightarrow{I} H\|\widetilde{G}}$

P2 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ H \xrightarrow{\Delta} \widetilde{H}}{G\|H \xrightarrow{\Gamma+\Delta} \widetilde{G}\|\widetilde{H}}$

P2i $\dfrac{G \xrightarrow{I} \widetilde{G},\ H \xrightarrow{J} \widetilde{H}}{G\|H \xrightarrow{I+J} \widetilde{G}\|\widetilde{H}}$

L $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{G[f] \xrightarrow{f(\Upsilon)} \widetilde{G}[f]}$

Rs $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G},\ a,\hat{a} \notin \mathcal{A}(\Upsilon)}{G \text{ rs } a \xrightarrow{\Upsilon} \widetilde{G} \text{ rs } a}$

I1 $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{[G*E*F] \xrightarrow{\Upsilon} [\widetilde{G}*E*F]}$

I2 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ \neg init(G) \vee (init(G) \wedge tang([\overline{F}]_{\approx}))}{[E*G*F] \xrightarrow{\Gamma} [E*\widetilde{G}*F]}$

I2i $\dfrac{G \xrightarrow{I} \widetilde{G}}{[E*G*F] \xrightarrow{I} [E*\widetilde{G}*F]}$

I3 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ \neg init(G) \vee (init(G) \wedge tang([\overline{F}]_{\approx}))}{[E*F*G] \xrightarrow{\Gamma} [E*F*\widetilde{G}]}$

I3i $\dfrac{G \xrightarrow{I} \widetilde{G}}{[E*F*G] \xrightarrow{I} [E*F*\widetilde{G}]}$

Sy1 $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{G \text{ sy } a \xrightarrow{\Upsilon} \widetilde{G} \text{ sy } a}$

Sy2 $\dfrac{G \text{ sy } a \xrightarrow{\Gamma'+\{(\alpha,\rho)\}+\{(\beta,\chi)\}} \widetilde{G} \text{ sy } a,\ a\in\alpha,\ \hat{a}\in\beta}{G \text{ sy } a \xrightarrow{\Gamma'+\{(\alpha\oplus_a\beta,\rho\cdot\chi)\}} \widetilde{G} \text{ sy } a}$

Sy2i $\dfrac{G \text{ sy } a \xrightarrow{I'+\{(\alpha,\natural_l)\}+\{(\beta,\natural_m)\}} \widetilde{G} \text{ sy } a,\ a\in\alpha,\ \hat{a}\in\beta}{G \text{ sy } a \xrightarrow{I'+\{(\alpha\oplus_a\beta,\natural_{l+m})\}} \widetilde{G} \text{ sy } a}$

RULECMP: Comparison of inaction, action and empty loop rules

Rules	State change	Time progress	Activities execution
Inaction rules	$-$	$-$	$-$
Action rules (stochastic multiactions)	$\pm$	$+$	$+$
Action rules (immediate multiactions)	$\pm$	$-$	$+$
Empty loop rule	$-$	$+$	$-$

Transition systems

Definition 10 *The derivation set $DR(G)$ of a dynamic expression G is the minimal set:*

- $[G]_{\approx} \in DR(G)$;
- *if* $[H]_{\approx} \in DR(G)$ *and* $\exists \Upsilon\ H \xrightarrow{\Upsilon} \widetilde{H}$ *then* $[\widetilde{H}]_{\approx} \in DR(G)$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

The set of *all multisets of activities executable from s* is $Exec(s) = \{\Upsilon \mid \exists H \in s\ \exists \widetilde{H}\ H \xrightarrow{\Upsilon} \widetilde{H}\}$.

The state s is *tangible*, $tang(s)$, if $Exec(s) \subseteq \mathbb{N}^{\mathcal{SL}}_{fin}$.
For tangible states we always have $\emptyset \in Exec(s)$, and we may have $Exec(s) = \{\emptyset\}$.

The state s is *vanishing*, $vanish(s)$, if $Exec(s) \subseteq \mathbb{N}^{\mathcal{IL}}_{fin} \setminus \{\emptyset\}$.

The set of *all tangible states from $DR(G)$* is $DR_T(G)$.

The set of *all vanishing states from $DR(G)$* is $DR_V(G)$.

Obviously, $DR(G) = DR_T(G) \uplus DR_V(G)$.

Let $\Upsilon \in Exec(s) \setminus \{\emptyset\}$. The *probability of the multiset of stochastic multiactions* or the *weight of the multiset of immediate multiactions* Υ *which is ready for execution in* s:

$$PF(\Upsilon, s) = \begin{cases} \prod_{(\alpha,\rho)\in\Upsilon} \rho \cdot \prod_{\{\{(\beta,\chi)\}\in Exec(s)|(\beta,\chi)\notin\Upsilon\}} (1-\chi), & s \in DR_T(G); \\ \sum_{(\alpha,\natural_l)\in\Upsilon} l, & s \in DR_V(G). \end{cases}$$

In the case $\Upsilon = \emptyset$ and $s \in DR_T(G)$ we define

$$PF(\emptyset, s) = \begin{cases} \prod_{\{(\beta,\chi)\}\in Exec(s)} (1-\chi), & Exec(s) \neq \{\emptyset\}; \\ 1, & Exec(s) = \{\emptyset\}. \end{cases}$$

Let $\Upsilon \in Exec(s)$. The *probability to execute the multiset of activities* Υ *in* s:

$$PT(\Upsilon, s) = \frac{PF(\Upsilon, s)}{\sum_{\Xi \in Exec(s)} PF(\Xi, s)}.$$

If s is tangible, then $PT(\emptyset, s) \in (0; 1]$: the residence time in s is ≥ 1.

The *probability to move from* s *to* $\tilde{s}$ *by executing any multiset of activities*:

$$PM(s, \tilde{s}) = \sum_{\{\Upsilon | \exists H \in s \ \exists \widetilde{H} \in \tilde{s} \ H \xrightarrow{\Upsilon} \widetilde{H}\}} PT(\Upsilon, s).$$

TRPROBIM: Calculation of the probability functions PF, PT, PM for $s_1 \in DR(\overline{E})$ and $E = (\{a\}, \rho)[](\{a\}, \chi)$

$s_1 \backslash \Upsilon$	$\emptyset$	$\{(\{a\}, \rho)\}$	$\{(\{a\}, \chi)\}$	Σ
PF	$(1-\rho)(1-\chi)$	$\rho(1-\chi)$	$\chi(1-\rho)$	$1-\rho\chi$
PT	$\frac{(1-\rho)(1-\chi)}{1-\rho\chi}$	$\frac{\rho(1-\chi)}{1-\rho\chi}$	$\frac{\chi(1-\rho)}{1-\rho\chi}$	1
PM	$\frac{(1-\rho)(1-\chi)}{1-\rho\chi}$ (s_1)	$\frac{\rho+\chi-2\rho\chi}{1-\rho\chi}$ (s_2)		1

TRPROBIM1: Calculation of the probability functions PF, PT, PM for $s'_1 \in DR(\overline{E}')$ and $E' = (\{a\}, \natural_l)[](\{a\}, \natural_m)$

$s'_1 \backslash \Upsilon$	$\{(\{a\}, \natural_l)\}$	$\{(\{a\}, \natural_m)\}$	Σ
PF	l	m	$l+m$
PT	$\frac{l}{l+m}$	$\frac{m}{l+m}$	1
PM	1 (s'_2)		1

Definition 11 *The* (labeled probabilistic) transition system *of a dynamic expression* G *is* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, *where*

- *the set of* states *is* $S_G = DR(G)$;
- *the set of* labels *is* $L_G = I\!N_{fin}^{\mathcal{SIL}} \times (0;1]$;
- *the set of* transitions *is* $\mathcal{T}_G = \{(s, (\Upsilon, PT(\Upsilon, s)), \tilde{s}) \mid s, \tilde{s} \in DR(G),\ \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Upsilon} \widetilde{H}\}$;
- *the* initial state *is* $s_G = [G]_{\approx}$.

A transition $(s, (\Upsilon, \mathcal{P}), \tilde{s}) \in \mathcal{T}_G$ is written as $s \xrightarrow{\Upsilon}_{\mathcal{P}} \tilde{s}$.

We write $s \xrightarrow{\Upsilon} \tilde{s}$ if $\exists \mathcal{P}\ s \xrightarrow{\Upsilon}_{\mathcal{P}} \tilde{s}$ and $s \rightarrow \tilde{s}$ if $\exists \Upsilon\ s \xrightarrow{\Upsilon} \tilde{s}$.

Definition 12 *Let G, G' be dynamic expressions and $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, $TS(G') = (S_{G'}, L_{G'}, \mathcal{T}_{G'}, s_{G'})$ be their transition systems. A mapping $\beta : S_G \to S_{G'}$ is an* isomorphism *between $TS(G)$ and $TS(G')$, $\beta : TS(G) \simeq TS(G')$, if*

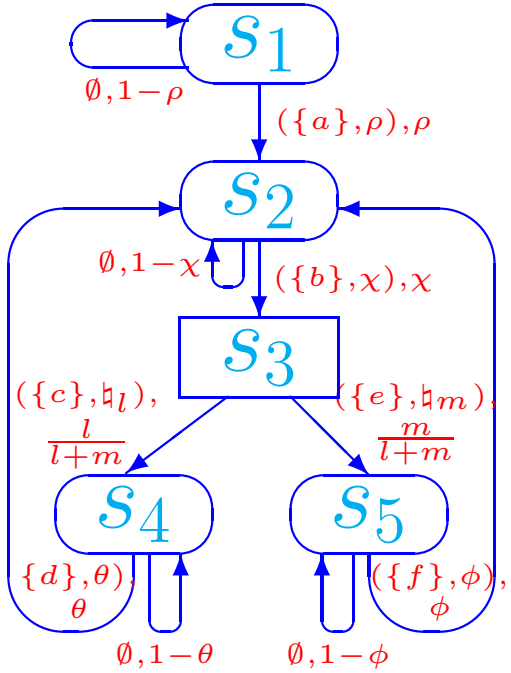
1. *β is a bijection s.t. $\beta(s_G) = s_{G'}$;*
2. *$\forall s, \tilde{s} \in S_G \; \forall \Upsilon \; s \xrightarrow{\Upsilon}_{\mathcal{P}} \tilde{s} \;\Leftrightarrow\; \beta(s) \xrightarrow{\Upsilon}_{\mathcal{P}} \beta(\tilde{s})$.*

$TS(G)$ and $TS(G')$ are isomorphic*, $TS(G) \simeq TS(G')$, if $\exists \beta : TS(G) \simeq TS(G')$.*

For $E \in RegStatExpr$, let $TS(E) = TS(\overline{E})$.

Definition 13 *G and G' are* equivalent w.r.t. transition systems*, $G =_{ts} G'$, if $TS(G) \simeq TS(G')$.*

$TS(\overline{E})$



TS: The transition system of $\overline{E}$ for $E = [(\{a\},\rho) * ((\{b\},\chi);((((\{c\},\natural_l);(\{d\},\theta))[]((\{e\},\natural_m);(\{f\},\phi)))) * \text{Stop}]$

Stop $= (\{c\}, \frac{1}{2})$ rs c is the process that performs empty loops with probability 1 and never terminates.

$DR(\overline{E})$ consists of:

$$s_1 = [[\overline{(\{a\},\rho)} * ((\{b\},\chi);((((\{c\},\natural_l);(\{d\},\theta))[]((\{e\},\natural_m);(\{f\},\phi)))) * \mathsf{Stop}]]_{\approx},$$

$$s_2 = [[(\{a\},\rho) * (\overline{(\{b\},\chi)};((((\{c\},\natural_l);(\{d\},\theta))[]((\{e\},\natural_m);(\{f\},\phi)))) * \mathsf{Stop}]]_{\approx},$$

$$s_3 = [[(\{a\},\rho) * ((\{b\},\chi);\overline{((((\{c\},\natural_l);(\{d\},\theta))[]((\{e\},\natural_m);(\{f\},\phi))))} * \mathsf{Stop}]]_{\approx},$$

$$s_4 = [[(\{a\},\rho) * ((\{b\},\chi);((((\{c\},\natural_l);\overline{(\{d\},\theta))}[]((\{e\},\natural_m);(\{f\},\phi)))) * \mathsf{Stop}]]_{\approx},$$

$$s_5 = [[(\{a\},\rho) * ((\{b\},\chi);((((\{c\},\natural_l);(\{d\},\theta))[]((\{e\},\natural_m);\overline{(\{f\},\phi))})) * \mathsf{Stop}]]_{\approx}.$$

$DR_T(\overline{E}) = \{s_1, s_2, s_4, s_5\}$ and $DR_V(\overline{E}) = \{s_3\}$.

Denotational semantics

Labeled DTSIPNs

Definition 14 *A* labeled discrete time stochastic and immediate Petri net (LDTSIPN) *is* $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$, *where*

- P_N *and* $T_N = Ts_N \uplus Ti_N$ *are finite sets of* places *and* stochastic and immediate transitions, *s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$;
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbb{N}$ *is the* arc weight *function;*
- Ω_N *is the* transition probability and weight *function s.t.*
  - $\Omega_N|_{Ts_N} : Ts_N \to (0;1)$ *(it associates stochastic transitions with* probabilities*);*
  - $\Omega_N|_{Ti_N} : Ti_N \to \mathbb{R}_{>0}$ *(it associates immediate transitions with* weights*);*
- $L_N : T_N \to \mathcal{L}$ *is the* transition labeling *function;*
- $M_N \in \mathbb{N}^{P_N}_{fin}$ *is the* initial marking.

Concurrent transition firings at discrete time moments.

LDTSIPNs have *step* semantics.

Let N be an LDTSIPN and $M, \widetilde{M} \in I\!N_{fin}^{P_N}$.

Immediate transitions have a priority over stochastic ones:

immediate transitions always fire first, if they can.

A transition $t \in T_N$ is *enabled* at M if ${}^\bullet t \subseteq M$. $Ena(M)$ is the set of *all transitions enabled at* M.

A set of transitions $U \subseteq Ena(M)$ is *fireable* at M, if ${}^\bullet U \subseteq M$ and one of the following holds:

1. $\emptyset \neq U \subseteq Ti_N$; or
2. $U \subseteq Ts_N$ and $Ena(M) \subseteq Ts_N$.

$Fire(M)$ is the set of *all transition sets fireable at* M.

$Fire(M) \subseteq 2^{Ti_N} \setminus \{\emptyset\}$ or $Fire(M) \subseteq 2^{Ts_N}$.

The marking M is *tangible*, $tang(M)$, if $Fire(M) \subseteq 2^{Ts_N}$ (we always have $\emptyset \in Fire(M)$).

The marking M is *vanishing*, $vanish(M)$, if $Fire(M) \subseteq 2^{Ti_N} \setminus \{\emptyset\}$.

A transition $t \in Ena(M)$ is *fireable* at M, $t \in Fire(M)$, if $\{t\} \in Fire(M)$.

If $stang(M)$ then a stochastic transition $t \in Fire(M)$ fires with probability $\Omega_N(t)$,
if no different stochastic transition is fireable in Q, i.e. $Fire(Q) = \{\emptyset, \{t\}\}$.

By the definition of fireability, $\forall U \in Fire(Q)\ 2^U \setminus \{\emptyset\} \subseteq Fire(Q)$.

Let $U \in Fire(M)$ and $U \neq \emptyset$. The *probability of the set of stochastic transitions* or the *weight of the set of immediate transitions U which is ready for firing at M* is

$$PF(U,M) = \begin{cases} \prod_{t \in U} \Omega_N(t) \cdot \prod_{\{u \in Fire(M) | u \notin U\}} (1 - \Omega_N(u)), & tang(M); \\ \sum_{t \in U} \Omega_N(t), & vanish(M). \end{cases}$$

In the case $U = \emptyset$ and $tang(M)$ we define

$$PF(\emptyset,M) = \begin{cases} \prod_{u \in Fire(M)} (1 - \Omega_N(u)), & Fire(M) \neq \{\emptyset\}; \\ 1, & Fire(M) = \{\emptyset\}. \end{cases}$$

Let $U \in Fire(Q)$. The *probability that the set of transitions U fires at M*:

$$PT(U,M) = \frac{PF(U,M)}{\sum_{V \in Fire(M)} PF(V,M)}.$$

If $U = \emptyset$ and $tang(M)$ then $M = \widetilde{M}$.

If $tang(M)$ then $PT(\emptyset, M) \in (0;1]$: the residence time in M is ≥ 1.

Firing of U changes M to $\widetilde{M} = M - {}^{\bullet}U + U^{\bullet}$, $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT(U,M)$.

The *probability to move from M to $\widetilde{M}$ by firing any set of transitions*:

$$PM(M,\widetilde{M}) = \sum_{\{U \mid M \xrightarrow{U} \widetilde{M}\}} PT(U,M).$$

We write $M \xrightarrow{U} \widetilde{M}$ if $\exists \mathcal{P}\ M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$ and $M \rightarrow \widetilde{M}$ if $\exists U\ M \xrightarrow{U} \widetilde{M}$.

Definition 15 *Let N be an LDTSIPN.*

- *The reachability set $RS(N)$ is the minimal set of markings s.t.*
  - $M_N \in RS(N)$;
  - *if $M \in RS(N)$ and $M \rightarrow \widetilde{M}$ then $\widetilde{M} \in RS(N)$.*
- *The reachability graph $RG(N)$ is a directed labeled graph with*
  - *the set of nodes $RS(N)$;*
  - *an arc labeled by $(U, \mathcal{P})$ from node M to $\widetilde{M}$ if $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$.*

The set of *all tangible markings from $RS(N)$* is $RS_T(N)$.

The set of *all vanishing markings from $RS(N)$* is $RS_V(N)$.

$RS(N) = RS_T(N) \cup RS_V(N)$.

Algebra of dtsi-boxes

Definition 16 *A* discrete time stochastic and immediate Petri box (dtsi-box) *is* $N = (P_N, T_N, W_N, \Lambda_N)$, *where:*

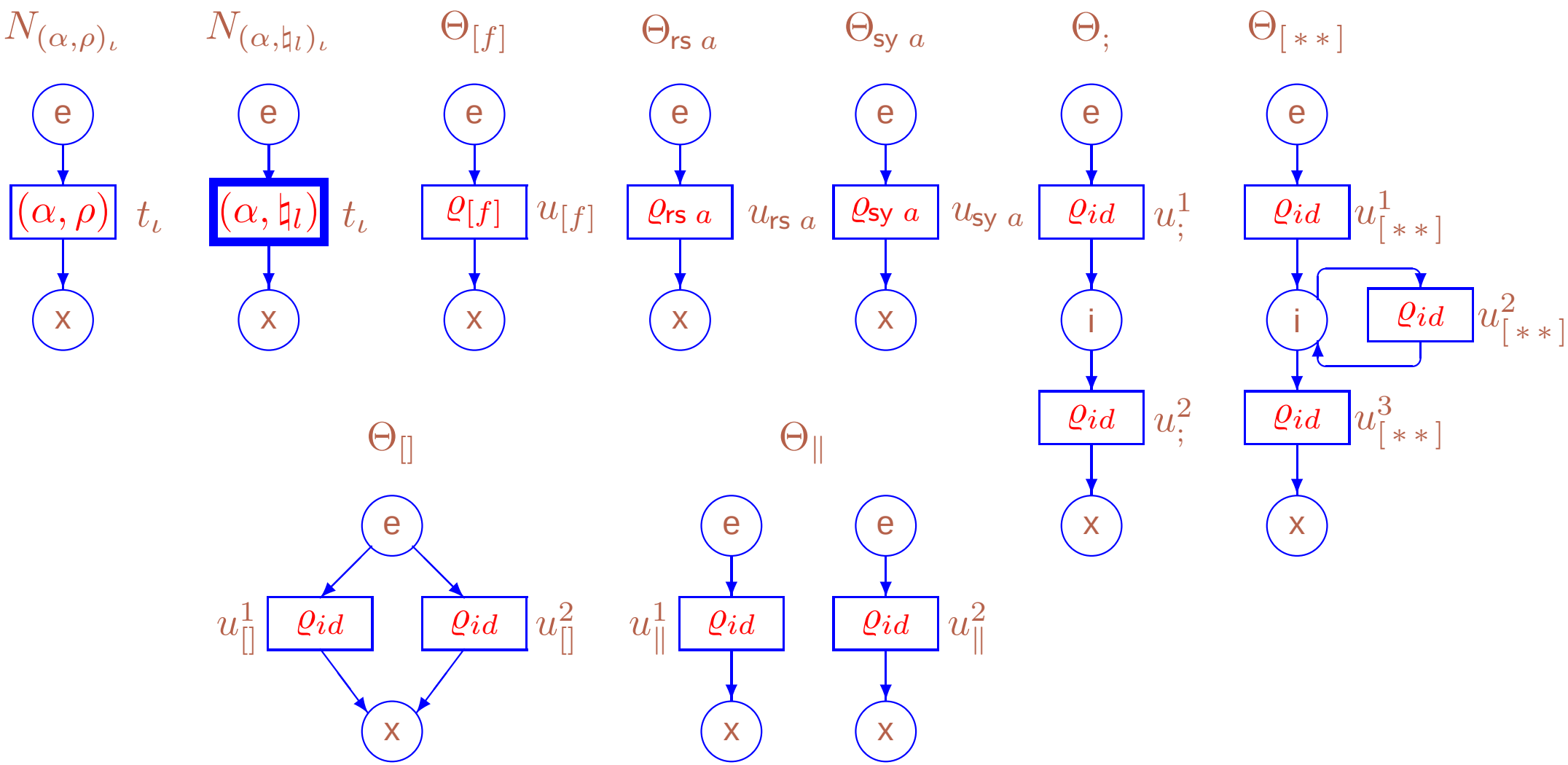
- P_N *and* T_N *are finite sets of* places *and* transitions, *s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$;
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbb{N}$ *is a function of the* weights of arcs *between places and transitions and vice versa;*
- Λ_N *is the* place and transition labeling *function s.t.*
  - $\Lambda_N|_{P_N} : P_N \to \{\mathsf{e}, \mathsf{i}, \mathsf{x}\}$ *(it specifies* entry, internal *and* exit *places);*
  - $\Lambda_N|_{T_N} : T_N \to \{\varrho \mid \varrho \subseteq \mathbb{N}^{\mathcal{SL}}_{fin} \times \mathcal{SL}\}$ *(it associates transitions with the* relabeling relations*).*

Moreover, $\forall t \in T_N \; {}^\bullet t \neq \emptyset \neq t^\bullet$.

For the set of entry *places of* N, ${}^\circ N = \{p \in P_N \mid \Lambda_N(p) = \mathsf{e}\}$, *and the set of* exit *places of* N, $N^\circ = \{p \in P_N \mid \Lambda_N(p) = \mathsf{x}\}$, *it holds:* ${}^\circ N \neq \emptyset \neq N^\circ$ *and* ${}^\bullet({}^\circ N) = \emptyset = (N^\circ)^\bullet$.

A dtsi-box is *plain* if $\forall t \in T_N \; \Lambda_N(t) = \varrho_{(\alpha,\kappa)}$, where $\varrho_{(\alpha,\kappa)} = \{(\emptyset, (\alpha, \kappa))\}$ is a *constant relabeling*, identified with (α, κ).

A *marked plain dtsi-box* is a pair (N, M_N), where N is a plain dtsi-box and $M_N \in \mathbb{N}^{P_N}_{fin}$ is its marking. Let $\overline{N} = (N, {}^\circ N)$ and $\underline{N} = (N, N^\circ)$.



BOXOPS: The plain and operator dtsi-boxes

Definition 17 *Let* $(\alpha, \kappa) \in \mathcal{SIL}$, $a \in Act$ *and* $E, F, K \in RegStatExpr$. *The* denotational semantics *of* $dtsiPBC$ *is a mapping* Box_{dtsi} *from* $RegStatExpr$ *into plain dtsi-boxes:*

1. $Box_{dtsi}((\alpha, \kappa)_\iota) = N_{(\alpha,\kappa)_\iota}$;
2. $Box_{dtsi}(E \circ F) = \Theta_\circ(Box_{dtsi}(E), Box_{dtsi}(F))$, $\circ \in \{;, [], \|\}$;
3. $Box_{dtsi}(E[f]) = \Theta_{[f]}(Box_{dtsi}(E))$;
4. $Box_{dtsi}(E \circ a) = \Theta_{\circ a}(Box_{dtsi}(E))$, $\circ \in \{\mathsf{rs}, \mathsf{sy}\}$;
5. $Box_{dtsi}([E * F * K]) = \Theta_{[**]}(Box_{dtsi}(E), Box_{dtsi}(F), Box_{dtsi}(K))$.

For $E \in RegStatExpr$, let $Box_{dtsi}(\overline{E}) = \overline{Box_{dtsi}(E)}$ and $Box_{dtsi}(\underline{E}) = \underline{Box_{dtsi}(E)}$.

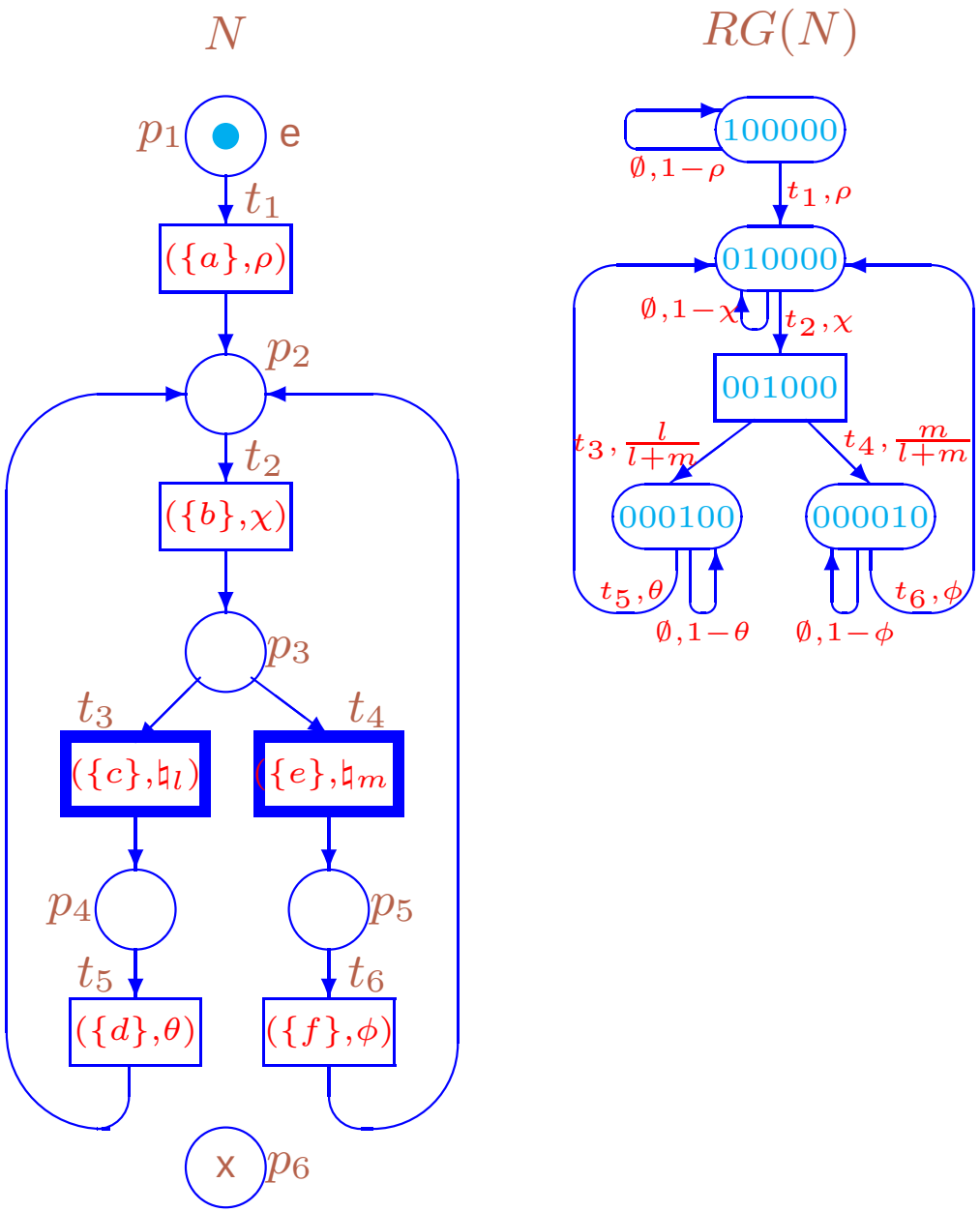
We denote isomorphism of transition systems by $\simeq$,

and the same symbol denotes isomorphism of reachability graphs and DTMCs

as well as isomorphism between transition systems and reachability graphs.

Theorem 1 *(OPDNSEM) For any static expression* E

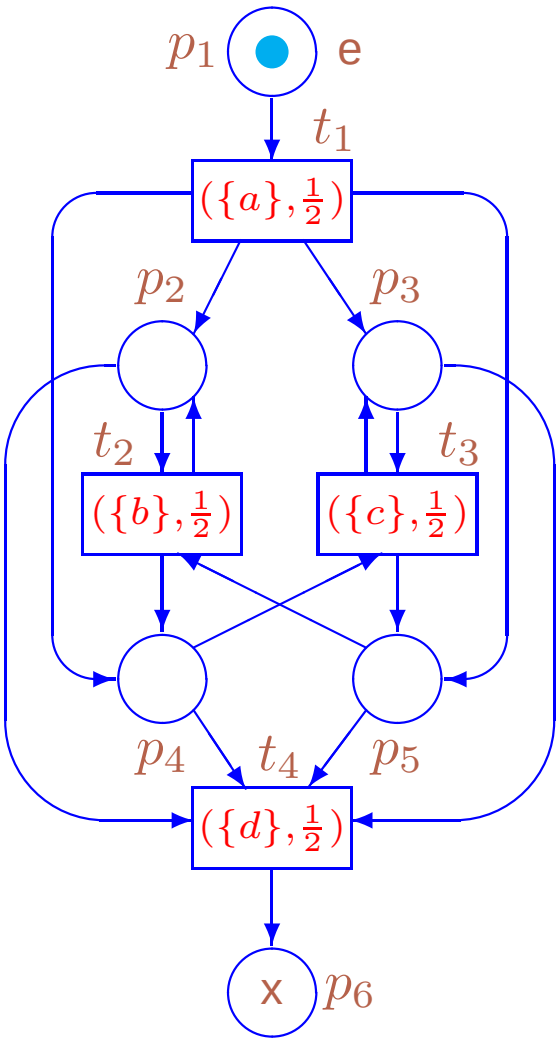
$$TS(\overline{E}) \simeq RG(Box_{dtsi}(\overline{E})).$$



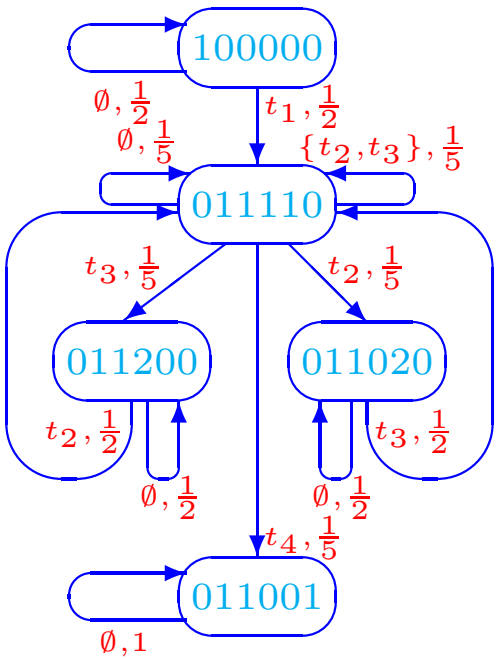
BOXRG: The marked dtsi-box $N = Box_{dtsi}(\overline{E})$ for

$E = [(\{a\},\rho) * ((\{b\},\chi);((((\{c\},\lightning_l);(\{d\},\theta))[]((\{e\},\lightning_m);(\{f\},\phi)))) * \mathsf{Stop}]$ and its reachability graph

N



$RG(N)$



NRBOXRG: The marked dtsi-box $N = Box_{dtsi}(\overline{E})$ for $E = [((\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}) \| (\{c\},\frac{1}{2})) * (\{d\},\frac{1}{2})]$

and its reachability graph

$M_1 = (1, 0, 0, 0, 0, 0)$ is the initial marking.

$M_2 = (0, 1, 1, 1, 1, 0)$ is obtained from M_1 by firing t_1.

$M_3 = (0, 1, 1, 2, 0, 0)$ is obtained from M_2 by firing t_2 and has 2 tokens in the place p_4.

$M_4 = (0, 1, 1, 0, 2, 0)$ is obtained from M_2 by firing t_3 and has 2 tokens in the place p_5.

Concurrency in the second argument of iteration in $\overline{E}$ can lead to non-safeness of the corresponding marked dtsi-box N, but it is 2-bounded in the worst case.

The origin of the problem: N has as a self-loop with two subnets which can function independently.

Performance evaluation

Analysis of the underlying SMC

For a dynamic expression G, a discrete random variable is associated with every tangible state from $DR_T(G)$.

The random variables (residence time in the tangible states) are geometrically distributed:
the probability to stay in the tangible state $s \in DR_T(G)$ for $k-1$ moments
and leave it at the moment $k \geq 1$ is $PM(s,s)^{k-1}(1-PM(s,s))$.

The mean value formula: the *average sojourn time in the tangible state* s is $\frac{1}{1-PM(s,s)}$.

The *average sojourn time in the vanishing state* s is 0.

The *average sojourn time in the state* s is

$$SJ(s) = \begin{cases} \frac{1}{1-PM(s,s)}, & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

The *average sojourn time vector* SJ of G has the elements $SJ(s),\ s \in DR(G)$.

The *sojourn time variance in the state* s is

$$VAR(s) = \begin{cases} \frac{PM(s,s)}{(1-PM(s,s))^2}, & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

The *sojourn time variance vector* VAR of G has the elements $VAR(s),\ s \in DR(G)$.

The stochastic process associated with a dynamic expression G: the *underlying semi-Markov chain (SMC)* of G, $SMC(G)$.

$SMC(G)$ is analyzed by extracting the *embedded (absorbing) discrete time Markov chain (EDTMC)* of G, $EDTMC(G)$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

Let $s \to s$. The *probability to stay in* s *due to* k $(k \geq 1)$ *self-loops* is $PM(s,s)^k$.

The *self-loops abstraction factor in the state* s is

$$SL(s) = \begin{cases} \frac{1}{1-PM(s,s)}, & s \to s; \\ 1, & \text{otherwise}. \end{cases}$$

The *self-loops abstraction vector* SL of G has the elements $SL(s),\ s \in DR(G)$.

Let $s \to \tilde{s}$ and $s \neq \tilde{s}$, i.e. $PM(s,s) < 1$. The *probability to move from* s *to* $\tilde{s}$ *by executing any multiset of activities after possible self-loops* is

$$PM^*(s,\tilde{s}) = \left\{ \begin{array}{ll} PM(s,\tilde{s}) \sum_{k=0}^{\infty} PM(s,s)^k = \frac{PM(s,\tilde{s})}{1-PM(s,s)}, & s \to s; \\ PM(s,\tilde{s}), & \text{otherwise}; \end{array} \right\} = SL(s)PM(s,\tilde{s}).$$

We have $\forall s \in DR_T(G)\ SL(s) = \frac{1}{1-PM(s,s)} = SJ(s)$, hence,
$\forall s \in DR_T(G)$ with $PM(s,s) < 1$ it holds $PM^*(s,\tilde{s}) = SJ(s)PM(s,\tilde{s})$.

Definition 18 *Let G be a dynamic expression. The embedded (absorbing) discrete time Markov chain (EDTMC) of G, $EDTMC(G)$, has the state space $DR(G)$, the initial state $[G]_{\approx}$ and the transitions $s \twoheadrightarrow_{\mathcal{P}} \tilde{s}$, if $s \to \tilde{s}$ and $s \neq \tilde{s}$, where $\mathcal{P} = PM^*(s, \tilde{s})$; or $s \twoheadrightarrow_1 s$, if $PM(s, s) = 1$.*

The underlying SMC of G, $SMC(G)$, has the EDTMC $EDTMC(G)$ and the sojourn time in every $s \in DR_T(G)$ is geometrically distributed with the parameter $1 - PM(s, s)$ while the sojourn time in every $s \in DR_V(G)$ is equal to zero.

For $E \in RegStatExpr$, let $EDTMC(E) = EDTMC(\overline{E})$ and $SMC(E) = SMC(\overline{E})$.

Let G be a dynamic expression. The elements $\mathcal{P}^*_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of *(one-step) transition probability matrix (TPM)* $\mathbf{P}^*$ *for* $EDTMC(G)$:

$$\mathcal{P}^*_{ij} = \begin{cases} PM^*(s_i, s_j), & s_i \to s_j,\ i \neq j; \\ 1, & PM(s_i, s_i) = 1,\ i = j; \\ 0, & \text{otherwise}. \end{cases}$$

The *transient (k-step, $k \in I\!N$) probability mass function (PMF)* $\psi^*[k] = (\psi^*[k](s_1), \ldots, \psi^*[k](s_n))$ *for* $EDTMC(G)$ is calculated as

$$\psi^*[k] = \psi^*[0](\mathbf{P}^*)^k,$$

where $\psi^*[0] = (\psi^*[0](s_1), \ldots, \psi^*[0](s_n))$ is the *initial PMF*:

$$\psi^*[0](s_i) = \begin{cases} 1, & s_i = [G]_{\approx}; \\ 0, & \text{otherwise}. \end{cases}$$

We have $\psi^*[k+1] = \psi^*[k]\mathbf{P}^*$ $(k \in I\!N)$.

The *steady-state PMF* $\psi^* = (\psi^*(s_1), \ldots, \psi^*(s_n))$ *for* $EDTMC(G)$ is a solution of

$$\begin{cases} \psi^*(\mathbf{P}^* - \mathbf{I}) = \mathbf{0} \\ \psi^* \mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of order n and $\mathbf{0}$ is a row vector of n values 0, $\mathbf{1}$ is that of n values 1.

When $EDTMC(G)$ has the single steady state, $\psi^* = \lim_{k\to\infty} \psi^*[k]$.

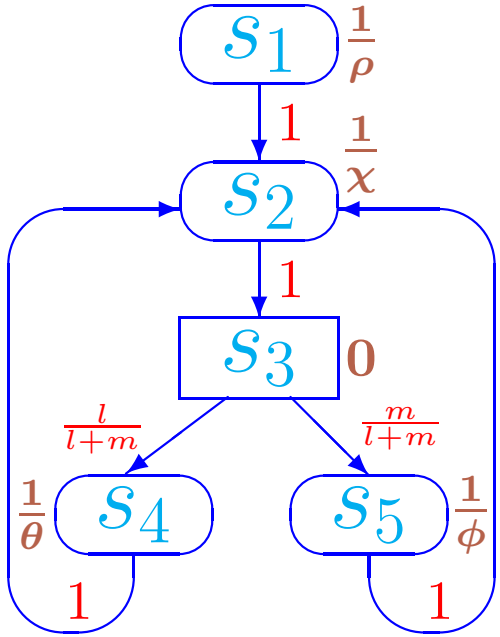
The *steady-state PMF* $\varphi = (\varphi(s_1), \ldots, \varphi(s_n))$ *for* $SMC(G)$:

$$\varphi(s_i) = \begin{cases} \frac{\psi^*(s_i)SJ(s_i)}{\sum_{j=1}^{n} \psi^*(s_j)SJ(s_j)}, & s_i \in DR_T(G); \\ 0, & s_i \in DR_V(G). \end{cases}$$

To calculate φ, we apply abstracting from self-loops with probability less than 1 to get $\mathbf{P}^*$ and ψ^*, followed by weighting by SJ and normalization.

$EDTMC(G)$ has no self-loops with probability less than 1, unlike $SMC(G)$, hence, the behaviour of $EDTMC(G)$ stabilizes quicker than that of $SMC(G)$, since $\mathbf{P}^*$ has only zero (excepting the states having self-loops with probability 1) elements at the main diagonal.

$SMC(\overline{E})$



EXPRSMC: The underlying SMC of $\overline{E}$ for

$$E = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta))[]((\{e\}, \natural_m); (\{f\}, \phi)))) * \mathsf{Stop}]$$

The average sojourn time vector of $\overline{E}$:

$$SJ = \left(\frac{1}{\rho}, \frac{1}{\chi}, 0, \frac{1}{\theta}, \frac{1}{\phi}\right).$$

The sojourn time variance vector of $\overline{E}$:

$$VAR = \left(\frac{1-\rho}{\rho^2}, \frac{1-\chi}{\chi^2}, 0, \frac{1-\theta}{\theta^2}, \frac{1-\phi}{\phi^2}\right).$$

The TPM for $EDTMC(\overline{E})$:

$$\mathbf{P}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC(\overline{E})$:

$$\psi^* = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{l}{3(l+m)}, \frac{m}{3(l+m)}\right).$$

The steady-state PMF ψ^* weighted by SJ:

$$\left(0, \frac{1}{3\chi}, 0, \frac{l}{3\theta(l+m)}, \frac{m}{3\phi(l+m)}\right).$$

We normalize the steady-state weighted PMF dividing it by the sum of its components:

$$\psi^* SJ^T = \frac{\theta\phi(l+m) + \chi(\phi l + \theta m)}{3\chi\theta\phi(l+m)}.$$

Thus, the steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \frac{1}{\theta\phi(l+m)+\chi(\phi l+\theta m)}(0, \theta\phi(l+m), 0, \chi\phi l, \chi\theta m).$$

The case $l = m$ and $\theta = \phi$:

$$\varphi = \frac{1}{2(\chi+\theta)}(0, 2\theta, 0, \chi, \chi).$$

Let G be a dynamic expression and $s, \tilde{s} \in DR(G),\ S, \widetilde{S} \subseteq DR(G)$.

The following performance indices (measures) are based on the steady-state PMF for $SMC(G)$.

- The *average recurrence (return) time in the state* s (the number of discrete time units or steps required for this) is $\frac{1}{\varphi(s)}$.
- The *fraction of residence time in the state* s is $\varphi(s)$.
- The *fraction of residence time in the set of states* $S \subseteq DR(G)$ or the *probability of the event determined by a condition that is true for all states from* S is $\sum_{s \in S} \varphi(s)$.
- The *relative fraction of residence time in the set of states* S *w.r.t. that in* $\widetilde{S}$ is $\frac{\sum_{s \in S} \varphi(s)}{\sum_{\tilde{s} \in \widetilde{S}} \varphi(\tilde{s})}$.
- The *rate of leaving the state* s is $\frac{\varphi(s)}{SJ(s)}$.
- The *steady-state probability to perform a step with a multiset of activities* Ξ is $\sum_{s \in DR(G)} \varphi(s) \sum_{\{\Upsilon | \Xi \subseteq \Upsilon\}} PT(\Upsilon, s)$.
- The *probability of the event determined by a reward function* r *on the states* is $\sum_{s \in DR(G)} \varphi(s) r(s)$, where $\forall s \in DR(G)\ 0 \leq r(s) \leq 1$.

Let $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$ be a LDTSIPN and $M, \widetilde{M} \in I\!N_{fin}^{P_N}$.

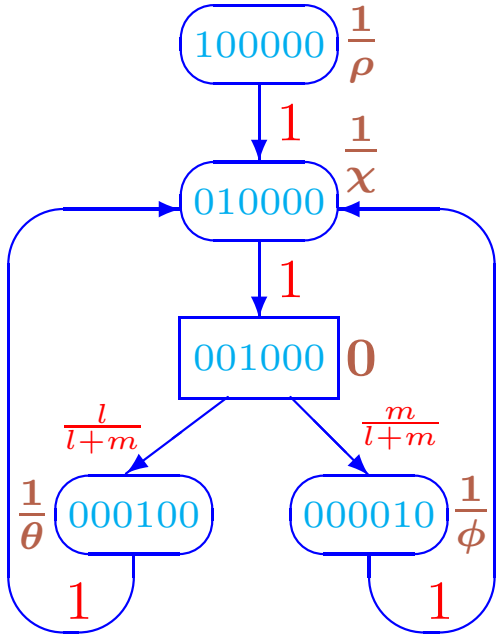
The average sojourn time $SJ(M)$, the sojourn time variance $VAR(M)$, the probabilities $PM^*(M, \widetilde{M})$, the transition relation $M \twoheadrightarrow_{\mathcal{P}} \widetilde{M}$, the *EDTMC* $EDTMC(N)$, the *underlying SMC* $SMC(N)$ and the steady-state PMF for it are defined like for dynamic expressions.

We denote isomorphism of SMCs by $\simeq$.

Proposition 1 *(SMCS) For any static expression* E

$$SMC(\overline{E}) \simeq SMC(Box_{dtsi}(\overline{E})).$$

$SMC(N)$



BOXSMC: The underlying SMC of $N = Box_{dtsi}(\overline{E})$ for

$$E = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta))[]((\{e\}, \natural_m); (\{f\}, \phi)))) * \mathsf{Stop}]$$

Analysis of the DTMC

Definition 19 *Let G be a dynamic expression. The discrete time Markov chain (DTMC) of G, $DTMC(G)$, has the state space $DR(G)$, the initial state $[G]_{\approx}$ and the transitions $s \rightarrow_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = PM(s, \tilde{s})$.*

For $E \in RegStatExpr$, let $DTMC(E) = DTMC(\overline{E})$.

Let G be a dynamic expression. The elements $\mathcal{P}_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of (one-step) transition probability matrix (TPM) $\mathbf{P}$ for $DTMC(G)$ are

$$\mathcal{P}_{ij} = \begin{cases} PM(s_i, s_j), & s_i \rightarrow s_j; \\ 0, & \text{otherwise}. \end{cases}$$

The steady-state PMF ψ for $DTMC(G)$ is defined like that for $EDTMC(G)$.

Theorem 2 *(PMFS) Let G be a dynamic expression and SL be its self-loops abstraction vector. Then the steady-state PMFs ψ for $DTMC(G)$ and ψ^* for $EDTMC(G)$ are related as: $\forall s \in DR(G)$*

$$\psi(s) = \frac{\psi^*(s)SL(s)}{\sum_{\tilde{s}\in DR(G)} \psi^*(\tilde{s})SL(\tilde{s})}.$$

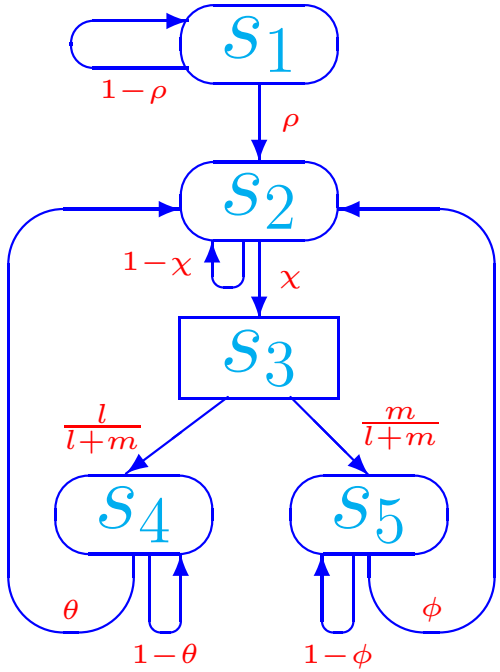
Proposition 2 *(PMFSMC) Let G be a dynamic expression, φ be the steady-state PMF for $SMC(G)$ and ψ be the steady-state PMF for $DTMC(G)$. Then $\forall s \in DR(G)$*

$$\varphi(s) = \begin{cases} \frac{\psi(s)}{\sum_{\tilde{s}\in DR_T(G)} \psi(\tilde{s})}, & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

To calculate φ, we apply normalization to some elements of ψ (corresponding to the tangible states), instead of abstracting from self-loops with probability less than 1 to get $\mathbf{P}^*$ and ψ^*, followed by weighting by SJ and normalization.

Using $DTMC(G)$ instead of $EDTMC(G)$ allows one to avoid multistage analysis. $DTMC(G)$ may have self-loops with probability less than 1, unlike $EDTMC(G)$, hence, the behaviour of $DTMC(G)$ stabilizes slower than that of $EDTMC(G)$ and $\mathbf{P}$ is denser matrix than $\mathbf{P}^*$, since $\mathbf{P}$ may have additional non-zero elements at the main diagonal.

$DTMC(\overline{E})$



EXPRDTMC: The DTMC of $\overline{E}$ for

$$E = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta))[]((\{e\}, \natural_m); (\{f\}, \phi)))) * \mathsf{Stop}]$$

The TPM for $DTMC(\overline{E})$:

$$\mathbf{P} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 & 0 \\ 0 & 1-\chi & \chi & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & \theta & 0 & 1-\theta & 0 \\ 0 & \phi & 0 & 0 & 1-\phi \end{pmatrix}.$$

The steady-state PMF for $DTMC(\overline{E})$:

$$\psi = \frac{1}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}(0, \theta\phi(l+m), \chi\theta\phi(l+m), \chi\phi l, \chi\theta m).$$

Since $DR_T(\overline{E}) = \{s_1, s_2, s_4, s_5\}$, $DR_V(\overline{E}) = \{s_3\}$ and by Proposition PMFSMC:

$$\sum_{\tilde{s} \in DR_T(\overline{E})} \psi(\tilde{s}) = \psi(s_1) + \psi(s_2) + \psi(s_4) + \psi(s_5) = \frac{\theta\phi(l+m) + \chi(\phi l + \theta m)}{\theta\phi(1+\chi)(l+m) + \chi(\phi l + \theta m)}.$$

$$\begin{aligned}
\varphi(s_1) &= 0 \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = 0,\\
\varphi(s_2) &= \frac{\theta\phi(l+m)}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)} \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = \frac{\theta\phi(l+m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)},\\
\varphi(s_3) &= 0,\\
\varphi(s_4) &= \frac{\chi\phi l}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)} \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = \frac{\chi\phi l}{\theta\phi(l+m)+\chi(\phi l+\theta m)},\\
\varphi(s_5) &= \frac{\chi\theta m}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)} \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = \frac{\chi\theta m}{\theta\phi(l+m)+\chi(\phi l+\theta m)}.
\end{aligned}$$

The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \frac{1}{\theta\phi(l+m) + \chi(\phi l + \theta m)}(0, \theta\phi(l+m), 0, \chi\phi l, \chi\theta m).$$

This coincides with the result obtained with the use of ψ^* and SJ.

Analysis of the reduced DTMC

Let G be a dynamic expression and $\mathbf{P}$ be the TPM for $DTMC(G)$.

Reordering the states from $DR(G)$: the first rows and columns of $\mathbf{P}$ correspond to the states from $DR_V(G)$ and the last ones correspond to the states from $DR_T(G)$.

Let $|DR(G)| = n$ and $|DR_T(G)| = m$. The resulting matrix is decomposed as:

$$\mathbf{P} = \begin{pmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{pmatrix}.$$

The elements of the $(n-m) \times (n-m)$ submatrix $\mathbf{C}$: the probabilities to move from vanishing to vanishing states.

The elements of the $(n-m) \times m$ submatrix $\mathbf{D}$: the probabilities to move from vanishing to tangible states.

The elements of the $m \times (n-m)$ submatrix $\mathbf{E}$: the probabilities to move from tangible to vanishing states.

The elements of the $m \times m$ submatrix $\mathbf{F}$: the probabilities to move from tangible to tangible states.

The TPM $\mathbf{P}^{\diamond}$ for $RDTMC(G)$ is the $m \times m$ matrix:

$$\mathbf{P}^{\diamond} = \mathbf{F} + \mathbf{EGD},$$

where the elements of the matrix $\mathbf{G}$ are the probabilities to move from vanishing to vanishing states in any number of state transitions, without traversal of tangible states:

$$\mathbf{G} = \sum_{k=0}^{\infty} \mathbf{C}^k = \begin{cases} \sum_{k=0}^{l} \mathbf{C}^k, & \exists l \in I\!N\ \forall k > l\ \mathbf{C}^k = \mathbf{0}, \quad \text{no loops among vanishing states;} \\ (\mathbf{I} - \mathbf{C})^{-1}, & \lim_{k \to \infty} \mathbf{C}^k = \mathbf{0}, \quad \text{loops among vanishing states;} \end{cases}$$

where $\mathbf{0}$ is the square matrix consisting only of zeros and $\mathbf{I}$ is the identity matrix, both of size $n - m$.

For $1 \leq i, j \leq m$ and $1 \leq k, l \leq n - m$, let

$\mathcal{F}_{ij}$ be the elements of the matrix $\mathbf{F}$, $\mathcal{E}_{ik}$ be those of $\mathbf{E}$, $\mathcal{G}_{kl}$ be those of $\mathbf{G}$ and $\mathcal{D}_{lj}$ be those of $\mathbf{D}$.

The elements $\mathcal{P}^{\diamond}_{ij}$ of the matrix $\mathbf{P}^{\diamond}$ are

$$\mathcal{P}^{\diamond}_{ij} = \mathcal{F}_{ij} + \sum_{k=1}^{n-m} \sum_{l=1}^{n-m} \mathcal{E}_{ik}\mathcal{G}_{kl}\mathcal{D}_{lj} = \mathcal{F}_{ij} + \sum_{k=1}^{n-m} \mathcal{E}_{ik} \sum_{l=1}^{n-m} \mathcal{G}_{kl}\mathcal{D}_{lj} = \mathcal{F}_{ij} + \sum_{l=1}^{n-m} \mathcal{D}_{lj} \sum_{k=1}^{n-m} \mathcal{E}_{ik}\mathcal{G}_{kl},$$

i.e. $\mathcal{P}^{\diamond}_{ij}$ $(1 \leq i, j \leq m)$ is the total probability to move from the tangible state s_i to the tangible state s_j in any number of steps, without traversal of tangible states, but possibly going through vanishing states.

Let $s, \tilde{s} \in DR_T(G)$ such that $s = s_i,\ \tilde{s} = s_j$.

The *probability to move from s to $\tilde{s}$ in any number of steps, without traversal of tangible states* is

$$PM^{\diamond}(s, \tilde{s}) = \mathcal{P}^{\diamond}_{ij}.$$

Definition 20 *Let G be a dynamic expression and $[G]_{\approx} \in DR_T(G)$.*

The reduced discrete time Markov chain (RDTMC) of G, denoted by $RDTMC(G)$, has the state space $DR_T(G)$, the initial state $[G]_{\approx}$ and the transitions $s \hookrightarrow_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = PM^{\diamond}(s, \tilde{s})$.

RDTMCs of static expressions can be defined as well. For $E \in RegStatExpr$, let $RDTMC(E) = RDTMC(\overline{E})$.

Let $DR_T(G) = \{s_1, \ldots, s_m\}$ and $[G]_{\approx} \in DR_T(G)$. The transient ($k$-step, $k \in I\!N$) probability mass function (PMF) $\psi^{\diamond}[k] = (\psi^{\diamond}[k](s_1), \ldots, \psi^{\diamond}[k](s_m))$ for $RDTMC(G)$ is calculated as

$$\psi^{\diamond}[k] = \psi^{\diamond}[0](\mathbf{P}^{\diamond})^k,$$

where $\psi^{\diamond}[0] = (\psi^{\diamond}[0](s_1), \ldots, \psi^{\diamond}[0](s_m))$ is the initial PMF:

$$\psi^{\diamond}[0](s_i) = \begin{cases} 1, & s_i = [G]_{\approx}; \\ 0, & \text{otherwise}. \end{cases}$$

We have $\psi^{\diamond}[k+1] = \psi^{\diamond}[k]\mathbf{P}^{\diamond}$ $(k \in I\!N)$.

The steady-state PMF $\psi^\diamond = (\psi^\diamond(s_1), \ldots, \psi^\diamond(s_m))$ for $RDTMC(G)$ is a solution of:

$$\begin{cases} \psi^\diamond(\mathbf{P}^\diamond - \mathbf{I}) = \mathbf{0} \\ \psi^\diamond \mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of size m and $\mathbf{0}$ is a row vector of m values 0, $\mathbf{1}$ is that of m values 1.

When $RDTMC(G)$ has the single steady state, $\psi^\diamond = \lim_{k\to\infty} \psi^\diamond[k]$.

Proposition 3 *(PMFSMCT) Let G be a dynamic expression, φ be the steady-state PMF for $SMC(G)$ and $\psi^\diamond$ be the steady-state PMF for $RDTMC(G)$. Then $\forall s \in DR(G)$*

$$\varphi(s) = \begin{cases} \psi^\diamond(s), & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

To calculate φ, we take all the elements of $\psi^\diamond$ as the steady-state probabilities of the tangible states, instead of abstracting from self-loops with probability less than 1 to get $\mathbf{P}^*$ and ψ^*, followed by weighting by SJ and normalization.

Using $RDTMC(G)$ instead of $EDTMC(G)$ allows one to avoid multistage analysis.

Constructing $\mathbf{P}^\diamond$ requires calculating matrix powers or inverse matrices.

$RDTMC(G)$ has self-loops, unlike $EDTMC(G)$, hence,
the behaviour of $RDTMC(G)$ may stabilize slower than that of $EDTMC(G)$.
$\mathbf{P}^{\diamond}$ is smaller and denser matrix than $\mathbf{P}^{*}$, since $\mathbf{P}^{\diamond}$ has non-zero elements
at the main diagonal and many of them outside it.

The complexity of the analytical calculation of $\psi^{\diamond}$ w.r.t. ψ^{*} depends on the model structure:
the number of vanishing states and loops among them.
Usually it is lower, since the matrix size reduction plays an important role.

The elimination of vanishing states.

- The system models with many immediate activities:

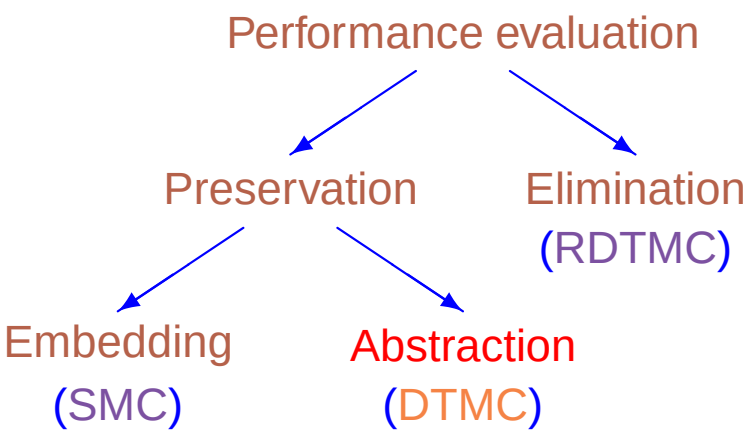
  significant simplification of the solution.

- The abstraction level of SMCs:

  decreases their impact to the solution complexity.

- The abstraction level of transition systems:

  allows immediate activities to specify logical structure.



PEVMETHS: Performance evaluation methods in $dtsiPBC$

$E = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta))[]((\{e\}, \natural_m); (\{f\}, \phi)))) * \mathsf{Stop}]$.

$DR_T(\overline{E}) = \{s_1, s_2, s_4, s_5\}$ and $DR_V(\overline{E}) = \{s_3\}$.

We reorder the states from $DR(\overline{E})$, by moving the vanishing states to the first positions:

s_3, s_1, s_2, s_4, s_5.

The reordered TPM for $DTMC(\overline{E})$:

$$\mathbf{P}_r = \begin{pmatrix} 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1-\rho & \rho & 0 & 0 \\ \chi & 0 & 1-\chi & 0 & 0 \\ 0 & 0 & \theta & 1-\theta & 0 \\ 0 & 0 & \phi & 0 & 1-\phi \end{pmatrix}.$$

The result of the decomposing $\mathbf{P}_r$:

$$\mathbf{C} = 0,\ \mathbf{D} = \left(0, 0, \frac{l}{l+m}, \frac{m}{l+m}\right),\ \mathbf{E} = \begin{pmatrix} 0 \\ \chi \\ 0 \\ 0 \end{pmatrix},\ \mathbf{F} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & 0 & 0 \\ 0 & \theta & 1-\theta & 0 \\ 0 & \phi & 0 & 1-\phi \end{pmatrix}.$$

Since $\mathbf{C}^1 = \mathbf{0}$, we have $\forall k > 0\ \mathbf{C}^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\mathbf{G} = \sum_{k=0}^{l} \mathbf{C}^k = \mathbf{C}^0 = \mathbf{I}.$$

The TPM for $RDTMC(\overline{E})$:

$$\mathbf{P}^{\diamond} = \mathbf{F} + \mathbf{EGD} = \mathbf{F} + \mathbf{EID} = \mathbf{F} + \mathbf{ED} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \frac{\chi l}{l+m} & \frac{\chi m}{l+m} \\ 0 & \theta & 1-\theta & 0 \\ 0 & \phi & 0 & 1-\phi \end{pmatrix}.$$

The steady-state PMF for $RDTMC(\overline{E})$:

$$\psi^{\diamond} = \frac{1}{\theta\phi(l+m) + \chi(\phi l + \theta m)}(0, \theta\phi(l+m), \chi\phi l, \chi\theta m).$$

Note that $\psi^{\diamond} = (\psi^{\diamond}(s_1), \psi^{\diamond}(s_2), \psi^{\diamond}(s_4), \psi^{\diamond}(s_5))$.

By Proposition PMFSMCT:

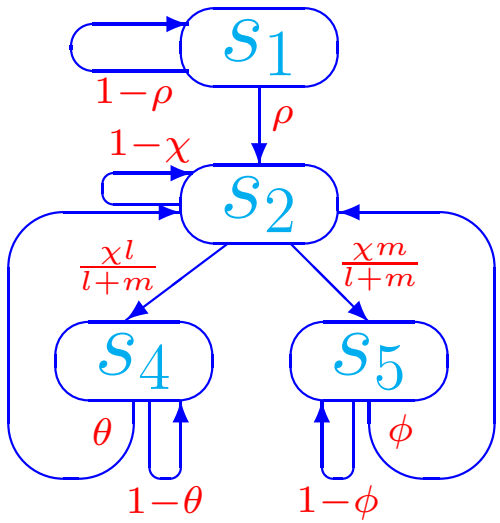
$$\varphi(s_1) = 0,$$
$$\varphi(s_2) = \frac{\theta\phi(l+m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)},$$
$$\varphi(s_3) = 0,$$
$$\varphi(s_4) = \frac{\chi\phi l}{\theta\phi(l+m)+\chi(\phi l+\theta m)},$$
$$\varphi(s_5) = \frac{\chi\theta m}{\theta\phi(l+m)+\chi(\phi l+\theta m)}.$$

The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \frac{1}{\theta\phi(l+m)+\chi(\phi l+\theta m)}(0, \theta\phi(l+m), 0, \chi\phi l, \chi\theta m).$$

This coincides with the result obtained with the use of ψ^* and SJ.

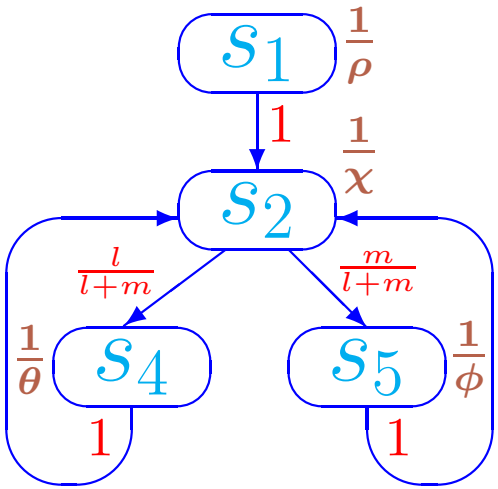
$RDTMC(\overline{E})$



EXPRRDTMC: The reduced DTMC of $\overline{E}$ for

$$E = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta))[]((\{e\}, \natural_m); (\{f\}, \phi)))) * \text{Stop}]$$

$RSMC(\overline{E})$



EXPRRSMC: The reduced SMC of $\overline{E}$ for

$$E = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta))[]((\{e\}, \natural_m); (\{f\}, \phi)))) * \text{Stop}]$$

Theorem 3 *(EREER) Let G be a dynamic expression, $(\mathbf{P}^{\diamond})^*$ results from embedding the TPM $\mathbf{P}^{\diamond}$ for $RDTMC(G)$, and $((\mathbf{P}^*)^{\diamond})^*$ results from reduction and final embedding the TPM $\mathbf{P}^*$ for $EDTMC(G)$. Then*

$$((\mathbf{P}^*)^{\diamond})^* = (\mathbf{P}^{\diamond})^*.$$

Let $E = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta))[]((\{e\}, \natural_m); (\{f\}, \phi)))) * \mathsf{Stop}]$.

The TPMs for $RDTMC(\overline{E})$ and $ERDTMC(\overline{E})$:

$$\mathbf{P}^{\diamond} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \frac{\chi l}{l+m} & \frac{\chi m}{l+m} \\ 0 & \theta & 1-\theta & 0 \\ 0 & \phi & 0 & 1-\phi \end{pmatrix}, \ (\mathbf{P}^{\diamond})^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The TPMs for $REDTMC(\overline{E})$ and $EREDTMC(\overline{E})$:

$$(\mathbf{P}^*)^{\diamond} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ ((\mathbf{P}^*)^{\diamond})^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The self-loops abstraction subvector of $\overline{E}$ for the submatrix $\mathbf{F}$: $SL_F = \left(\frac{1}{\rho}, \frac{1}{\chi}, \frac{1}{\theta}, \frac{1}{\phi}\right)$.

The self-loops abstraction vector of $\overline{E}$ in $REDTMC(\overline{E})$ (for the matrix $\mathbf{H}'$):
$(SL^*)^\diamond = SL_{H'} = (1,1,1,1)$.

The self-loops abstraction vector of $\overline{E}$ in $RDTMC(\overline{E})$:
$SL^\diamond = \mathbf{1}Diag(SL_F)Diag(SL_{H'}) = \left(\frac{1}{\rho}, \frac{1}{\chi}, \frac{1}{\theta}, \frac{1}{\phi}\right)$, where $\mathbf{1}$ is a row vector of n values 1.

The elements of $\mathbf{H}'$ are the probabilities to move from tangible to tangible states, via any *positive* number of vanishing states, without traversal of tangible states, in $EDTMC(G)$. $\mathbf{H}' = Diag(SL_F)\mathbf{H}$.

The elements of $\mathbf{H} = \mathbf{EGD}$ are the probabilities to move from tangible to tangible states, via any *positive* number of vanishing states, without traversal of tangible states, in $DTMC(G)$.

The matrices $\mathbf{H}$ and $\mathbf{H}'$:

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\chi l}{l+m} & \frac{\chi m}{l+m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{H}' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$((\mathbf{P}^*)^\diamond)^* = Diag(SL^\diamond)(\mathbf{P}^\diamond - \mathbf{I}) + \mathbf{I} = Diag(SL_{H'})Diag(SL_F)(\mathbf{P}^\diamond - \mathbf{I}) + \mathbf{I} = (\mathbf{P}^\diamond)^*.$$

Stochastic equivalences

Step stochastic bisimulation equivalence

For $\Upsilon \in I\!N_{fin}^{\mathcal{SIL}}$, we consider $\mathcal{L}(\Upsilon) \in I\!N_{fin}^{\mathcal{L}}$, i.e. (possibly empty) multisets of multiactions.

Let G be a dynamic expression and $\mathcal{H} \subseteq DR(G)$. For $s \in DR(G)$ and $A \in I\!N_{fin}^{\mathcal{L}}$ we write $s \xrightarrow{A}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM_A(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via steps with the multiaction part A*:

$$PM_A(s, \mathcal{H}) = \sum_{\{\Upsilon \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{\Upsilon} \tilde{s},\ \mathcal{L}(\Upsilon) = A\}} PT(\Upsilon, s).$$

We write $s \xrightarrow{A} \mathcal{H}$ if $\exists \mathcal{P}\ s \xrightarrow{A}_{\mathcal{P}} \mathcal{H}$.

We write $s \rightarrow_{\mathcal{P}} \mathcal{H}$ if $\exists A\ s \xrightarrow{A} \mathcal{H}$, where $\mathcal{P} = PM(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via any steps*:

$$PM(s, \mathcal{H}) = \sum_{\{\Upsilon \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{\Upsilon} \tilde{s}\}} PT(\Upsilon, s).$$

Definition 21 *Let G and G' be dynamic expressions. An equivalence relation $\mathcal{R} \subseteq (DR(G) \cup DR(G'))^2$ is a step stochastic bisimulation between G and G', $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, if:*

1. $([G]_{\approx}, [G']_{\approx}) \in \mathcal{R}$.
2. $(s_1, s_2) \in \mathcal{R} \;\Rightarrow\; \forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}\; \forall A \in \mathbb{N}^{\mathcal{L}}_{fin}$

$$s_1 \xrightarrow{A}_{\mathcal{P}} \mathcal{H} \;\Leftrightarrow\; s_2 \xrightarrow{A}_{\mathcal{P}} \mathcal{H}.$$

Two dynamic expressions G and G' are step stochastic bisimulation equivalent, $G \underline{\leftrightarrow}_{ss} G'$, if $\exists \mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$.

Proposition 4 *(BISSPL) Let G and G' be dynamic expressions and $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$. Then*

$$\mathcal{R} \subseteq (DR_T(G) \cup DR_T(G'))^2 \uplus (DR_V(G) \cup DR_V(G'))^2,$$

where $\uplus$ is disjoint union.

$\mathcal{R}_{ss}(G, G') = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G'\}$ is the *union of all step stochastic bisimulations* between G and G'.

Proposition 5 *(LARBIS) Let G and G' be dynamic expressions and $G \underline{\leftrightarrow}_{ss} G'$. Then $\mathcal{R}_{ss}(G, G')$ is the largest step stochastic bisimulation between G and G'.*

Interrelations of the stochastic equivalences

$$\underline{\leftrightarrow}_{ss} \longleftarrow =_{ts} \longleftarrow \approx$$

INTSTEQ: Interrelations of the stochastic equivalences

Theorem 4 *(INTSTEQ) Let* $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, ss, ts\}$*. For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \twoheadleftarrow\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph in Figure INTSTEQ there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\twoheadrightarrow_{\star\star}$*.*

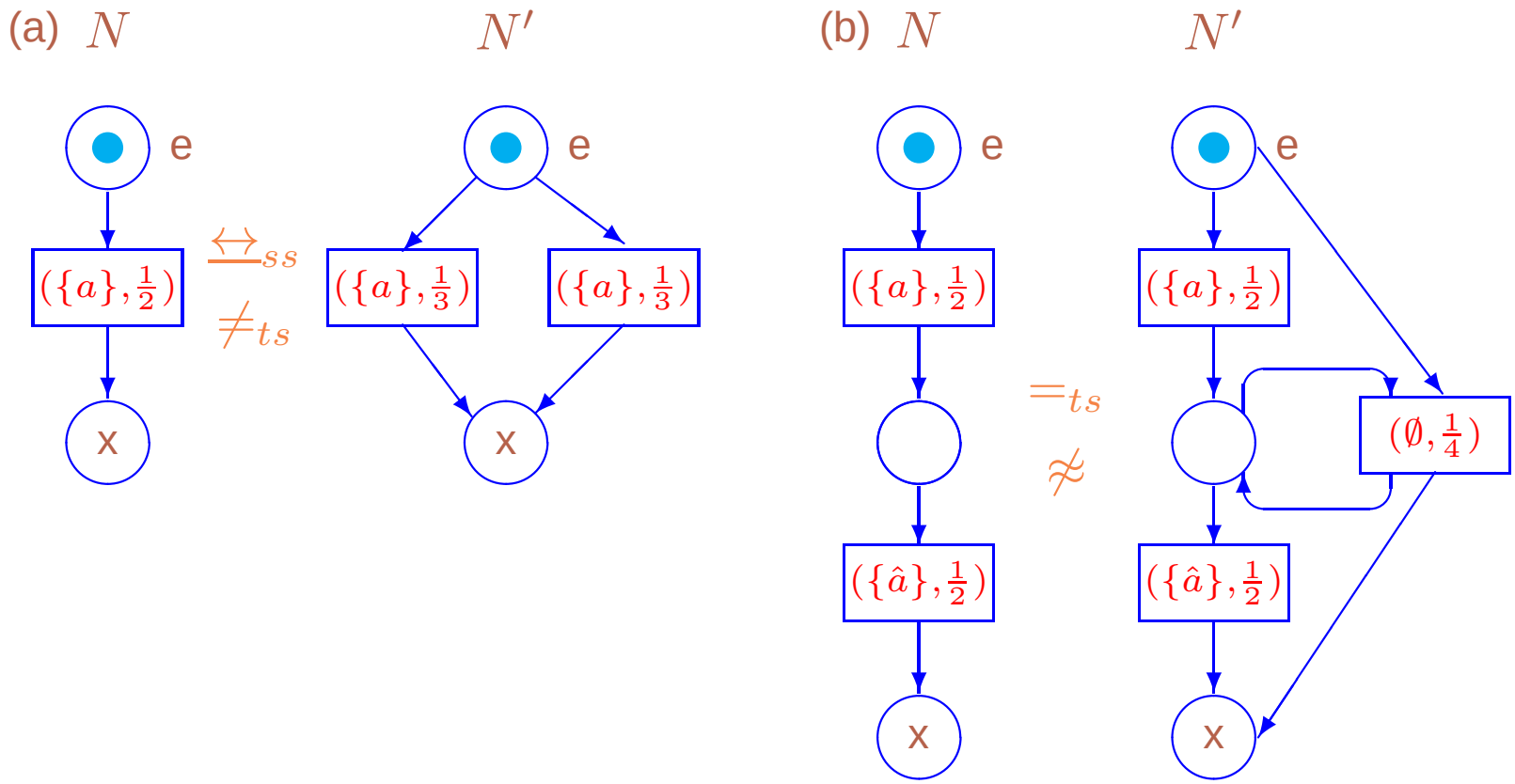
Validity of the implications

- The implication $=_{ts} \rightarrow \underline{\leftrightarrow}_{ss}$ is proved as follows. Let $\beta : G{=_{ts}}G'$. Then $\mathcal{R} : G\underline{\leftrightarrow}_{ss}G'$, where $\mathcal{R} = \{(s, \beta(s)) \mid s \in DR(G)\}$.
- The implication $\approx \rightarrow =_{ts}$ is valid, since the transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

(a) Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{3})_1[](\{a\}, \frac{1}{3})_2$. Then $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$, but $\overline{E}{\neq_{ts}}\overline{E'}$, since $TS(\overline{E})$ has only one transition from the initial to the final state while $TS(\overline{E'})$ has two such ones.

(b) Let $E = (\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E}{=_{ts}}\overline{E'}$, but $\overline{E}{\not\approx}\overline{E'}$, since $\overline{E}$ and $\overline{E'}$ cannot be reached from each other by applying inaction rules.



EXMSTEQ: Dtsi-boxes of the dynamic expressions from equivalence examples of the Theorem INTSTEQ

In Figure EXMSTEQ, $N = Box_{dtsi}(\overline{E})$ and $N' = Box_{dtsi}(\overline{E'})$ for each picture (a)–(b).

Reduction modulo equivalences

An *autobisimulation* is a bisimulation between an expression and itself.

For a dynamic expression G and a step stochastic autobisimulation $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G$,
let $\mathcal{K} \in DR(G)/_{\mathcal{R}}$ and $s_1, s_2 \in \mathcal{K}$.

We have $\forall \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}}\ \forall A \in I\!N^{\mathcal{L}}_{fin}\ s_1 \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}} \ \Leftrightarrow\ s_2 \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

The equality is valid for all $s_1, s_2 \in \mathcal{K}$, hence, we can rewrite it as $\mathcal{K} \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$, where
$\mathcal{P} = PM_A(\mathcal{K}, \widetilde{\mathcal{K}}) = PM_A(s_1, \widetilde{\mathcal{K}}) = PM_A(s_2, \widetilde{\mathcal{K}})$.

We write $\mathcal{K} \stackrel{A}{\rightarrow} \widetilde{\mathcal{K}}$ if $\exists \mathcal{P}\ \mathcal{K} \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$ and $\mathcal{K} \rightarrow \widetilde{\mathcal{K}}$ if $\exists A\ \mathcal{K} \stackrel{A}{\rightarrow} \widetilde{\mathcal{K}}$.

The similar arguments: we write $\mathcal{K} \rightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, where
$\mathcal{P} = PM(\mathcal{K}, \widetilde{\mathcal{K}}) = PM(s_1, \widetilde{\mathcal{K}}) = PM(s_2, \widetilde{\mathcal{K}})$.

Since $\mathcal{R} \subseteq (DR_T(G))^2 \uplus (DR_V(G))^2$, we have $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}}$,
all states from $\mathcal{K}$ are tangible, when $\mathcal{K} \in DR_T(G)/_{\mathcal{R}}$,
or all of them are vanishing, when $\mathcal{K} \in DR_V(G)/_{\mathcal{R}}$.

The *average sojourn time in the equivalence class (w.r.t. $\mathcal{R}$) of states $\mathcal{K}$* is

$$SJ_{\mathcal{R}}(\mathcal{K}) = \begin{cases} \frac{1}{1-PM(\mathcal{K},\mathcal{K})}, & \mathcal{K} \in DR_T(G)/_{\mathcal{R}}; \\ 0, & \mathcal{K} \in DR_V(G)/_{\mathcal{R}}. \end{cases}$$

The *average sojourn time vector for the equivalence classes (w.r.t. $\mathcal{R}$) of states* of G, $SJ_{\mathcal{R}}$, has the elements $SJ_{\mathcal{R}}(\mathcal{K}),\ \mathcal{K} \in DR(G)/_{\mathcal{R}}$.

The *sojourn time variance in the equivalence class (w.r.t. $\mathcal{R}$) of states $\mathcal{K}$* is

$$VAR_{\mathcal{R}}(\mathcal{K}) = \begin{cases} \frac{PM(\mathcal{K},\mathcal{K})}{(1-PM(\mathcal{K},\mathcal{K}))^2}, & \mathcal{K} \in DR_T(G)/_{\mathcal{R}}; \\ 0, & \mathcal{K} \in DR_V(G)/_{\mathcal{R}}. \end{cases}$$

The *sojourn time variance vector for the equivalence classes (w.r.t. $\mathcal{R}$) of states* of G, $VAR_{\mathcal{R}}$, has the elements $VAR_{\mathcal{R}}(\mathcal{K}),\ \mathcal{K} \in DR(G)/_{\mathcal{R}}$.

$\mathcal{R}_{ss}(G) = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G\}$ is the *largest step stochastic autobisimulation* on G.

Definition 22 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) (labeled probabilistic) transition system *of a dynamic expression* G *is* $TS_{\underline{\leftrightarrow}_{ss}}(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$, *where*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\mathcal{R}_{ss}(G)}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq (I\!N^{\mathcal{L}}_{fin}) \times (0;1]$;
- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{K}, (A, PM_A(\mathcal{K}, \widetilde{\mathcal{K}})), \widetilde{\mathcal{K}}) \mid \mathcal{K}, \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}_{ss}(G)},\ \mathcal{K} \stackrel{A}{\rightarrow} \widetilde{\mathcal{K}}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = [[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$.

The transition $(\mathcal{K}, (A, \mathcal{P}), \widetilde{\mathcal{K}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{K} \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

For $E \in RegStatExpr$, let $TS_{\underline{\leftrightarrow}_{ss}}(E) = TS_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Let F be an abstraction of E from the examples above, s.t. $c = e,\ d = f,\ \theta = \phi$:

$$F = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta)_1)[]((\{c\}, \natural_m); (\{d\}, \theta)_2))) * \mathsf{Stop}].$$

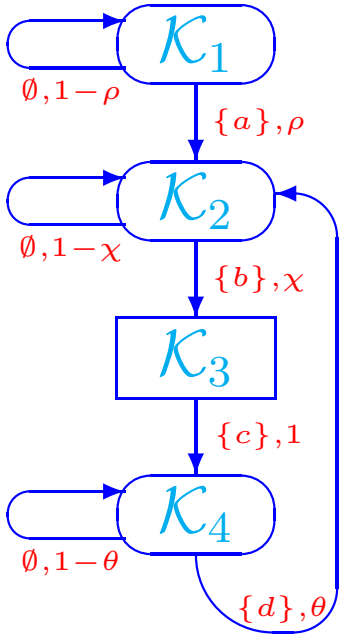
$DR(\overline{F}) = \{s_1, s_2, s_3, s_4, s_5\}$ is obtained from $DR(\overline{E})$
via substitution of $e,\ f,\ \phi$ by $c,\ d,\ \theta$, respectively.

$DR_T(\overline{F}) = \{s_1, s_2, s_4, s_5\}$ and $DR_V(\overline{F}) = \{s_3\}$.

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$,
where $\mathcal{K}_1 = \{s_1\},\ \mathcal{K}_2 = \{s_2\},\ \mathcal{K}_3 = \{s_3\},\ \mathcal{K}_4 = \{s_4, s_5\}$.

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3\}$.

$TS_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



QTS: The quotient transition system of $\overline{F}$ for

$$F = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \lightning_l); (\{d\}, \theta)_1)[]((\{c\}, \lightning_m); (\{d\}, \theta)_2))) * \mathsf{Stop}]$$

The *quotient (by $\underline{\leftrightarrow}_{ss}$) average sojourn time vector* of G is $SJ_{\underline{\leftrightarrow}_{ss}} = SJ_{\mathcal{R}_{ss}(G)}$.

The *quotient (by $\underline{\leftrightarrow}_{ss}$) sojourn time variance vector* of G is $VAR_{\underline{\leftrightarrow}_{ss}} = VAR_{\mathcal{R}_{ss}(G)}$.

Let $\mathcal{K} \to \widetilde{\mathcal{K}}$ and $\mathcal{K} \neq \widetilde{\mathcal{K}}$, i.e. $PM(\mathcal{K},\mathcal{K}) < 1$. The *probability to move from $\mathcal{K}$ to $\widetilde{\mathcal{K}}$ by executing any multiset of activities after possible self-loops* is

$$PM^*(\mathcal{K},\widetilde{\mathcal{K}}) = \begin{cases} PM(\mathcal{K},\widetilde{\mathcal{K}}) \sum_{k=0}^{\infty} PM(\mathcal{K},\mathcal{K})^k = \frac{PM(\mathcal{K},\widetilde{\mathcal{K}})}{1-PM(\mathcal{K},\mathcal{K})}, & \mathcal{K} \to \mathcal{K}; \\ PM(\mathcal{K},\widetilde{\mathcal{K}}), & \text{otherwise}. \end{cases}$$

We have $\forall \mathcal{K} \in DR_T(G)/_{\mathcal{R}_{ss}(G)}\ PM^*(\mathcal{K},\widetilde{\mathcal{K}}) = SJ_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) PM(\mathcal{K},\widetilde{\mathcal{K}})$.

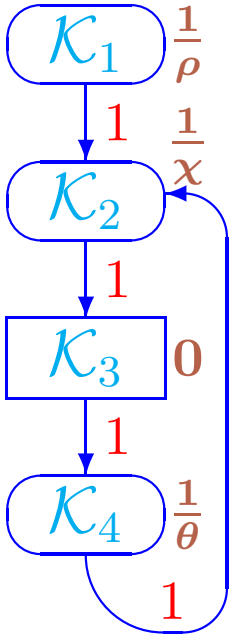
Definition 23 *The quotient (by $\underline{\leftrightarrow}_{ss}$) EDTMC of a dynamic expression G, $EDTMC_{\underline{\leftrightarrow}_{ss}}(G)$, has the state space $DR(G)/_{\mathcal{R}_{ss}(G)}$, the initial state $[[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$ and the transitions $\mathcal{K} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, if $\mathcal{K} \rightarrow \widetilde{\mathcal{K}}$ and $\mathcal{K} \neq \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}})$; or $\mathcal{K} \twoheadrightarrow_1 \mathcal{K}$, if $PM(\mathcal{K}, \mathcal{K}) = 1$.*

The quotient (by $\underline{\leftrightarrow}_{ss}$) underlying SMC of G, $SMC_{\underline{\leftrightarrow}_{ss}}(G)$, has the EDTMC $EDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ and the sojourn time in every $\mathcal{K} \in DR_T(G)/_{\mathcal{R}_{ss}(G)}$ is geometrically distributed with the parameter $1 - PM(\mathcal{K}, \mathcal{K})$ while the sojourn time in every $\mathcal{K} \in DR_V(G)/_{\mathcal{R}_{ss}(G)}$ is equal to zero.

For $E \in RegStatExpr$, let $SMC_{\underline{\leftrightarrow}_{ss}}(E) = SMC_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

The steady-state PMFs $\psi^*_{\underline{\leftrightarrow}_{ss}}$ for $EDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ and $\varphi_{\underline{\leftrightarrow}_{ss}}$ for $SMC_{\underline{\leftrightarrow}_{ss}}(G)$ are defined like ψ^* for $EDTMC(G)$ and φ for $SMC(G)$.

$SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



EXPRQSMC: The quotient underlying SMC of $\overline{F}$ for

$$F = [(\{a\}, \rho) * ((\{b\}, \chi); ((((\{c\}, \natural_l); (\{d\}, \theta)_1)[]((\{c\}, \natural_m); (\{d\}, \theta)_2))) * \mathsf{Stop}]$$

The quotient average sojourn time vector of $\overline{F}$:

$$SJ_{\underline{\leftrightarrow}_{ss}} = \left(\frac{1}{\rho}, \frac{1}{\chi}, 0, \frac{1}{\theta}\right).$$

The quotient sojourn time variance vector of $\overline{F}$:

$$VAR_{\underline{\leftrightarrow}_{ss}} = \left(\frac{1-\rho}{\rho^2}, \frac{1-\chi}{\chi^2}, 0, \frac{1-\theta}{\theta^2}\right).$$

The TPM for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}^*_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi^*_{\underline{\leftrightarrow}_{ss}} = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

The steady-state PMF $\psi^*_{\underline{\leftrightarrow}_{ss}}$ weighted by $SJ_{\underline{\leftrightarrow}_{ss}}$:

$$\left(0, \frac{1}{3\chi}, 0, \frac{l}{3\theta}\right).$$

We normalize the steady-state weighted PMF by dividing it by the sum of its components

$$\psi^*_{\underline{\leftrightarrow}_{ss}} SJ^T_{\underline{\leftrightarrow}_{ss}} = \frac{\chi+\theta}{3\chi\theta}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

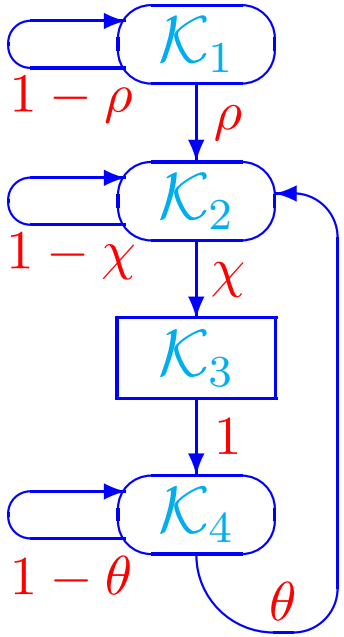
$$\varphi_{\underline{\leftrightarrow}_{ss}} = \frac{1}{\chi+\theta}(0, \theta, 0, \chi).$$

Definition 24 *Let G be a dynamic expression. The* quotient (by $\underline{\leftrightarrow}_{ss}$) DTMC *of G, $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$, has the state space $DR(G)/_{\mathcal{R}_{ss}(G)}$, the initial state $[[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$ and the transitions $\mathcal{K} \rightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM(\mathcal{K}, \widetilde{\mathcal{K}})$.*

For $E \in RegStatExpr$, let $DTMC_{\underline{\leftrightarrow}_{ss}}(E) = DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

The steady-state PMF $\psi_{\underline{\leftrightarrow}_{ss}}$ for $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$ is defined like ψ for $DTMC(G)$.

$DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



EXPRQDTMC: The quotient DTMC of $\overline{F}$ for

$$F = [(\{a\},\rho) * ((\{b\},\chi);((((\{c\},\natural_l);(\{d\},\theta)_1)[]((\{c\},\natural_m);(\{d\},\theta)_2))) * \mathsf{Stop}]$$

The TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \chi & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \theta & 0 & 1-\theta \end{pmatrix}.$$

The steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi_{\underline{\leftrightarrow}_{ss}} = \frac{1}{\chi+\theta+\chi\theta}(0,\theta,\chi\theta,\chi).$$

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(F)} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(F)} = \{\mathcal{K}_3\}$. Hence,

$$\sum_{\mathcal{K}\in DR_T(\overline{F})/_{\mathcal{R}_{ss}(F)}} \psi(\mathcal{K}) = \psi(\mathcal{K}_1) + \psi(\mathcal{K}_2) + \psi(\mathcal{K}_4) = \frac{\chi+\theta}{\chi+\theta+\chi\theta}.$$

By the "quotient" analogue of Proposition PMFSMC:

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_1) = 0 \cdot \frac{\chi+\theta+\chi\theta}{\chi+\theta} = 0,$$

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_2) = \frac{\theta}{\chi+\theta+\chi\theta} \cdot \frac{\chi+\theta+\chi\theta}{\chi+\theta} = \frac{\theta}{\chi+\theta},$$

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_3) = 0,$$

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_4) = \frac{\chi}{\chi+\theta+\chi\theta} \cdot \frac{\chi+\theta+\chi\theta}{\chi+\theta} = \frac{\chi}{\chi+\theta}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi_{\underline{\leftrightarrow}_{ss}} = \frac{1}{\chi+\theta}(0, \theta, 0, \chi).$$

This coincides with the result obtained with the use of $\psi^*_{\underline{\leftrightarrow}_{ss}}$ and $SJ_{\underline{\leftrightarrow}_{ss}}$.

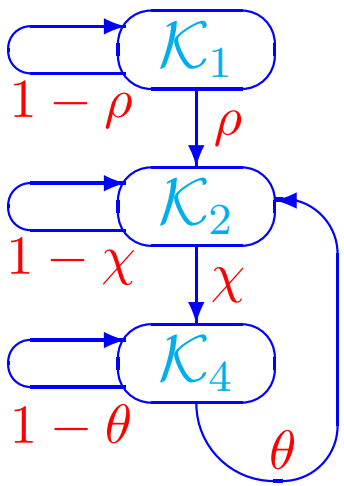
Definition 25 *The reduced quotient (by $\underline{\leftrightarrow}_{ss}$) DTMC of G, denoted by $RDTMC_{\underline{\leftrightarrow}_{ss}}(G)$, is defined like $RDTMC(G)$, but it is constructed from $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$ instead of $DTMC(G)$.*

For $E \in RegStatExpr$, let $RDTMC_{\underline{\leftrightarrow}_{ss}}(E) = RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

The steady-state PMF $\psi^{\diamond}_{\underline{\leftrightarrow}_{ss}}$ for $RDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ is defined like $\psi^{\diamond}$ for $RDTMC(G)$.

The relationships between the steady-state PMFs $\psi_{\underline{\leftrightarrow}_{ss}}$ and $\psi^{*}_{\underline{\leftrightarrow}_{ss}}$, $\varphi_{\underline{\leftrightarrow}_{ss}}$ and $\psi_{\underline{\leftrightarrow}_{ss}}$, $\varphi_{\underline{\leftrightarrow}_{ss}}$ and $\psi^{\diamond}_{\underline{\leftrightarrow}_{ss}}$ are the same as those between their "non-quotient" versions.

$RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



EXPRQRDTMC: The reduced quotient DTMC of $\overline{F}$ for

$$F = [(\{a\},\rho) * ((\{b\},\chi);((((\{c\},\natural_l);(\{d\},\theta)_1)[]((\{c\},\natural_m);(\{d\},\theta)_2))) * \mathsf{Stop}]$$

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(F)} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(F)} = \{\mathcal{K}_3\}$.

We reorder the states from $DR(\overline{F})/_{\mathcal{R}_{ss}(F)}$, by moving vanishing states to the first positions: $\mathcal{K}_3, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4$.

The reordered TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}_{r\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1-\rho & \rho & 0 \\ \chi & 0 & 1-\chi & 0 \\ 0 & 0 & \theta & 1-\theta \end{pmatrix}.$$

The result of the decomposing $\mathbf{P}_{r\underline{\leftrightarrow}_{ss}}$:

$$\mathbf{C}_{\underline{\leftrightarrow}_{ss}} = 0,\ \mathbf{D}_{\underline{\leftrightarrow}_{ss}} = (0,0,1),\ \mathbf{E}_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 0 \\ \chi \\ 0 \end{pmatrix},\ \mathbf{F}_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 1-\rho & \rho & 0 \\ 0 & 1-\chi & 0 \\ 0 & \theta & 1-\theta \end{pmatrix}.$$

Since $\mathbf{C}^1_{\underline{\leftrightarrow}_{ss}} = 0$, we have $\forall k > 0\ \mathbf{C}^k_{\underline{\leftrightarrow}_{ss}} = 0$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\mathbf{G}_{\underline{\leftrightarrow}_{ss}} = \sum_{k=0}^{l} \mathbf{C}^k_{\underline{\leftrightarrow}_{ss}} = \mathbf{C}^0_{\underline{\leftrightarrow}_{ss}} = \mathbf{I}.$$

The TPM for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}^{\diamond}_{\underline{\leftrightarrow}_{ss}} = \mathbf{F}_{\underline{\leftrightarrow}_{ss}} + \mathbf{E}_{\underline{\leftrightarrow}_{ss}}\mathbf{G}_{\underline{\leftrightarrow}_{ss}}\mathbf{D}_{\underline{\leftrightarrow}_{ss}} = \mathbf{F}_{\underline{\leftrightarrow}_{ss}} + \mathbf{E}_{\underline{\leftrightarrow}_{ss}}\mathbf{I}\mathbf{D}_{\underline{\leftrightarrow}_{ss}} = \mathbf{F}_{\underline{\leftrightarrow}_{ss}} + \mathbf{E}_{\underline{\leftrightarrow}_{ss}}\mathbf{D}_{\underline{\leftrightarrow}_{ss}} =$$

$$\begin{pmatrix} 1-\rho & \rho & 0 \\ 0 & 1-\chi & \chi \\ 0 & \theta & 1-\theta \end{pmatrix}.$$

The steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi^{\diamond}_{\underline{\leftrightarrow}_{ss}} = \frac{1}{\chi+\theta}(0, \theta, \chi).$$

Note that $\psi^{\diamond}_{\underline{\leftrightarrow}_{ss}} = (\psi^{\diamond}_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_1), \psi^{\diamond}_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_2), \psi^{\diamond}_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_4))$.

By the "quotient" analogue of Proposition PMFSMCT:

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_1) = 0,$$

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_2) = \frac{\theta}{\chi+\theta},$$

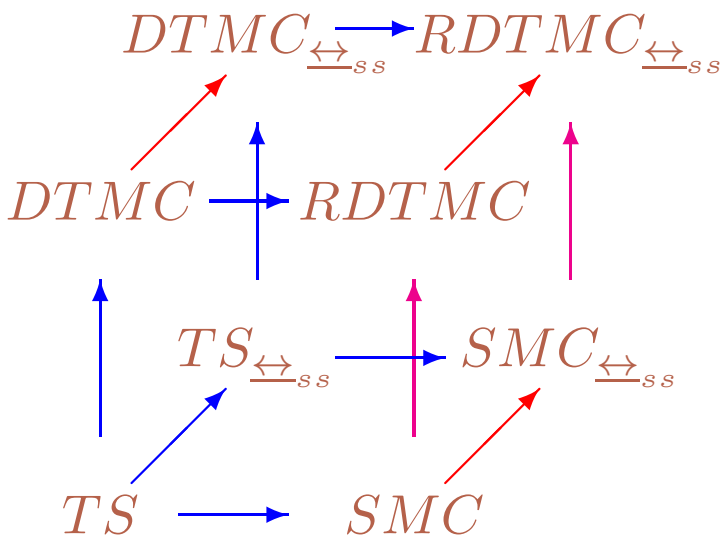
$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_3) = 0,$$

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}_4) = \frac{\chi}{\chi+\theta}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi_{\underline{\leftrightarrow}_{ss}} = \frac{1}{\chi+\theta}(0, \theta, 0, \chi).$$

This coincides with the result obtained with the use of $\psi^{*}_{\underline{\leftrightarrow}_{ss}}$ and $SJ_{\underline{\leftrightarrow}_{ss}}$.



CUBTSMCQ: The cube of interrelations for standard and quotient transition systems and Markov chains of expressions (the red arrows are correct by Propositions QXXQ, EQEEQ and QRRQ; the magenta arrows are correct by Theorem EREER and its “quotient” analogue)

Proposition 6 *(QXXQ) Let G be a dynamic expression, $\mathbf{P}_{\underline{\leftrightarrow}_{ss}}$ be the TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$ and $(\mathbf{P})_{\underline{\leftrightarrow}_{ss}}$ results from quotienting (by $\underline{\leftrightarrow}_{ss}$) the TPM $\mathbf{P}$ for $DTMC(G)$. Then*

$$(\mathbf{P})_{\underline{\leftrightarrow}_{ss}} = \mathbf{P}_{\underline{\leftrightarrow}_{ss}}.$$

The TPMs for $DTMC(\overline{F})$ and $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 & 0 \\ 0 & 1-\chi & \chi & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & \theta & 0 & 1-\theta & 0 \\ 0 & \theta & 0 & 0 & 1-\theta \end{pmatrix}, \ \mathbf{P}_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \chi & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \theta & 0 & 1-\theta \end{pmatrix}.$$

The TPM for the quotient of $DTMC(\overline{F})$:

$$(\mathbf{P})_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \chi & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \theta & 0 & 1-\theta \end{pmatrix}.$$

It is clear that $(\mathbf{P})_{\underline{\leftrightarrow}_{ss}} = \mathbf{P}_{\underline{\leftrightarrow}_{ss}}$.

Proposition 7 *(EQEEQ) Let G be a dynamic expression, $\mathbf{P}^*_{\underline{\leftrightarrow}_{ss}}$ be the TPM for $EDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ and $(\mathbf{P}^*)^*_{\underline{\leftrightarrow}_{ss}}$ results from quotienting (by $\underline{\leftrightarrow}_{ss}$) and final embedding the TPM $\mathbf{P}^*$ for $EDTMC(G)$. Then*

$$(\mathbf{P}^*)^*_{\underline{\leftrightarrow}_{ss}} = \mathbf{P}^*_{\underline{\leftrightarrow}_{ss}}.$$

The TPMs for $EDTMC(\overline{F})$ and $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{P}^*_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The TPMs for the quotient of $EDTMC(\overline{F})$ and EDTMC of the quotient of $EDTMC(\overline{F})$ ($EDTMC'(\overline{F})$):

$$(\mathbf{P}^*)_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ (\mathbf{P}^*)^*_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is clear that $(\mathbf{P}^*)^*_{\underline{\leftrightarrow}_{ss}} = \mathbf{P}^*_{\underline{\leftrightarrow}_{ss}}$.

The TPMs for $DTMC(\overline{F})$ and $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 & 0 \\ 0 & 1-\chi & \chi & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & \theta & 0 & 1-\theta & 0 \\ 0 & \theta & 0 & 0 & 1-\theta \end{pmatrix}, \ \mathbf{P}_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \chi & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \theta & 0 & 1-\theta \end{pmatrix}.$$

The collector matrix $\mathbf{V}$ for $\mathcal{R}_{ss}(\overline{F})$ and the distributor matrix $\mathbf{W}$ for $\mathbf{V}$:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{W} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

It is easy to check that $\mathbf{WPV} = \mathbf{P}_{\underline{\leftrightarrow}_{ss}}$.

Proposition 8 *(QRRQ) Let G be a dynamic expression, $\mathbf{P}^{\diamond}_{\underline{\leftrightarrow}_{ss}}$ be the TPM for $RDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ and $(\mathbf{P}^{\diamond})_{\underline{\leftrightarrow}_{ss}}$ results from quotienting (by $\underline{\leftrightarrow}_{ss}$) the TPM $\mathbf{P}^{\diamond}$ for $RDTMC(G)$. Then*

$$(\mathbf{P}^{\diamond})_{\underline{\leftrightarrow}_{ss}} = \mathbf{P}^{\diamond}_{\underline{\leftrightarrow}_{ss}}.$$

The reordered TPMs for $DTMC(\overline{F})$ and $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}_r = \begin{pmatrix} 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1-\rho & \rho & 0 & 0 \\ \chi & 0 & 1-\chi & 0 & 0 \\ 0 & 0 & \theta & 1-\theta & 0 \\ 0 & 0 & \theta & 0 & 1-\theta \end{pmatrix}, \ \mathbf{P}_{r\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1-\rho & \rho & 0 \\ \chi & 0 & 1-\chi & 0 \\ 0 & 0 & \theta & 1-\theta \end{pmatrix}.$$

The reordered collector matrix $\mathbf{V}_r$ for $\mathcal{R}_{ss}(\overline{F})$ and the reordered distributor matrix $\mathbf{W}_r$ for $\mathbf{V}_r$:

$$\mathbf{V}_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{W}_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

It is easy to check that $\mathbf{W}_r \mathbf{P}_r \mathbf{V}_r = \mathbf{P}_{r_{\underline{\leftrightarrow}_{ss}}}$.

The TPMs for $RDTMC(\overline{F})$ and $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}^{\diamond} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \frac{\chi l}{l+m} & \frac{\chi m}{l+m} \\ 0 & \theta & 1-\theta & 0 \\ 0 & \theta & 0 & 1-\theta \end{pmatrix}, \ \mathbf{P}^{\diamond}_{\underline{\leftrightarrow}_{ss}} = \begin{pmatrix} 1-\rho & \rho & 0 \\ 0 & 1-\chi & \chi \\ 0 & \theta & 1-\theta \end{pmatrix}.$$

The result of the decomposing the reordered collector matrix $\mathbf{V}_r$ for $\mathcal{R}_{ss}(\overline{F})$ and the reordered distributor matrix $\mathbf{W}_r$ for $\mathbf{V}_r$:

$$\mathbf{V}_C = 1, \ \mathbf{V}_F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{W}_C = 1, \ \mathbf{W}_F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

It is easy to check that $(\mathbf{P}^{\diamond})_{\underline{\leftrightarrow}_{ss}} = \mathbf{W}_F \mathbf{P}^{\diamond} \mathbf{V}_F = \mathbf{P}^{\diamond}_{\underline{\leftrightarrow}_{ss}}$.

Stationary behaviour

Steady state and equivalences

Proposition 9 *(STPROB) Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ and φ be the steady-state PMF for $SMC(G)$, φ' be the steady-state PMF for $SMC(G')$. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$\sum_{s \in \mathcal{H} \cap DR(G)} \varphi(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \varphi'(s').$$

Let G be a dynamic expression and φ be the steady-state PMF for $SMC(G)$, $\varphi_{\underline{\leftrightarrow}_{ss}}$ be the steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(G)$.

By Proposition STPROB: $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)}$

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) = \sum_{s \in \mathcal{K}} \varphi(s).$$

Definition 26 *A derived step trace of a dynamic expression G is $\Sigma = A_1 \cdots A_n \in (\mathbb{N}^{\mathcal{L}}_{fin})^*$, where $\exists s \in DR(G)\ s \xrightarrow{\Upsilon_1} s_1 \xrightarrow{\Upsilon_2} \cdots \xrightarrow{\Upsilon_n} s_n,\ \mathcal{L}(\Upsilon_i) = A_i\ (1 \leq i \leq n)$.*

The probability to execute the derived step trace Σ in s:

$$PT(\Sigma, s) = \sum_{\{\Upsilon_1,\ldots,\Upsilon_n \mid s=s_0 \xrightarrow{\Upsilon_1} s_1 \xrightarrow{\Upsilon_2} \cdots \xrightarrow{\Upsilon_n} s_n,\ \mathcal{L}(\Upsilon_i)=A_i\ (1\leq i\leq n)\}} \prod_{i=1}^{n} PT(\Upsilon_i, s_{i-1}).$$

Theorem 5 *(STTRAC) Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ and φ be the steady-state PMF for $SMC(G)$, φ' be the steady-state PMF for $SMC(G')$ and Σ be a derived step trace of G and G'. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$\sum_{s \in \mathcal{H} \cap DR(G)} \varphi(s) PT(\Sigma, s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \varphi'(s') PT(\Sigma, s').$$

By Theorem STTRAC: $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)}$

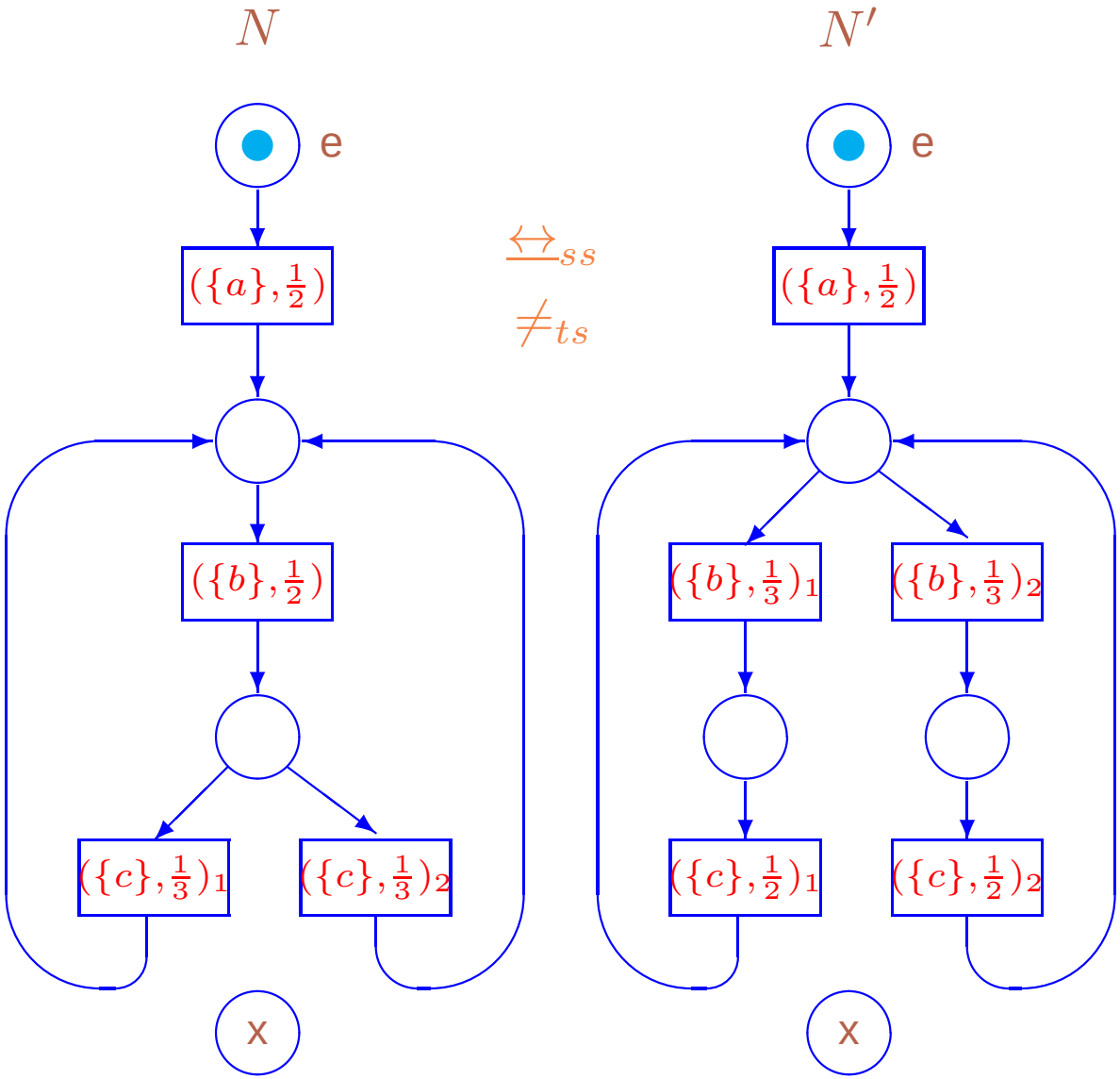
$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K})PT(\Sigma,\mathcal{K}) = \sum_{s\in\mathcal{K}} \varphi(s)PT(\Sigma,s),$$

where $\forall s \in \mathcal{K}\ PT(\Sigma,\mathcal{K}) = PT(\Sigma,s)$.

Proposition 10 *(SJAVVA) Let G, G' be dynamic expressions with $\mathcal{R} : G\underline{\leftrightarrow}_{ss}G'$. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$SJ_{\mathcal{R}\cap(DR(G))^2}(\mathcal{H} \cap DR(G)) = SJ_{\mathcal{R}\cap(DR(G'))^2}(\mathcal{H} \cap DR(G')),$$

$$VAR_{\mathcal{R}\cap(DR(G))^2}(\mathcal{H} \cap DR(G)) = VAR_{\mathcal{R}\cap(DR(G'))^2}(\mathcal{H} \cap DR(G')).$$



SSBSSP: $\underline{\leftrightarrow}_{ss}$ preserves steady-state behaviour and sojourn time properties in the equivalence classes

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2)) * \mathsf{Stop}]$ and
$E' = [(\{a\}, \frac{1}{2}) * ((((\{b\}, \frac{1}{3})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{3})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$.

We have $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$.

$DR(\overline{E})$ consists of

$s_1 = [[\overline{(\{a\}, \frac{1}{2})} * (\underline{(\{b\}, \frac{1}{2})}; ((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2)) * \mathsf{Stop}]]_{\approx}$,

$s_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; \underline{((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2)}) * \mathsf{Stop}]]_{\approx}$,

$s_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); \overline{((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2)}) * \mathsf{Stop}]]_{\approx}$.

$DR(\overline{E'})$ consists of

$s'_1 = [[\overline{(\{a\}, \frac{1}{2})} * \underline{((((\{b\}, \frac{1}{3})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{3})_2; (\{c\}, \frac{1}{2})_2))} * \mathsf{Stop}]]_{\approx}$,

$s'_2 = [[(\{a\}, \frac{1}{2}) * \overline{((((\{b\}, \frac{1}{3})_1; \underline{(\{c\}, \frac{1}{2})_1})[]((\{b\}, \frac{1}{3})_2; (\{c\}, \frac{1}{2})_2))} * \mathsf{Stop}]]_{\approx}$,

$s'_3 = [[(\{a\}, \frac{1}{2}) * ((((\{b\}, \frac{1}{3})_1; \overline{(\{c\}, \frac{1}{2})_1})[]((\{b\}, \frac{1}{3})_2; \underline{(\{c\}, \frac{1}{2})_2)}) * \mathsf{Stop}]]_{\approx}$,

$s'_4 = [[(\{a\}, \frac{1}{2}) * ((((\{b\}, \frac{1}{3})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{3})_2; \overline{(\{c\}, \frac{1}{2})_2)}) * \mathsf{Stop}]]_{\approx}$.

The steady-state PMFs φ for $SMC(\overline{E})$ and φ' for $SMC(\overline{E'})$ are

$$\varphi = \left(0, \frac{1}{2}, \frac{1}{2}\right), \ \varphi' = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Consider $\mathcal{H} = \{s_3, s'_3, s'_4\}$. The steady-state probabilities for $\mathcal{H}$ coincide:
$\sum_{s \in \mathcal{H} \cap DR(\overline{E})} \varphi(s) = \varphi(s_3) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \varphi'(s'_3) + \varphi'(s'_4) = \sum_{s' \in \mathcal{H} \cap DR(\overline{E'})} \varphi'(s')$.

Let $\Sigma = \{\{c\}\}$. The steady-state probabilities to enter into the equivalence class $\mathcal{H}$ and start the derived step trace Σ from it coincide: $\varphi(s_3)(PT(\{(\{c\}, \frac{1}{3})_1\}, s_3) + PT(\{(\{c\}, \frac{1}{3})_2\}, s_3)) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4} = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \varphi'(s'_3)PT(\{(\{c\}, \frac{1}{2})_1\}, s'_3) + \varphi'(s'_4)PT(\{(\{c\}, \frac{1}{2})_2\}, s'_4)$.

In Figure SSBSSP, $N = Box_{dtsi}(\overline{E})$ and $N' = Box_{dtsi}(\overline{E'})$.

The sojourn time averages in the equivalence class $\mathcal{H}$ coincide:

$$SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\mathcal{H}\cap DR(G)) = SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\{s_3\}) = \frac{1}{1-PM(\{s_3\},\{s_3\})} = \frac{1}{1-PM(s_3,s_3)} = \frac{1}{1-\frac{1}{2}} = 2 = \frac{1}{1-\frac{1}{2}} = \frac{1}{1-PM(s'_3,s'_3)} = \frac{1}{1-PM(s'_4,s'_4)} = \frac{1}{1-PM(\{s'_3,s'_4\},\{s'_3,s'_4\})} = SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\{s'_3, s'_4\}) = SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\mathcal{H}\cap DR(G')).$$

The sojourn time variances in the equivalence class $\mathcal{H}$ coincide:

$$VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\mathcal{H}\cap DR(G)) = VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\{s_3\}) = \frac{PM(\{s_3\},\{s_3\})}{(1-PM(\{s_3\},\{s_3\}))^2} = \frac{PM(s_3,s_3)}{(1-PM(s_3,s_3))^2} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2 = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{PM(s'_3,s'_3)}{(1-PM(s'_3,s'_3))^2} = \frac{PM(s'_4,s'_4)}{(1-PM(s'_4,s'_4))^2} = \frac{PM(\{s'_3,s'_4\},\{s'_3,s'_4\})}{(1-PM(\{s'_3,s'_4\},\{s'_3,s'_4\}))^2} = VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\{s'_3, s'_4\}) = VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\mathcal{H}\cap DR(G')).$$

Simplification of performance analysis

The method of performance analysis simplification.

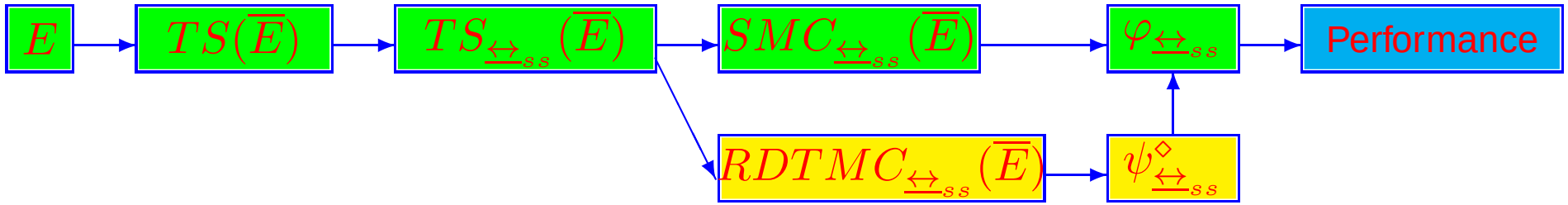
1. The investigated system is specified by a static expression of $dtsiPBC$.
2. The transition system of the expression is constructed.
3. After treating the transition system for self-similarity,
   a step stochastic autobisimulation equivalence for the expression is determined.
4. The quotient underlying SMC is constructed from the quotient transition system.
5. Stationary probabilities and performance indices are calculated using the SMC.

Simplification of the steps 4 and 5:

constructing the reduced quotient DTMC from the quotient transition system,

calculating the stationary probabilities of the quotient underlying SMC using this DTMC

and obtaining the performance indices.



EQPEVA: Equivalence-based simplification of performance evaluation

The limitation of the method: the expressions with underlying SMCs containing one closed communication class of states, which is ergodic, to ensure uniqueness of the stationary distribution.

If an SMC contains several closed communication classes of states that are all ergodic: several stationary distributions may exist, depending on the initial PMF.

The general steady-state probabilities are then calculated as the sum of the stationary probabilities of all the ergodic classes of states, weighted by the probabilities to enter these classes, starting from the initial state and passing through transient states.

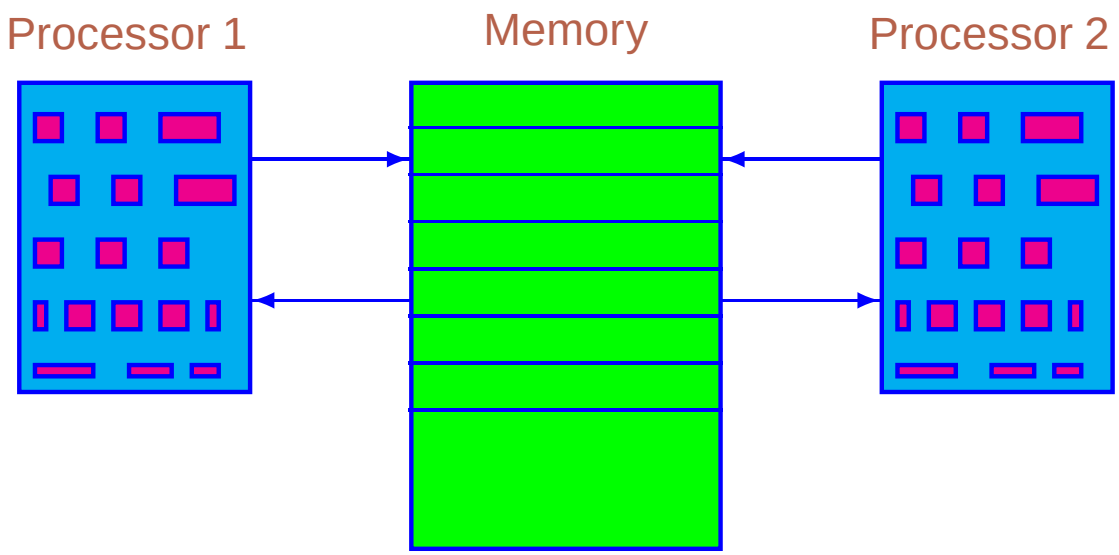
The underlying SMC of each process expression has one initial PMF (that at the time moment 0): the stationary distribution is unique.

It is worth applying the method to the systems with similar subprocesses.

Shared memory system

A model of two processors accessing a common shared memory [MBCDF95]

The standard system



SHMDIA: The diagram of the shared memory system

After activation of the system (turning the computer on), two processors are active, and the common memory is available. Each processor can request an access to the memory after which the instantaneous decision is made.

When the decision is made in favour of a processor, it starts an acquisition of the memory, and another processor waits until the former one ends its operations, and the system returns to the state with both active processors and the available memory.

a corresponds to the system activation.

r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i.

d_i correspond to the (instantaneous) decision on the memory allocation in favour of the processor i.

m_i represent the common memory access of processor i.

The other actions are used for communication purpose only.

The static expression of the first processor is

$E_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r_1\}, \frac{1}{2}); (\{d_1, y_1\}, \natural_1); (\{m_1, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$E_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r_2\}, \frac{1}{2}); (\{d_2, y_2\}, \natural_1); (\{m_2, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$E_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * ((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the shared memory system with two processors is

$E = (E_1 \| E_2 \| E_3)\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2$.

Effect of synchronization

The synchronization of $(\{d_i, y_i\}, \natural_1)$ and $(\{\widehat{y_i}\}, \natural_1)$ produces $(\{d_i\}, \natural_2)$ $(1 \leq i \leq 2)$.

The synchronization of $(\{m_i, z_i\}, \frac{1}{2})$ and $(\{\widehat{z_i}\}, \frac{1}{2})$ produces $(\{m_i\}, \frac{1}{4})$ $(1 \leq i \leq 2)$.

The synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a, \widehat{x_2}\}, \frac{1}{4})$,

Synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_2\}, \frac{1}{2})$ produces $(\{a, \widehat{x_1}\}, \frac{1}{4})$.

Synchronization of $(\{a, \widehat{x_2}\}, \frac{1}{4})$ and $(\{x_2\}, \frac{1}{2})$, as well as $(\{a, \widehat{x_1}\}, \frac{1}{4})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a\}, \frac{1}{8})$.

$DR(\overline{E})$ consists of

$$s_1 = [([\overline{(\{x_1\}, \tfrac{1}{2})} * ((\{r_1\}, \tfrac{1}{2}); (\{d_1, y_1\}, \natural_1); (\{m_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_2\}, \tfrac{1}{2})} * ((\{r_2\}, \tfrac{1}{2}); (\{d_2, y_2\}, \natural_1); (\{m_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[\overline{(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2})} * ((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_2 = [([(\{x_1\}, \tfrac{1}{2}) * \overline{((\{r_1\}, \tfrac{1}{2})}; (\{d_1, y_1\}, \natural_1); (\{m_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * \overline{((\{r_2\}, \tfrac{1}{2})}; (\{d_2, y_2\}, \natural_1); (\{m_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * \overline{((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \tfrac{1}{2})))} * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_3 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); \overline{(\{d_1, y_1\}, \natural_1)}; (\{m_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * \overline{((\{r_2\}, \tfrac{1}{2})}; (\{d_2, y_2\}, \natural_1); (\{m_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((\overline{(\{\widehat{y_1}\}, \natural_1)}; (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$
\begin{aligned}
s_4 = {} & [([(\{x_1\}, \tfrac{1}{2}) * (\overline{(\{r_1\}, \tfrac{1}{2})}; (\{d_1, y_1\}, \natural_1); (\{m_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{d_2, y_2\}, \natural_1)}; (\{m_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]\overline{((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \tfrac{1}{2}))}) * \mathsf{Stop}]) \\
& \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}
$$

$$
\begin{aligned}
s_5 = {} & [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{d_1, y_1\}, \natural_1); \overline{(\{m_1, z_1\}, \tfrac{1}{2})}) * \mathsf{Stop}] \\
& \| [(\{x_2\}, \tfrac{1}{2}) * (\overline{(\{r_2\}, \tfrac{1}{2})}; (\{d_2, y_2\}, \natural_1); (\{m_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((((\{\widehat{y_1}\}, \natural_1); \overline{(\{\widehat{z_1}\}, \tfrac{1}{2})})[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}]) \\
& \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}
$$

$$
\begin{aligned}
s_6 = {} & [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); \overline{(\{d_1, y_1\}, \natural_1)}; (\{m_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{d_2, y_2\}, \natural_1)}; (\{m_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * \overline{((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \tfrac{1}{2})))} * \mathsf{Stop}]) \\
& \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}
$$

$$
\begin{aligned}
s_7 = {} & [([(\{x_1\}, \tfrac{1}{2}) * (\overline{(\{r_1\}, \tfrac{1}{2})}; (\{d_1, y_1\}, \natural_1); (\{m_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{d_2, y_2\}, \natural_1); \overline{(\{m_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}] \\
& \| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}]) \\
& \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}
$$

$$
\begin{aligned}
s_8 = {} & [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{d_1, y_1\}, \natural_1); \overline{(\{m_1, z_1\}, \tfrac{1}{2})}) * \mathsf{Stop}] \\
& \| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{d_2, y_2\}, \natural_1)}; (\{m_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((((\{\widehat{y_1}\}, \natural_1); \overline{(\{\widehat{z_1}\}, \tfrac{1}{2})})[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}]) \\
& \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}
$$

$$
\begin{aligned}
s_9 = {} & [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); \overline{(\{d_1, y_1\}, \natural_1)}; (\{m_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
& \| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{d_2, y_2\}, \natural_1); \overline{(\{m_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}] \\
& \| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}]) \\
& \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx}.
\end{aligned}
$$

Interpretation of the states

$DR_T(\overline{E}) = \{s_1, s_2, s_5, s_7, s_8, s_9\}$ and $DR_V(\overline{E}) = \{s_3, s_4, s_6\}$.

s_1: the initial state,

s_2: the system is activated and the memory is not requested,

s_3: the memory is requested by the first processor,

s_4: the memory is requested by the second processor,

s_5: the memory is allocated to the first processor,

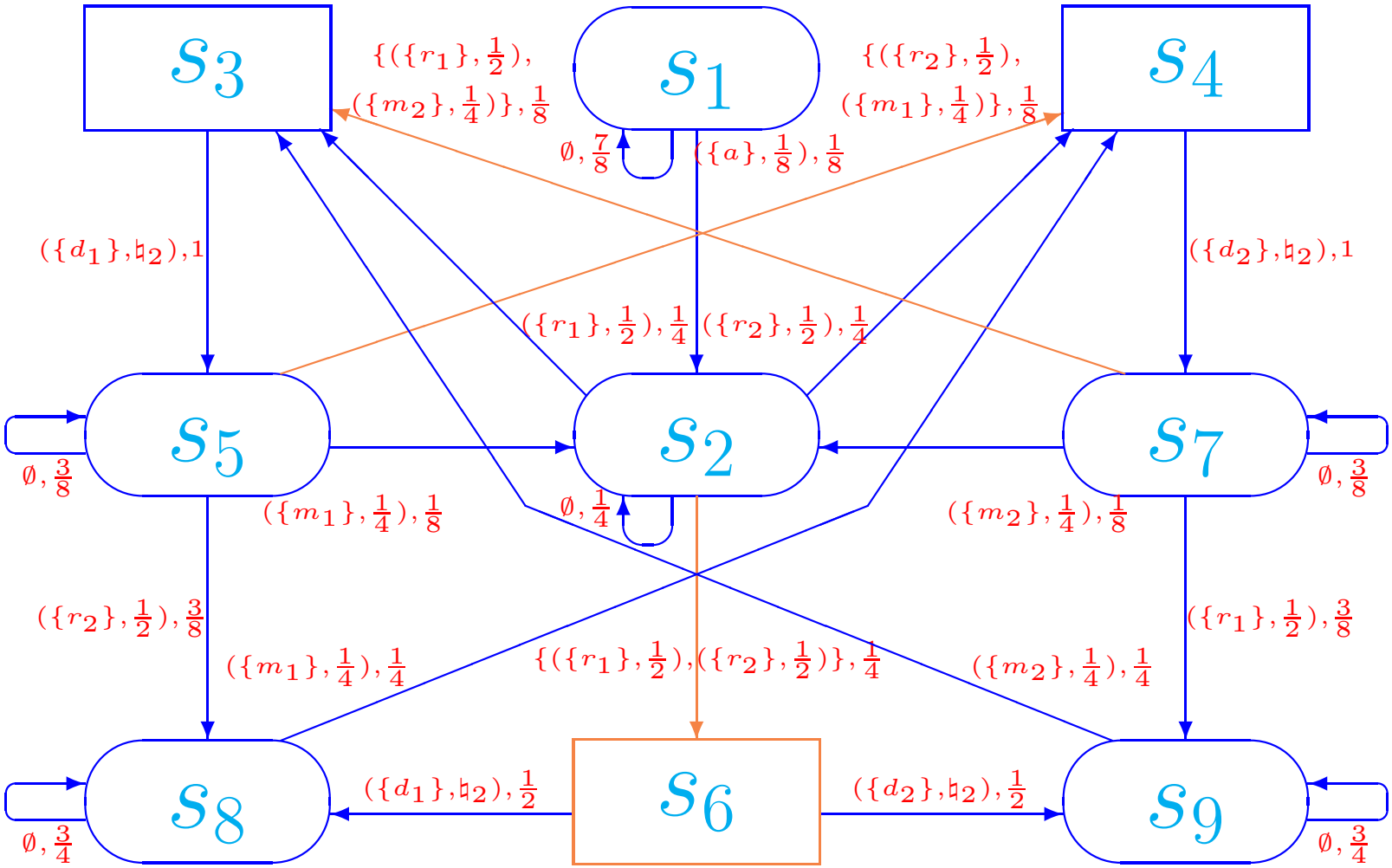
s_6: the memory is requested by two processors,

s_7: the memory is allocated to the second processor,

s_8: the memory is allocated to the first processor and the memory is requested by the second processor,

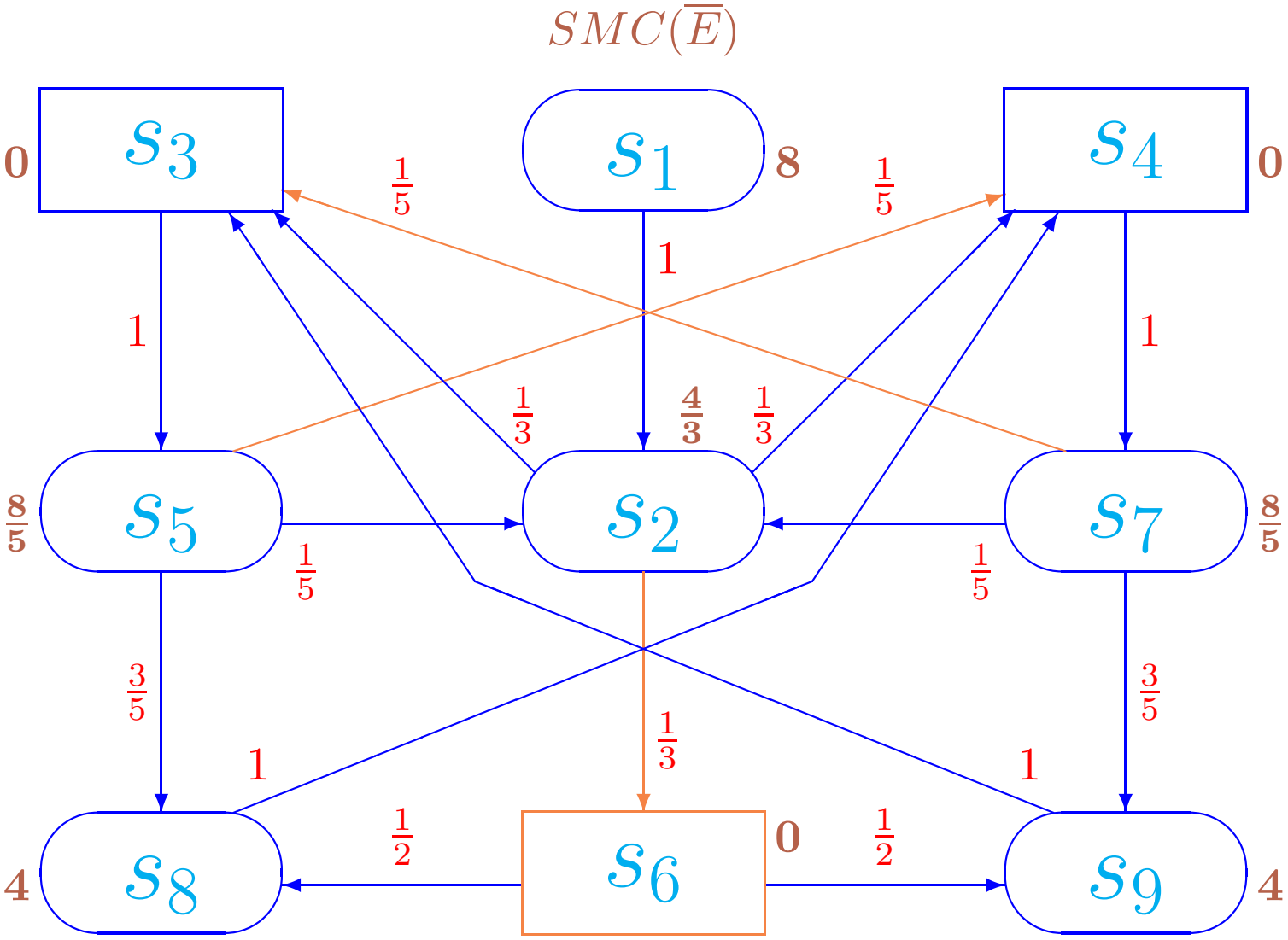
s_9: the memory is allocated to the second processor and the memory is requested by the first processor.

$TS(\overline{E})$



SHMTS: The transition system of the shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)



SHMSMC: The underlying SMC of the shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The average sojourn time vector of $\overline{E}$:

$$SJ = \left(8, \frac{4}{3}, 0, 0, \frac{8}{5}, 0, \frac{8}{5}, 4, 4\right).$$

The sojourn time variance vector of $\overline{E}$:

$$VAR = \left(56, \frac{4}{9}, 0, 0, \frac{24}{25}, 0, \frac{24}{25}, 12, 12\right).$$

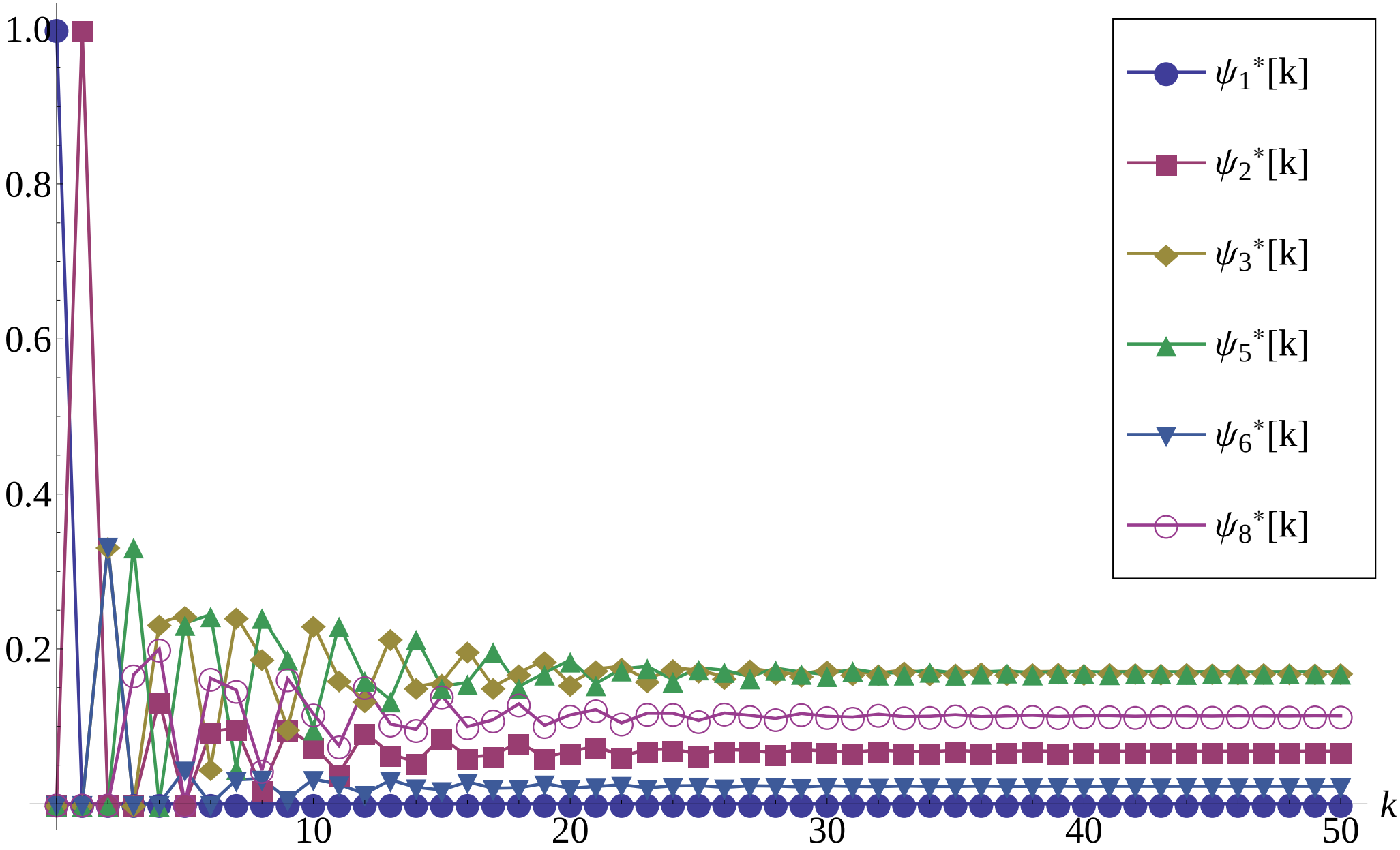
The TPM for $EDTMC(\overline{E})$:

$$\mathbf{P}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

SHMTP: Transient and steady-state probabilities for the EDTMC of the shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2^*[k]$	0	0	0.0754	0.0859	0.0677	0.0641	0.0680	0.0691	0.0683	0.0680	0.0681	0.0682
$\psi_3^*[k]$	0	0.2444	0.2316	0.1570	0.1554	0.1726	0.1741	0.1702	0.1696	0.1705	0.1707	0.1705
$\psi_5^*[k]$	0	0.2333	0.0982	0.1516	0.1859	0.1758	0.1672	0.1690	0.1711	0.1708	0.1703	0.1705
$\psi_6^*[k]$	0	0.0444	0.0323	0.0179	0.0202	0.0237	0.0234	0.0226	0.0226	0.0228	0.0228	0.0227
$\psi_8^*[k]$	0	0	0.1163	0.1395	0.1147	0.1077	0.1130	0.1150	0.1139	0.1133	0.1136	0.1136

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.



SHMTP: Transient probabilities alteration diagram for the EDTMC of the shared memory system

The steady-state PMF for $EDTMC(\overline{E})$:

$$\psi^* = \left(0, \frac{3}{44}, \frac{15}{88}, \frac{15}{88}, \frac{15}{88}, \frac{1}{44}, \frac{15}{88}, \frac{5}{44}, \frac{5}{44}\right).$$

The steady-state PMF ψ^* weighted by SJ:

$$\left(0, \frac{1}{11}, 0, 0, \frac{3}{11}, 0, \frac{3}{11}, \frac{5}{11}, \frac{5}{11}\right).$$

We normalize the steady-state weighted PMF dividing it by the sum of its components $\psi^* SJ^T = \frac{17}{11}$.

The steady-state PMF for $SMC(\overline{E})$:

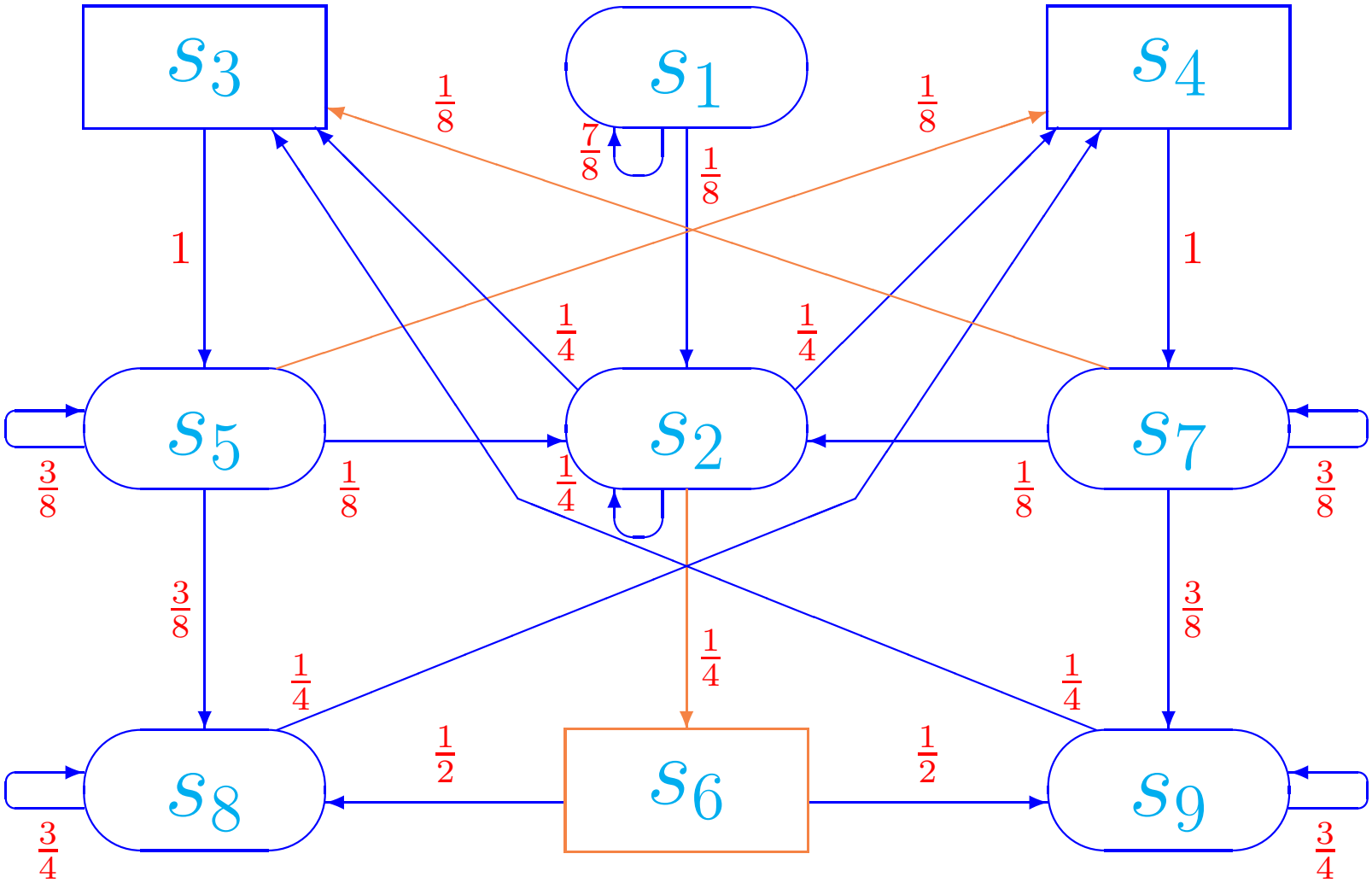
$$\varphi = \left(0, \frac{1}{17}, 0, 0, \frac{3}{17}, 0, \frac{3}{17}, \frac{5}{17}, \frac{5}{17}\right).$$

Otherwise, from $TS(\overline{E})$, we can construct $DTMC(\overline{E})$ and calculate φ using it.

The TPM for $DTMC(\overline{E})$:

$$\mathbf{P} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{3}{8} & 0 & 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

$DTMC(\overline{E})$



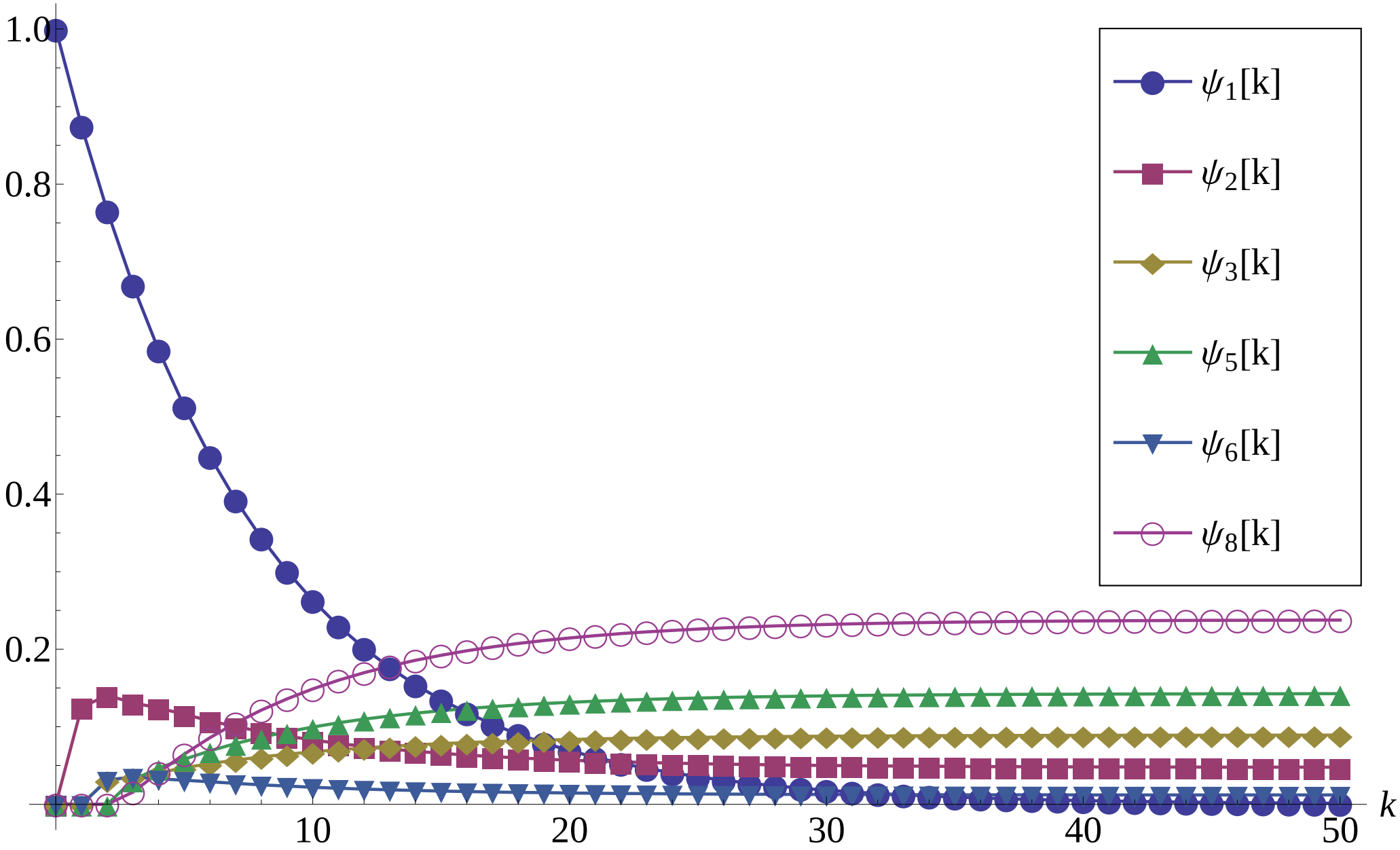
SHMDTMC: The DTMC of the shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

SHMTPDTMC: Transient and steady-state probabilities for the DTMC of the shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
$\psi_1[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	0
$\psi_2[k]$	0	0.1161	0.0829	0.0657	0.0569	0.0524	0.0501	0.0489	0.0483	0.0479	0.0478	0.0476
$\psi_3[k]$	0	0.0472	0.0677	0.0782	0.0836	0.0864	0.0878	0.0885	0.0889	0.0891	0.0892	0.0893
$\psi_5[k]$	0	0.0581	0.0996	0.1207	0.1315	0.1370	0.1399	0.1413	0.1421	0.1425	0.1427	0.1429
$\psi_6[k]$	0	0.0311	0.0220	0.0171	0.0146	0.0133	0.0126	0.0123	0.0121	0.0120	0.0120	0.0119
$\psi_8[k]$	0	0.0647	0.1487	0.1923	0.2146	0.2260	0.2319	0.2349	0.2365	0.2373	0.2377	0.2381

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.



SHMTPDTMC: Transient probabilities alteration diagram for the DTMC of the shared memory system

The steady-state PMF for $DTMC(\overline{E})$:

$$\psi = \left(0, \frac{1}{21}, \frac{5}{56}, \frac{5}{56}, \frac{1}{7}, \frac{1}{84}, \frac{1}{7}, \frac{5}{21}, \frac{5}{21}\right).$$

Remember that $DR_T(\overline{E}) = \{s_1, s_2, s_5, s_7, s_8, s_9\}$ and $DR_V(\overline{E}) = \{s_3, s_4, s_6\}$. Hence,

$$\sum_{s \in DR_T(\overline{E})} \psi(s) = \psi(s_1) + \psi(s_2) + \psi(s_5) + \psi(s_7) + \psi(s_8) + \psi(s_9) = \frac{17}{21}.$$

By Proposition PMFSMC:

$$\varphi(s_1) = 0 \cdot \frac{21}{17} = 0,$$
$$\varphi(s_2) = \frac{1}{21} \cdot \frac{21}{17} = \frac{1}{17},$$
$$\varphi(s_3) = 0,$$
$$\varphi(s_4) = 0,$$
$$\varphi(s_5) = \frac{1}{7} \cdot \frac{21}{17} = \frac{3}{17},$$
$$\varphi(s_6) = 0,$$
$$\varphi(s_7) = \frac{1}{7} \cdot \frac{21}{17} = \frac{3}{17},$$
$$\varphi(s_8) = \frac{5}{21} \cdot \frac{21}{17} = \frac{5}{17},$$
$$\varphi(s_9) = \frac{5}{21} \cdot \frac{21}{17} = \frac{5}{17}.$$

The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \left(0, \frac{1}{17}, 0, 0, \frac{3}{17}, 0, \frac{3}{17}, \frac{5}{17}, \frac{5}{17}\right).$$

This coincides with the result obtained with the use of ψ^* and SJ.

Alternatively, from $TS(\overline{E})$, we can construct $DTMC(\overline{E})$

and calculate φ using it.

$DR_T(\overline{E}) = \{s_1, s_2, s_5, s_7, s_8, s_9\}$ and $DR_V(\overline{E}) = \{s_3, s_4, s_6\}$.

We reorder the elements of $DR(\overline{E})$ by

moving vanishing states to the first positions: $s_3, s_4, s_6, s_1, s_2, s_5, s_7, s_8, s_9$.

The reordered TPM for $DTMC(\overline{E})$:

$$\mathbf{P}_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

The result of the decomposing $\mathbf{P}_r$:

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix},$$

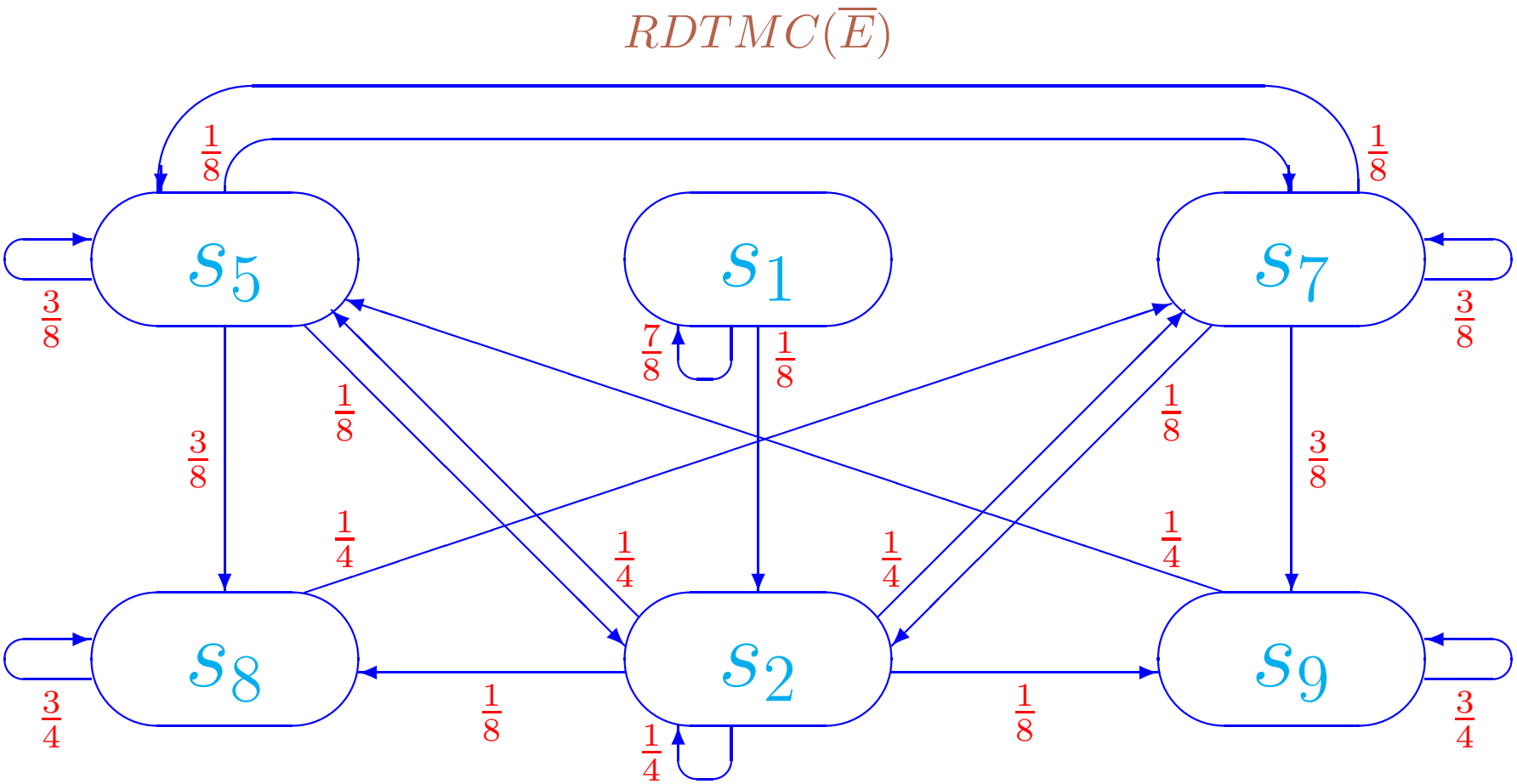
$$\mathbf{F} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} & 0 \\ 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

Since $\mathbf{C}^1 = \mathbf{0}$, we have $\forall k > 0,\ \mathbf{C}^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\mathbf{G} = \sum_{k=0}^{l} \mathbf{C}^k = \mathbf{C}^0 = \mathbf{I}.$$

The TPM for $RDTMC(\overline{E})$:

$$\mathbf{P}^{\diamond} = \mathbf{F} + \mathbf{EGD} = \mathbf{F} + \mathbf{EID} = \mathbf{F} + \mathbf{ED} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

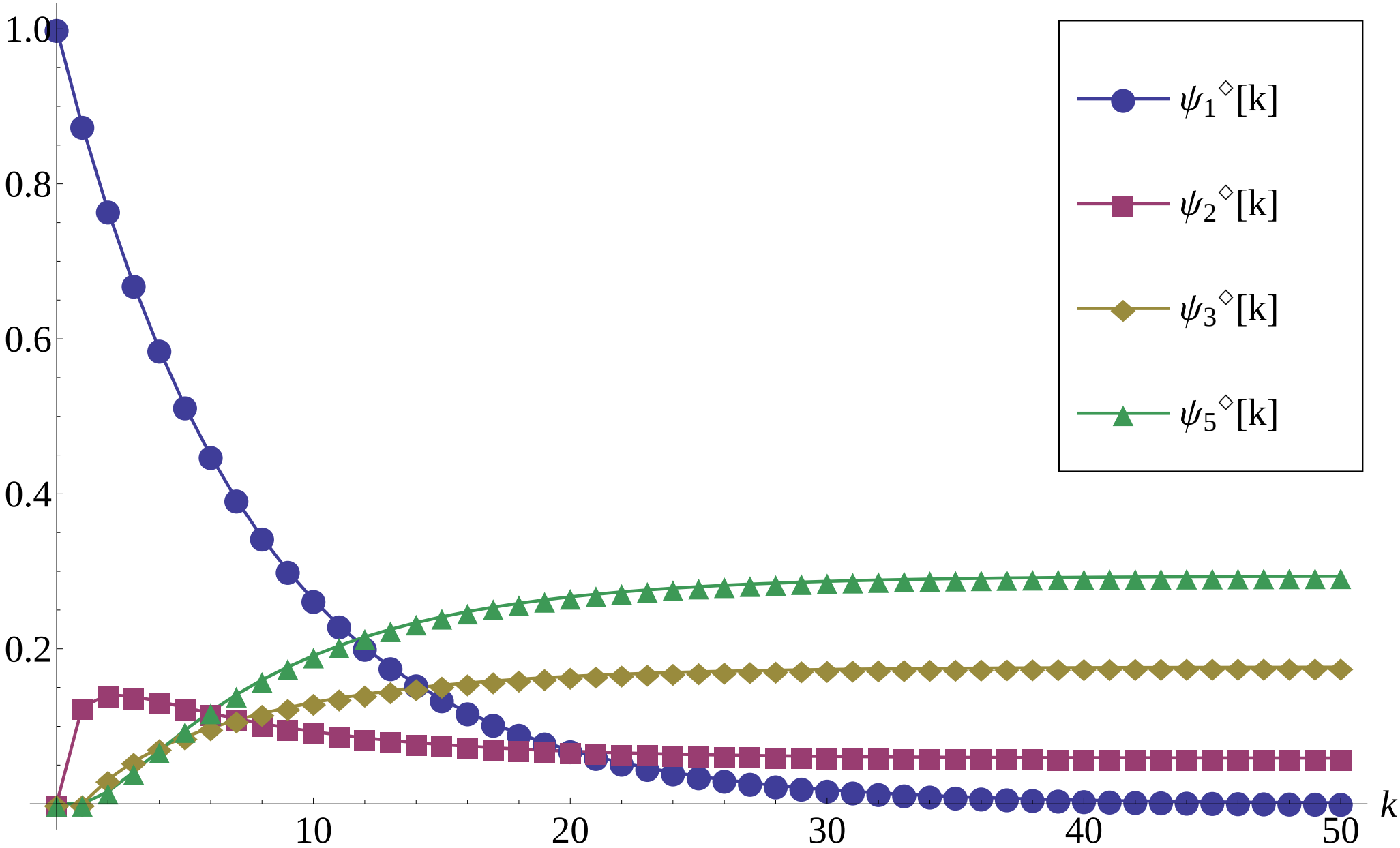


SHMRDTMC: The reduced DTMC of the shared memory system

SHMTRPR: Transient and steady-state probabilities for the RDTMC of the shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
$\psi_1^{\diamond}[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	0
$\psi_2^{\diamond}[k]$	0	0.1244	0.0931	0.0764	0.0679	0.0635	0.0612	0.0600	0.0594	0.0591	0.0590	0.0588
$\psi_3^{\diamond}[k]$	0	0.0863	0.1307	0.1530	0.1644	0.1703	0.1733	0.1748	0.1756	0.1760	0.1763	0.1765
$\psi_5^{\diamond}[k]$	0	0.0951	0.1912	0.2413	0.2670	0.2802	0.2870	0.2905	0.2922	0.2932	0.2936	0.2941

We depict the probabilities for states s_1, s_2, s_5, s_8 only, since the corresponding values coincide for s_5, s_7, as well as for s_8, s_9.



SHMTRPR: Transient probabilities alteration diagram for the RDTMC of the shared memory system

The steady-state PMF for $RDTMC(\overline{E})$:

$$\psi^{\diamond} = \left(0, \frac{1}{17}, \frac{3}{17}, \frac{3}{17}, \frac{5}{17}, \frac{5}{17}\right).$$

Note that $\psi^{\diamond} = (\psi^{\diamond}(s_1), \psi^{\diamond}(s_2), \psi^{\diamond}(s_5), \psi^{\diamond}(s_7), \psi^{\diamond}(s_8), \psi^{\diamond}(s_9))$.

By Proposition PMFSMCT:

$$\begin{array}{lllll} \varphi(s_1) = 0, & \varphi(s_2) = \frac{1}{17}, & \varphi(s_3) = 0, & \varphi(s_4) = 0, & \varphi(s_5) = \frac{3}{17}, \\ \varphi(s_6) = 0, & \varphi(s_7) = \frac{3}{17}, & \varphi(s_8) = \frac{5}{17}, & \varphi(s_9) = \frac{5}{17}. & \end{array}$$

The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \left(0, \frac{1}{17}, 0, 0, \frac{3}{17}, 0, \frac{3}{17}, \frac{5}{17}, \frac{5}{17}\right).$$

This coincides with the result obtained with the use of ψ^* and SJ.

Performance indices

- The average recurrence time in the state s_2, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\varphi_2} = 17$.

- The common memory is available only in the states s_2, s_3, s_4, s_6.

  The steady-state probability that the memory is available is
  $\varphi_2 + \varphi_3 + \varphi_4 + \varphi_6 = \frac{1}{17} + 0 + 0 + 0 = \frac{1}{17}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{1}{17} = \frac{16}{17}$.

- After activation of the system, we leave the state s_1 for ever, and the common memory is either requested or allocated in every remaining state, with exception of s_2.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving s_2, calculated as $\frac{\varphi_2}{SJ_2} = \frac{1}{17} \cdot \frac{3}{4} = \frac{3}{68}$.

- The parallel common memory request of two processors $\{(\{r_1\}, \frac{1}{2}), (\{r_2\}, \frac{1}{2})\}$ is only possible from the state s_2.

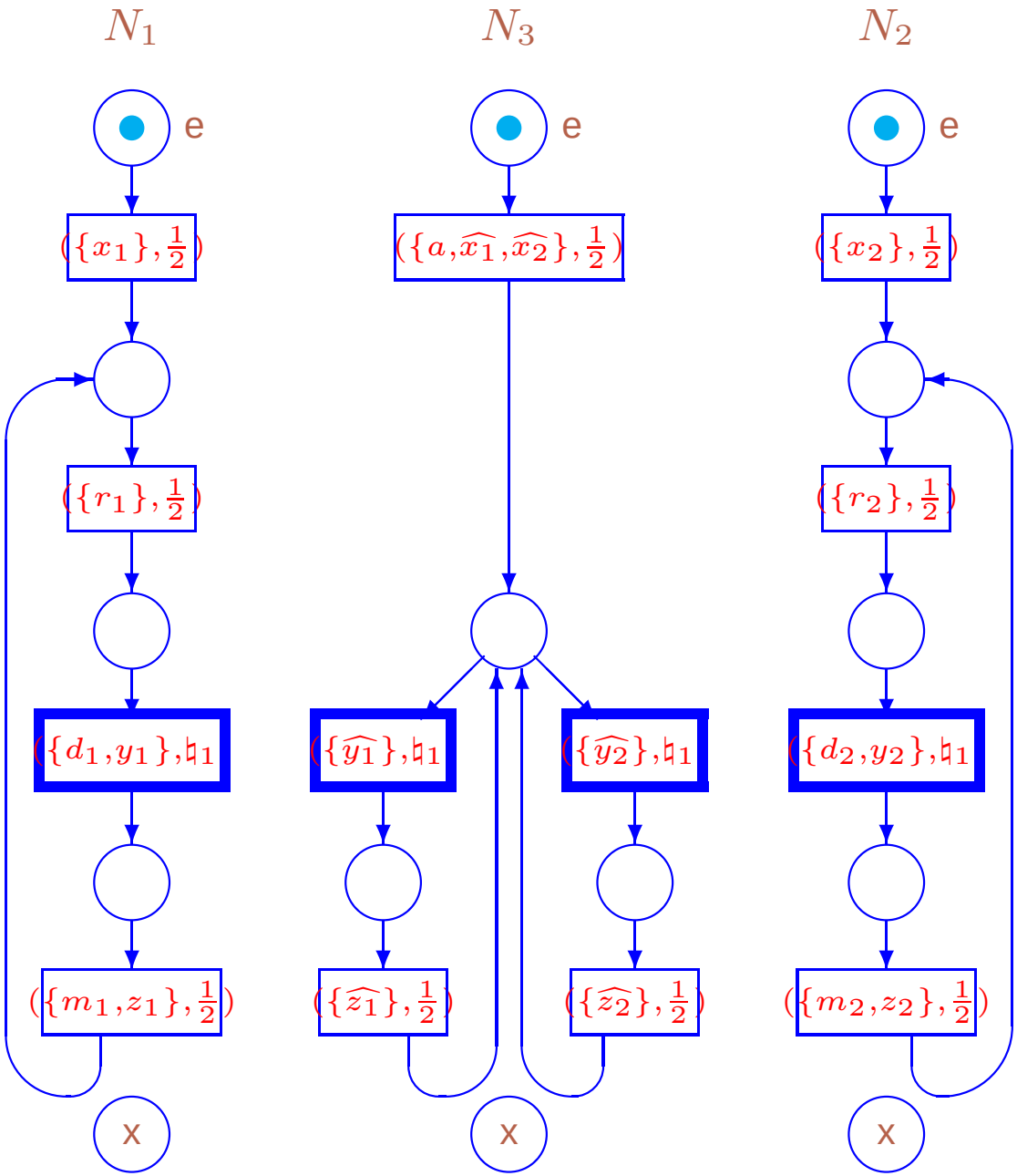
  The request probability in this state is the sum of the execution probabilities for all multisets of activities containing both $(\{r_1\}, \frac{1}{2})$ and $(\{r_2\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from two processors* is $\varphi_2 \sum_{\{\Upsilon \mid (\{(\{r_1\},\frac{1}{2}),(\{r_2\},\frac{1}{2})\} \subseteq \Upsilon\}} PT(\Upsilon, s_2) = \frac{1}{17} \cdot \frac{1}{4} = \frac{1}{68}$.

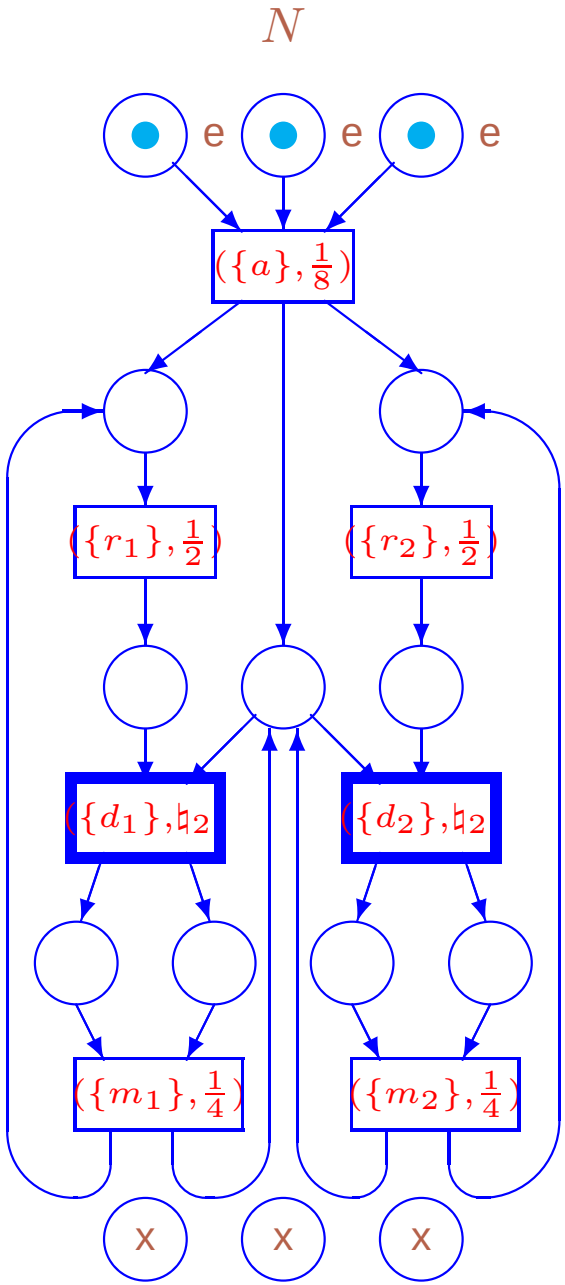
- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is only possible from the states s_2, s_7.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from the first processor* is $\varphi_2 \sum_{\{\Upsilon \mid (\{r_1\},\frac{1}{2}) \in \Upsilon\}} PT(\Upsilon, s_2) + \varphi_7 \sum_{\{\Upsilon \mid (\{r_1\},\frac{1}{2}) \in \Upsilon\}} PT(\Upsilon, s_7) =$
  $\frac{1}{17}\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{3}{17}\left(\frac{3}{8} + \frac{1}{8}\right) = \frac{2}{17}$.



SHMPMBOX: The marked dtsi-boxes of two processors and shared memory



SHMBOX: The marked dtsi-box of the shared memory system

The abstract system and its reduction

The static expression of the first processor is

$F_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{d, y_1\}, \natural_1); (\{m, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$F_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{d, y_2\}, \natural_1); (\{m, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$F_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * ((((\{\widehat{y_1}\}, \natural_1); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \natural_1); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract shared memory system with two processors is

$F = (F_1 \| F_2 \| F_3)\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2$.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS(\overline{F})$ is similar to $TS(\overline{E})$.

$SMC(\overline{F}) \simeq SMC(\overline{E})$, thus, the average sojourn time vectors of $\overline{F}$ and $\overline{E}$, the TPMs and the steady-state PMFs for $EDTMC(\overline{F})$ and $EDTMC(\overline{E})$ coincide.

Performance indices

The first, second, third and fourth performance indices are the same for the standard and abstract systems.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \frac{1}{2})$ is only possible from the states s_2, s_5, s_7.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from a processor* is
  $\varphi_2 \sum_{\{\Upsilon|(\{r\},\frac{1}{2})\in\Upsilon\}} PT(\Upsilon, s_2) + \varphi_5 \sum_{\{\Upsilon|(\{r\},\frac{1}{2})\in\Upsilon\}} PT(\Upsilon, s_5) +$
  $\varphi_7 \sum_{\{\Upsilon|(\{r\},\frac{1}{2})\in\Upsilon\}} PT(\Upsilon, s_7) = \frac{1}{17}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + \frac{3}{17}\left(\frac{3}{8}+\frac{1}{8}\right) + \frac{3}{17}\left(\frac{3}{8}+\frac{1}{8}\right) = \frac{15}{68}$.

The quotient of the abstract system

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6\}$, where

$\mathcal{K}_1 = \{s_1\}$ (the initial state),

$\mathcal{K}_2 = \{s_2\}$ (the system is activated and the memory is not requested),

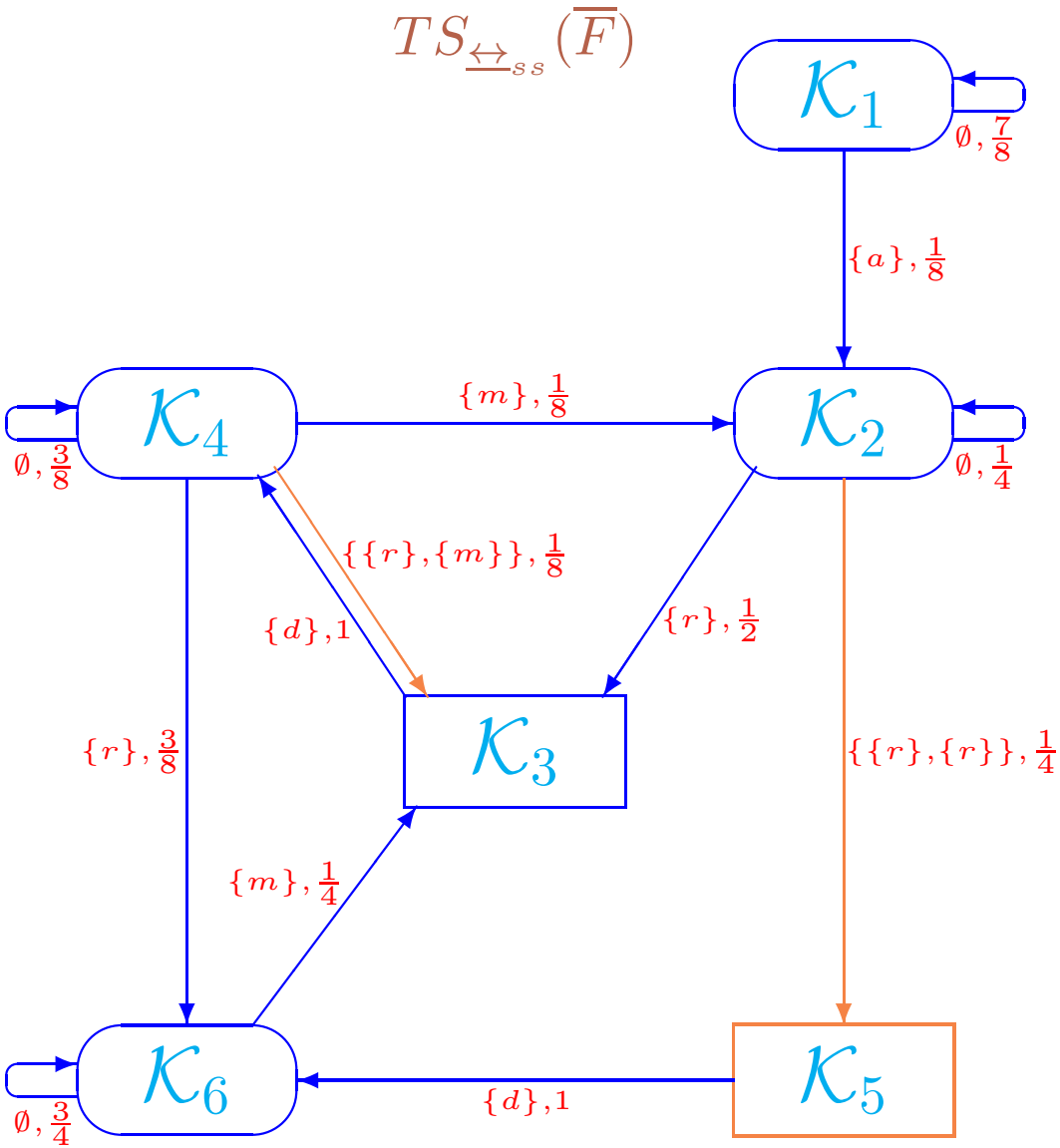
$\mathcal{K}_3 = \{s_3, s_4\}$ (the memory is requested by one processor),

$\mathcal{K}_4 = \{s_5, s_7\}$ (the memory is allocated to a processor),

$\mathcal{K}_5 = \{s_6\}$ (the memory is requested by two processors),

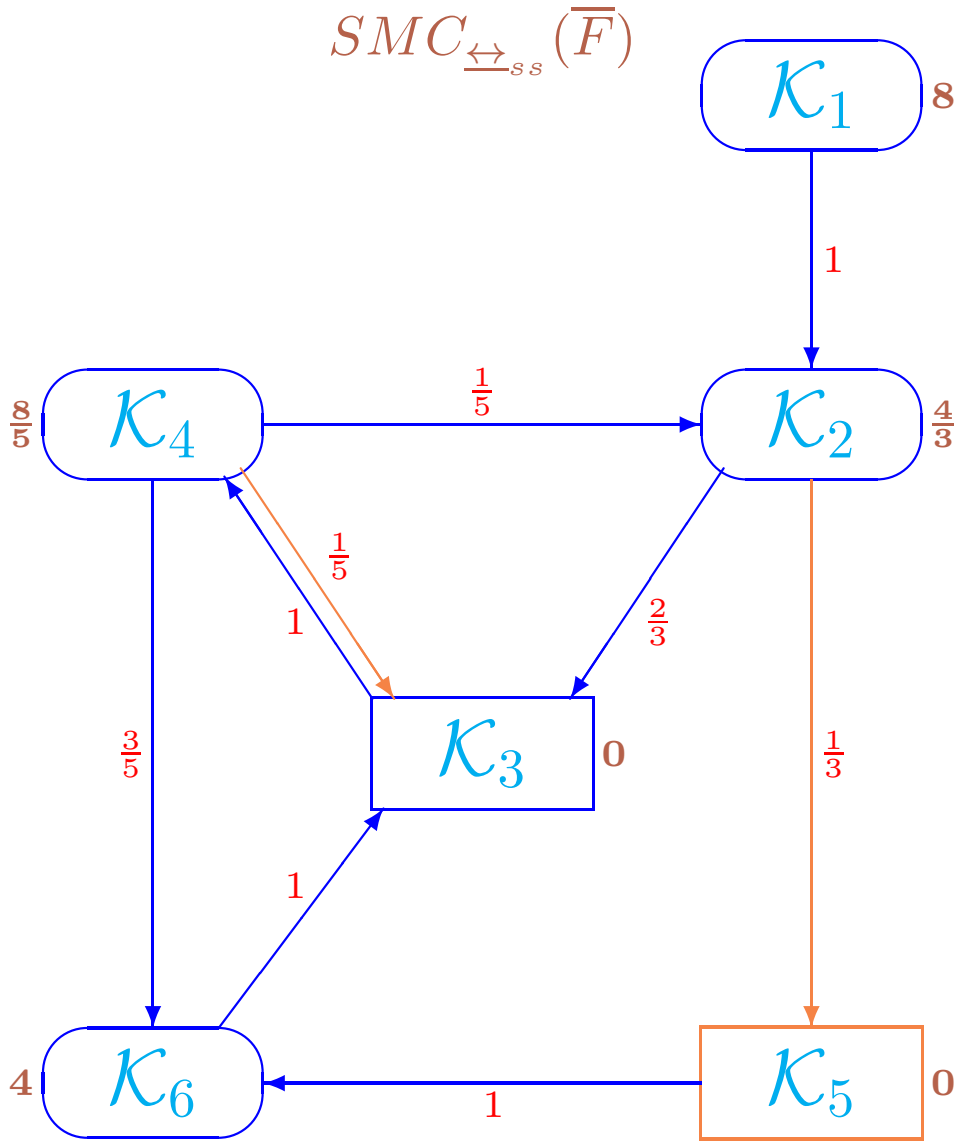
$\mathcal{K}_6 = \{s_8, s_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3, \mathcal{K}_5\}$.



SHMQTS: The quotient transition system of the abstract shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)



SHMQSMC: The quotient underlying SMC of the abstract shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The quotient average sojourn time vector of $\overline{F}$:

$$SJ' = \left(8, \frac{4}{3}, 0, \frac{8}{5}, 0, 4\right).$$

The quotient sojourn time variance vector of $\overline{F}$:

$$VAR' = \left(56, \frac{4}{9}, 0, \frac{24}{25}, 0, 12\right).$$

The TPM for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}'^{*} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi'^{*} = \left(0, \frac{3}{44}, \frac{15}{44}, \frac{15}{44}, \frac{1}{44}, \frac{5}{22}\right).$$

The steady-state PMF ψ'^{*} weighted by SJ':

$$\left(0, \frac{1}{11}, 0, \frac{6}{11}, 0, \frac{10}{11}\right).$$

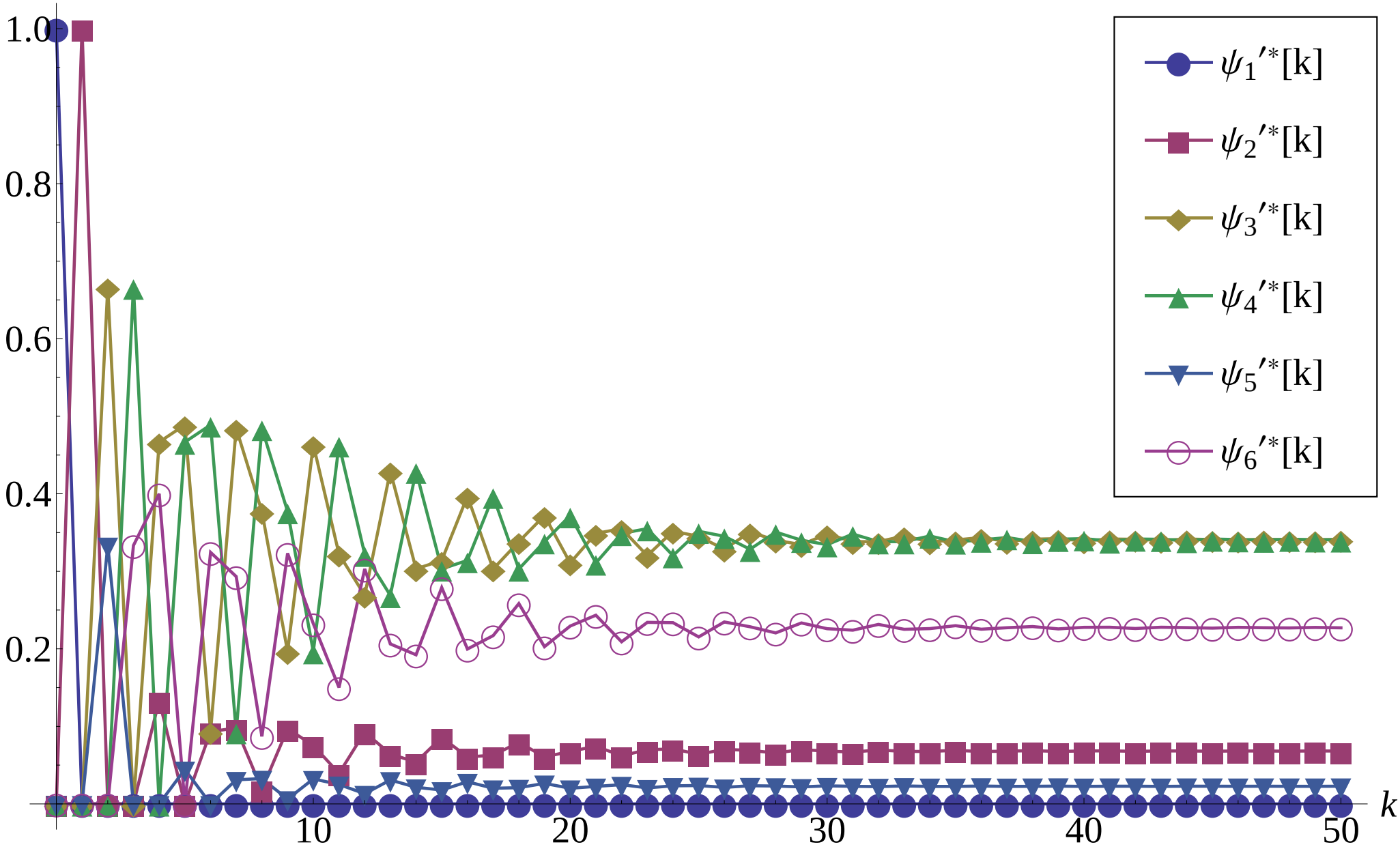
We normalize the steady-state weighted PMF dividing it by the sum of its components $\psi'^{*} SJ'^{T} = \frac{17}{11}$.

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

SHMQTP: Transient and steady-state probabilities for the quotient EDTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
$\psi_1'^{\,*}[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2'^{\,*}[k]$	0	0	0.0754	0.0859	0.0677	0.0641	0.0680	0.0691	0.0683	0.0680	0.0681	0.0682
$\psi_3'^{\,*}[k]$	0	0.4889	0.4633	0.3140	0.3108	0.3452	0.3482	0.3404	0.3392	0.3409	0.3413	0.3409
$\psi_4'^{\,*}[k]$	0	0.4667	0.1964	0.3031	0.3719	0.3517	0.3344	0.3380	0.3422	0.3417	0.3407	0.3409
$\psi_5'^{\,*}[k]$	0	0.0444	0.0323	0.0179	0.0202	0.0237	0.0234	0.0226	0.0226	0.0228	0.0228	0.0227
$\psi_6'^{\,*}[k]$	0	0	0.2325	0.2791	0.2294	0.2154	0.2260	0.2299	0.2277	0.2267	0.2271	0.2273



SHMQTP: Transient probabilities alteration diagram for the quotient EDTMC of the abstract shared memory system

The steady-state PMF for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi'^{*} = \left(0, \frac{3}{44}, \frac{15}{44}, \frac{15}{44}, \frac{1}{44}, \frac{5}{22}\right).$$

The steady-state PMF ψ'^{*} weighted by SJ':

$$\left(0, \frac{1}{11}, 0, \frac{6}{11}, 0, \frac{10}{11}\right).$$

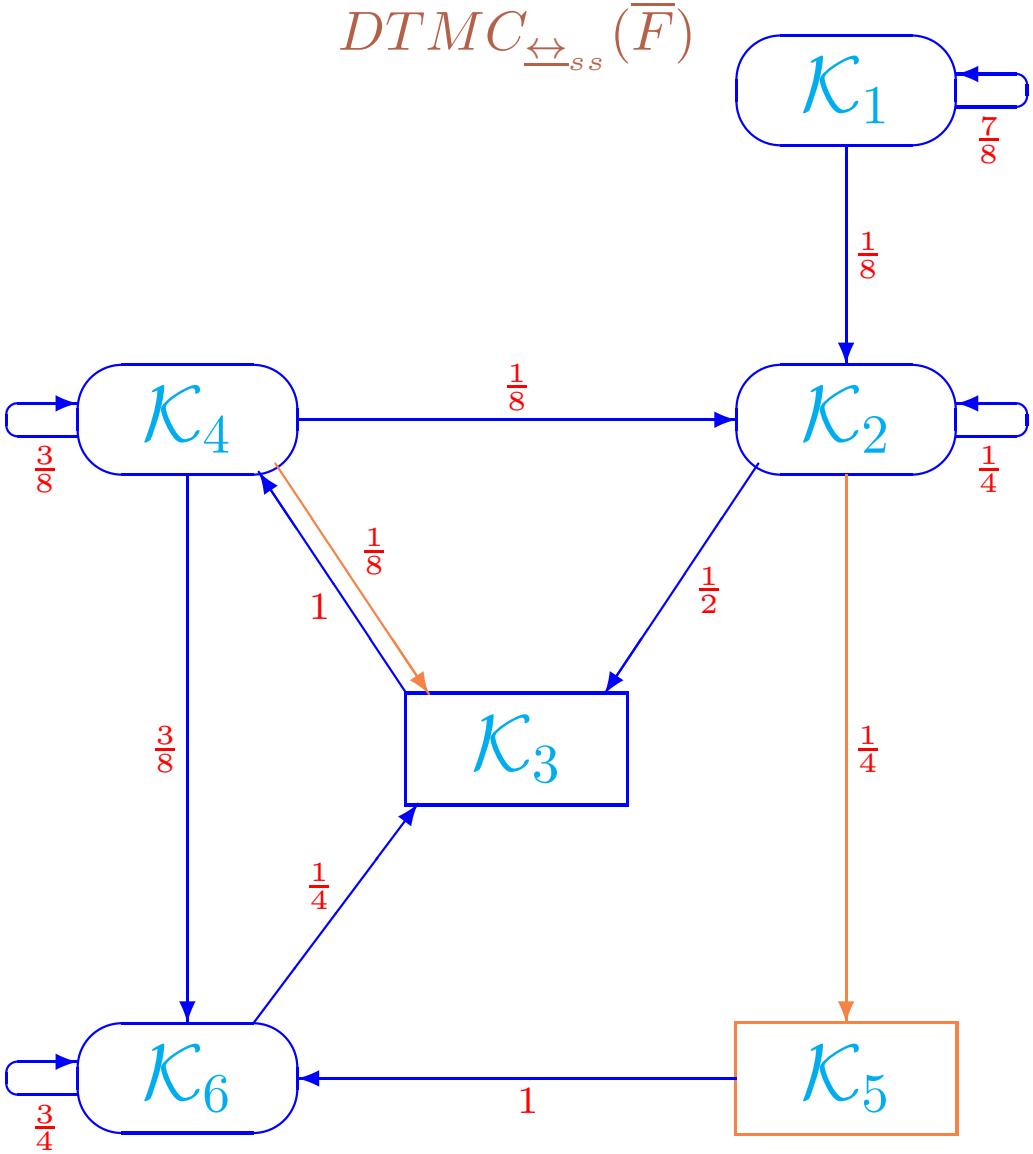
We normalize the steady-state weighted PMF dividing it by the sum of its components

$$\psi'^{*} SJ'^{T} = \frac{17}{11}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

Otherwise, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{F})$, we can construct the quotient DTMC of $\overline{F}$, $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$, and calculate φ' using it.

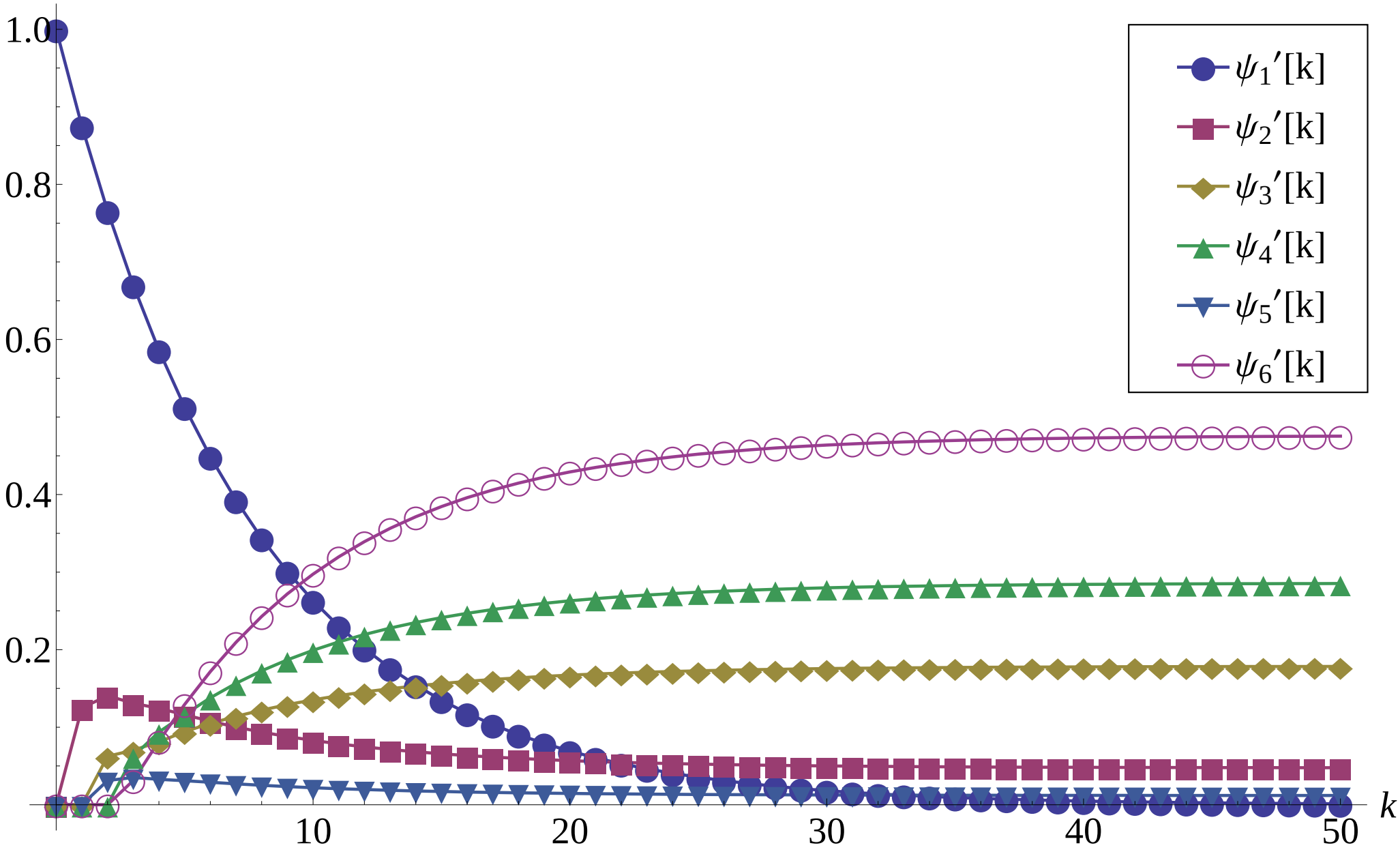


SHMQDTMC: The quotient DTMC of the abstract shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

SHMTPQDTMC: Transient and steady-state probabilities for the quotient DTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
$\psi'_1[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	0
$\psi'_2[k]$	0	0.1161	0.0829	0.0657	0.0569	0.0524	0.0501	0.0489	0.0483	0.0479	0.0478	0.0476
$\psi'_3[k]$	0	0.0944	0.1353	0.1564	0.1672	0.1727	0.1756	0.1770	0.1778	0.1782	0.1784	0.1786
$\psi'_4[k]$	0	0.1162	0.1992	0.2414	0.2630	0.2740	0.2797	0.2826	0.2841	0.2849	0.2853	0.2857
$\psi'_5[k]$	0	0.0311	0.0220	0.0171	0.0146	0.0133	0.0126	0.0123	0.0121	0.0120	0.0120	0.0119
$\psi'_6[k]$	0	0.1294	0.2974	0.3845	0.4292	0.4521	0.4638	0.4698	0.4729	0.4745	0.4753	0.4762



SHMTPQDTMC: Transient probabilities alteration diagram for the quotient DTMC of the abstract shared memory system

The TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}' = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

The steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi' = \left(0, \frac{1}{21}, \frac{5}{28}, \frac{2}{7}, \frac{1}{84}, \frac{10}{21}\right).$$

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3, \mathcal{K}_5\}$. Hence,

$$\sum_{\mathcal{K} \in DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})}} \psi'(\mathcal{K}) = \psi'(\mathcal{K}_1) + \psi'(\mathcal{K}_2) + \psi'(\mathcal{K}_4) + \psi'(\mathcal{K}_6) = \frac{17}{21}.$$

By the "quotient" analogue of Proposition PMFSMC:

$$\varphi'(\mathcal{K}_1) = 0 \cdot \tfrac{21}{17} = 0,$$
$$\varphi'(\mathcal{K}_2) = \tfrac{1}{21} \cdot \tfrac{21}{17} = \tfrac{1}{17},$$
$$\varphi'(\mathcal{K}_3) = 0,$$
$$\varphi'(\mathcal{K}_4) = \tfrac{2}{7} \cdot \tfrac{21}{17} = \tfrac{6}{17},$$
$$\varphi'(\mathcal{K}_5) = 0,$$
$$\varphi'(\mathcal{K}_6) = \tfrac{10}{21} \cdot \tfrac{21}{17} = \tfrac{10}{17}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

This coincides with the result obtained with the use of ψ'^{*} and SJ'.

Alternatively, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{F})$, we can construct $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ and calculate φ' using it.

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3, \mathcal{K}_5\}$.

We reorder the elements of $DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})}$ by moving the equivalence classes of vanishing states to the first positions: $\mathcal{K}_3, \mathcal{K}_5, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6$.

The reordered TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}'_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

The result of the decomposing $\mathbf{P}'_r$:

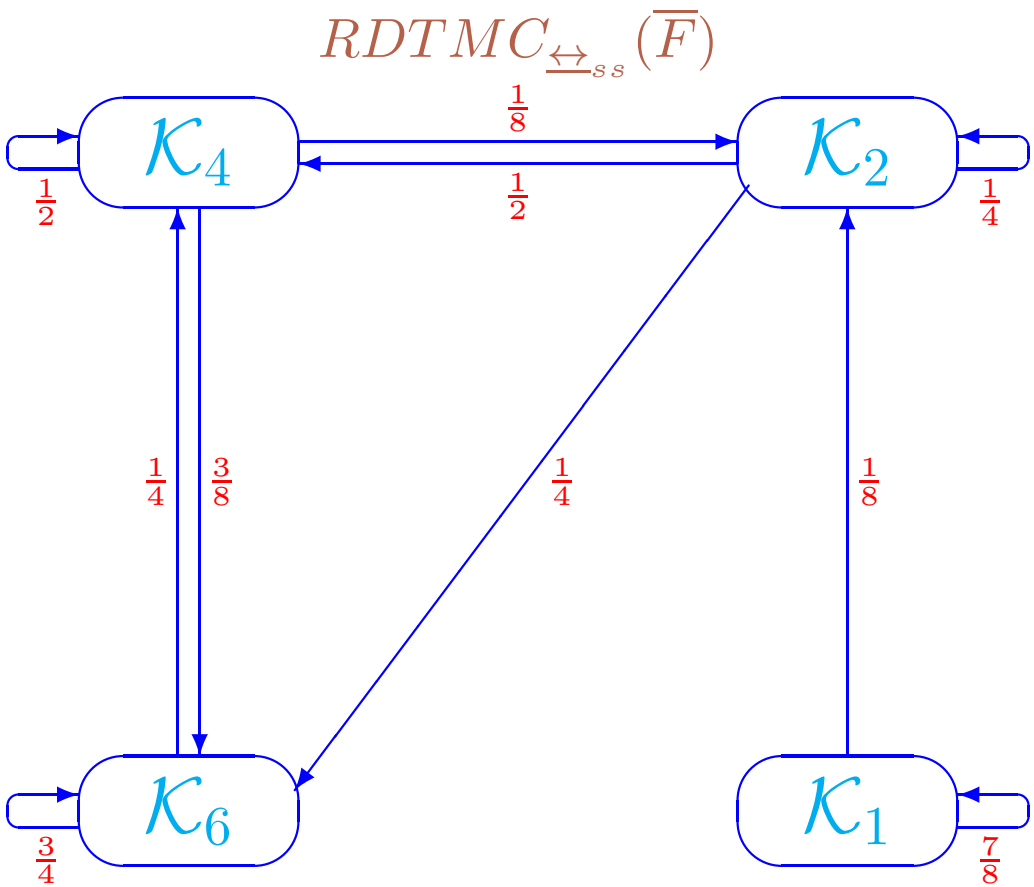
$$\mathbf{C}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{D}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{E}' = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & 0 \\ \frac{1}{4} & 0 \end{pmatrix}, \mathbf{F}' = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

Since $\mathbf{C}'^1 = \mathbf{0}$, we have $\forall k > 0,\ \mathbf{C}'^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\mathbf{G}' = \sum_{k=0}^{l} \mathbf{C}'^l = \mathbf{C}'^0 = \mathbf{I}.$$

The TPM for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

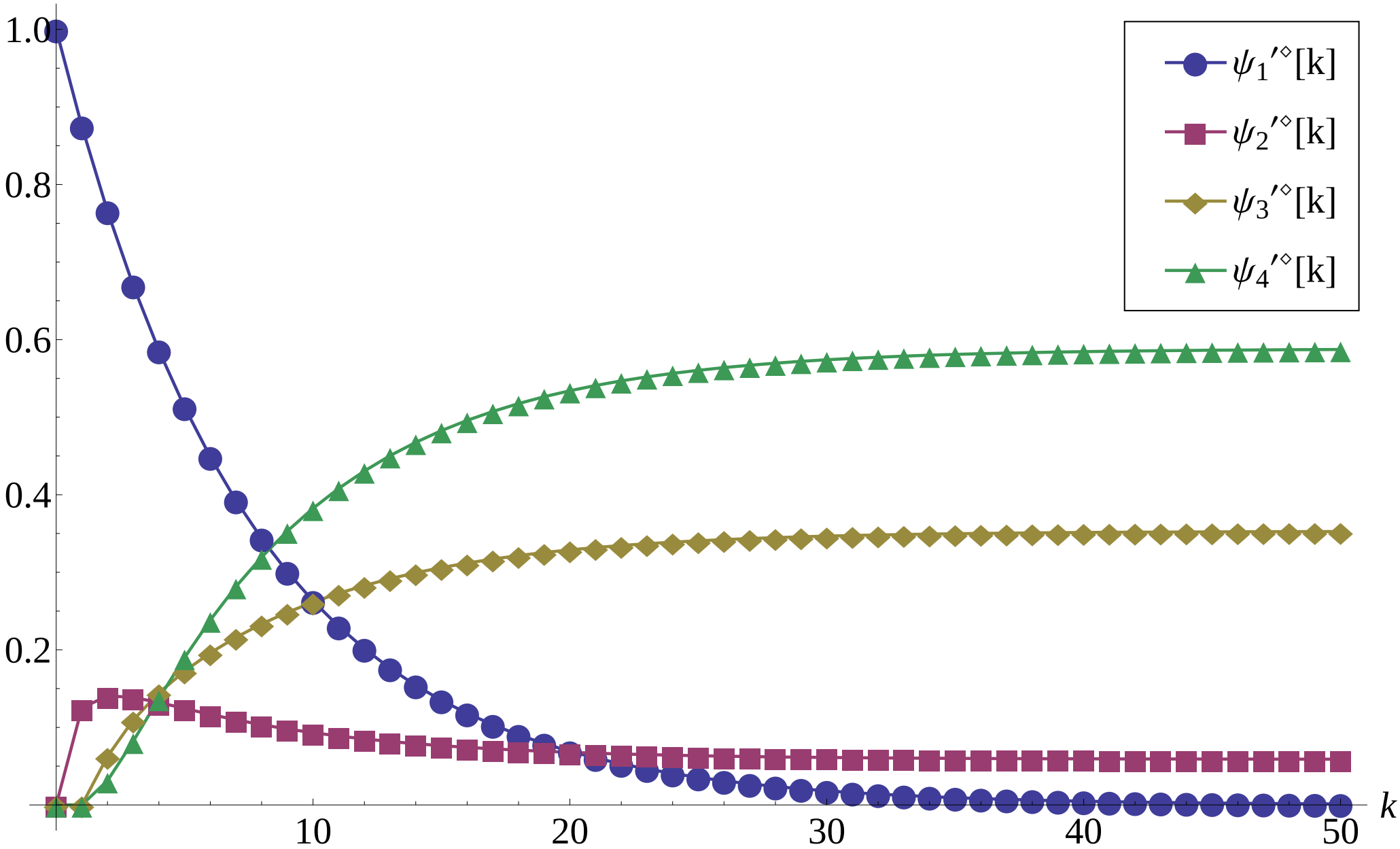
$$\mathbf{P}'^{\diamond} = \mathbf{F}' + \mathbf{E}'\mathbf{G}'\mathbf{D}' = \mathbf{F}' + \mathbf{E}'\mathbf{I}\mathbf{D}' = \mathbf{F}' + \mathbf{E}'\mathbf{D}' = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$



SHMQRDTMC: The reduced quotient DTMC of the abstract shared memory system

SHMQRTP: Transient and steady-state probabilities for the reduced quotient DTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
${\psi'_1}^{\diamond}[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	0
${\psi'_2}^{\diamond}[k]$	0	0.1244	0.0931	0.0764	0.0679	0.0635	0.0612	0.0600	0.0594	0.0591	0.0590	0.0588
${\psi'_3}^{\diamond}[k]$	0	0.1726	0.2614	0.3060	0.3289	0.3406	0.3466	0.3497	0.3513	0.3521	0.3525	0.3529
${\psi'_4}^{\diamond}[k]$	0	0.1901	0.3824	0.4826	0.5341	0.5605	0.5740	0.5810	0.5845	0.5863	0.5872	0.5882



SHMQRTP: Transient probabilities alteration diagram for the reduced quotient DTMC of the abstract shared memory system

The steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi'^{\diamond} = \left(0, \frac{1}{17}, \frac{6}{17}, \frac{10}{17}\right).$$

Note that $\psi'^{\diamond} = (\psi'^{\diamond}(\mathcal{K}_1), \psi'^{\diamond}(\mathcal{K}_2), \psi'^{\diamond}(\mathcal{K}_4), \psi'^{\diamond}(\mathcal{K}_6))$.

By the "quotient" analogue of Proposition PMFSMCT:

$$\varphi'(\mathcal{K}_1) = 0, \quad \varphi'(\mathcal{K}_2) = \tfrac{1}{17}, \quad \varphi'(\mathcal{K}_3) = 0, \quad \varphi'(\mathcal{K}_4) = \tfrac{6}{17}, \quad \varphi'(\mathcal{K}_5) = 0, \quad \varphi'(\mathcal{K}_6) = \tfrac{10}{17}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

This coincides with the result obtained with the use of ψ'^{*} and SJ'.

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\varphi'_2} = \frac{17}{1} = 17$.
- The common memory is available only in the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5$.

  The steady-state probability that the memory is available is $\varphi'_2 + \varphi'_3 + \varphi'_5 = \frac{1}{17} + 0 + 0 = \frac{1}{17}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{1}{17} = \frac{16}{17}$.
- After activation of the system, we leave the state $\mathcal{K}_1$ for all, and the common memory is either requested or allocated in every remaining state, with exception of $\mathcal{K}_2$.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving $\mathcal{K}_2$, calculated as $\frac{\varphi'_2}{SJ'_2} = \frac{1}{17} \cdot \frac{3}{4} = \frac{3}{68}$.

- The parallel common memory request of two processors $\{\{r\},\{r\}\}$ is only possible from the state $\mathcal{K}_2$.

  The request probability in this state is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$ twice.

  The *steady-state probability of the shared memory request from two processors* is $\varphi'_2 \sum_{\{A,\mathcal{K}|\{\{r\},\{r\}\}\subseteq A,\ \mathcal{K}_2 \xrightarrow{A} \mathcal{K}\}} PM_A(\mathcal{K}_2,\mathcal{K}) = \frac{1}{17}\cdot\frac{1}{4} = \frac{1}{68}$.

- The common memory request of a processor $\{r\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_4$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is $\varphi'_2 \sum_{\{A,\mathcal{K}|\{r\}\in A,\ \mathcal{K}_2 \xrightarrow{A} \mathcal{K}\}} PM_A(\mathcal{K}_2,\mathcal{K}) + \varphi'_4 \sum_{\{A,\mathcal{K}|\{r\}\in A,\ \mathcal{K}_4 \xrightarrow{A} \mathcal{K}\}} PM_A(\mathcal{K}_4,\mathcal{K}) = \frac{1}{17}\left(\frac{1}{2}+\frac{1}{4}\right) + \frac{6}{17}\left(\frac{3}{8}+\frac{1}{8}\right) = \frac{15}{68}$.

The performance indices are the same for the complete and quotient abstract shared memory systems.

The coincidence of the first and second performance indices illustrates Proposition STPROB.

The coincidence of the third performance index illustrates Proposition STPROB and Proposition SJAVVA.

The coincidence of the fourth performance index is by Theorem STTRAC:

one should apply its result to the derived step trace $\{\{r\},\{r\}\}$ of $\overline{F}$ and itself.

The coincidence of the fifth performance index is by Theorem STTRAC:

one should apply its result to the derived step traces $\{\{r\}\},\ \{\{r\},\{r\}\},\ \{\{r\},\{m\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

The generalized system

The static expression of the first processor is

$$K_1 = [(\{x_1\}, \rho) * ((\{r_1\}, \rho); (\{d_1, y_1\}, \natural_l); (\{m_1, z_1\}, \rho)) * \mathsf{Stop}].$$

The static expression of the second processor is

$$K_2 = [(\{x_2\}, \rho) * ((\{r_2\}, \rho); (\{d_2, y_2\}, \natural_l); (\{m_2, z_2\}, \rho)) * \mathsf{Stop}].$$

The static expression of the shared memory is

$$K_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \rho) * ((((\{\widehat{y_1}\}, \natural_l); (\{\widehat{z_1}\}, \rho))[]((\{\widehat{y_2}\}, \natural_l); (\{\widehat{z_2}\}, \rho))) * \mathsf{Stop}].$$

The static expression of the generalized shared memory system with two processors is

$$K = (K_1 \| K_2 \| K_3) \ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2.$$

Interpretation of the states

$DR_T(\overline{K}) = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9\}$ and $DR_V(\overline{K}) = \{\tilde{s}_3, \tilde{s}_4, \tilde{s}_6\}$.

$\tilde{s}_1$: the initial state,

$\tilde{s}_2$: the system is activated and the memory is not requested,

$\tilde{s}_3$: the memory is requested by the first processor,

$\tilde{s}_4$: the memory is requested by the second processor,

$\tilde{s}_5$: the memory is allocated to the first processor,

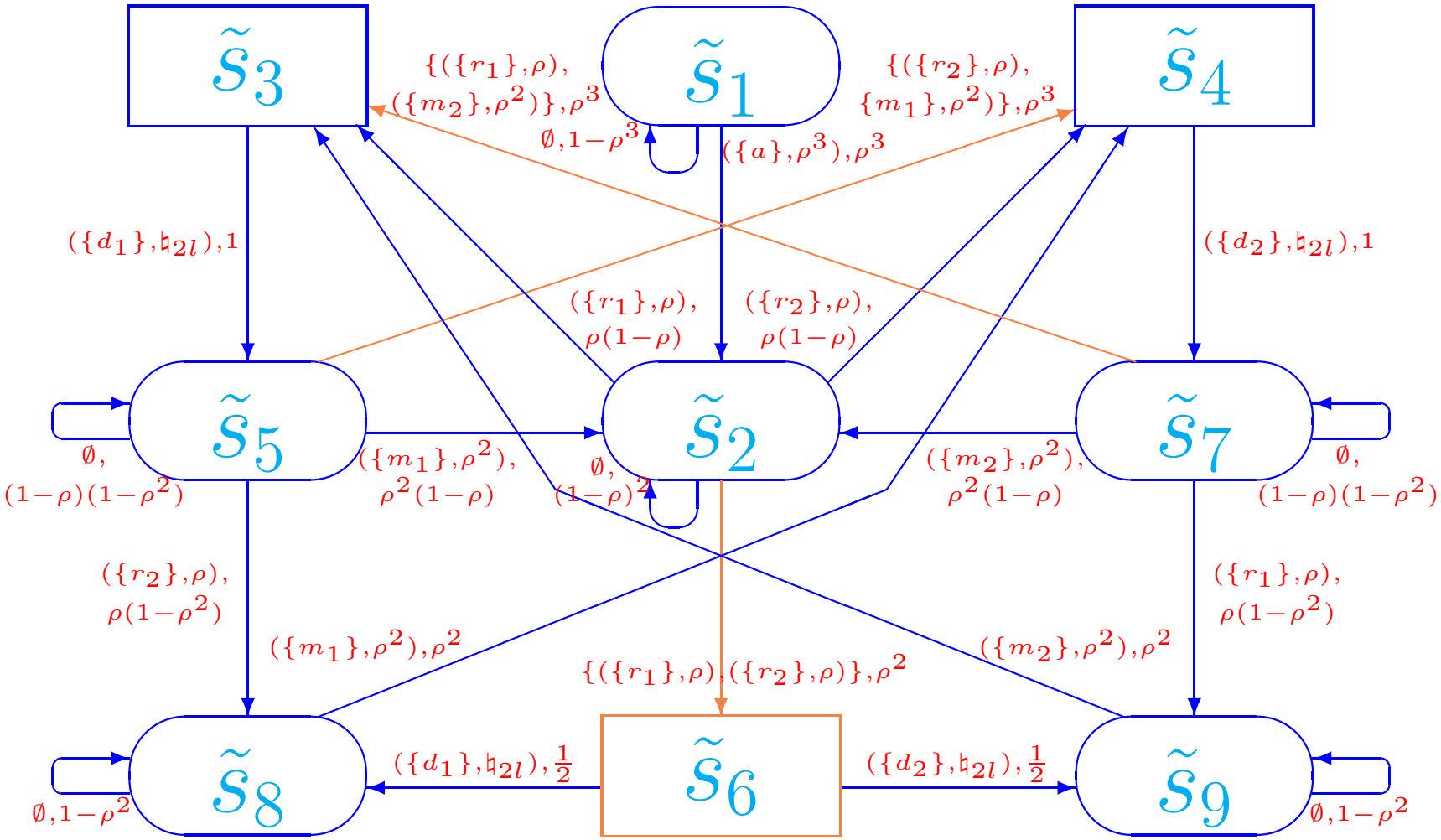
$\tilde{s}_6$: the memory is requested by two processors,

$\tilde{s}_7$: the memory is allocated to the second processor,

$\tilde{s}_8$: the memory is allocated to the first processor and the memory is requested by the second processor,

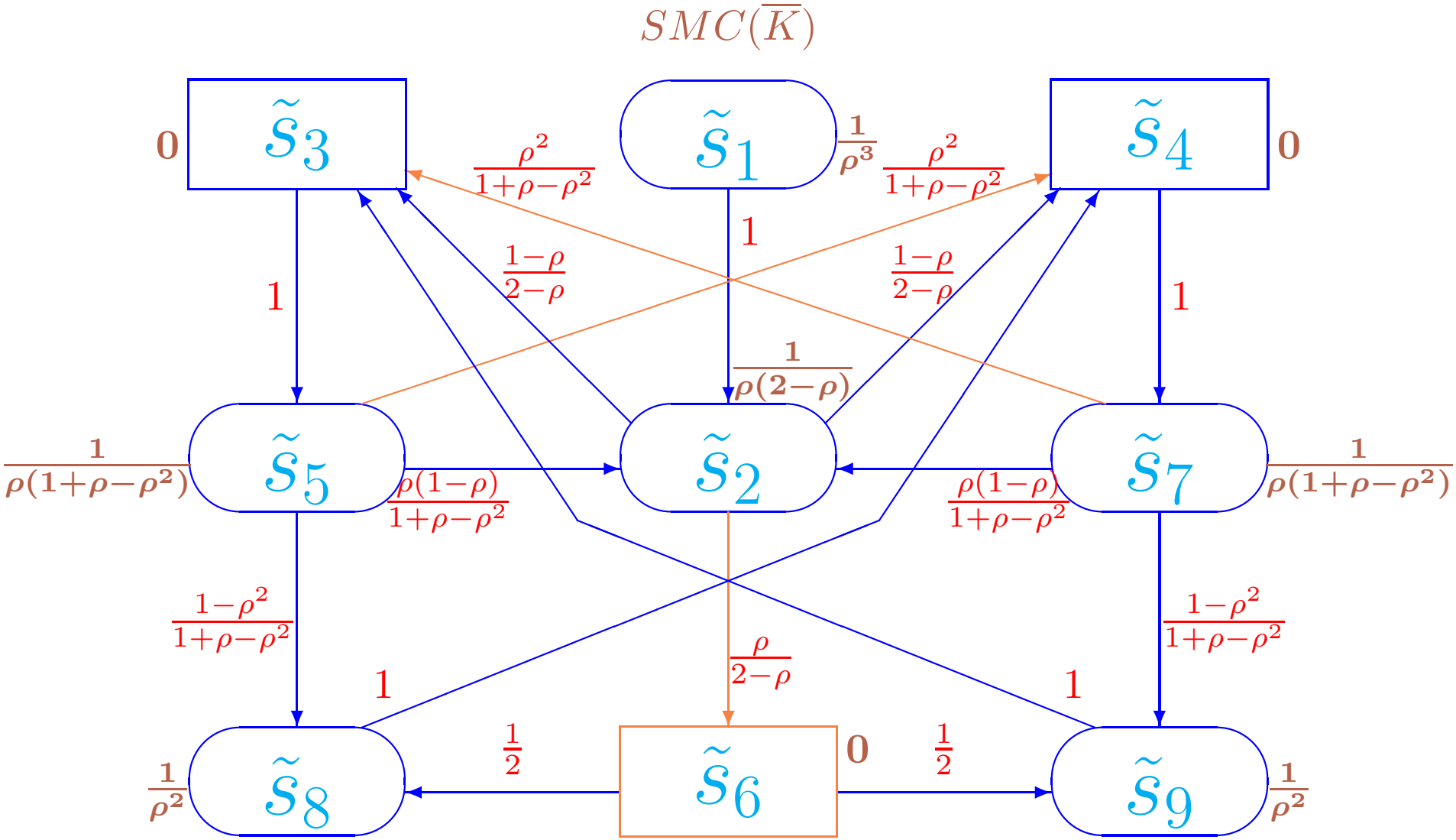
$\tilde{s}_9$: the memory is allocated to the second processor and the memory is requested by the first processor.

$TS(\overline{K})$



SHMGTS: The transition system of the generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)



SHMGSMC: The underlying SMC of the generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The average sojourn time vector of $\overline{K}$:

$$\widetilde{SJ} = \left(\frac{1}{\rho^3}, \frac{1}{\rho(2-\rho)}, 0, 0, \frac{1}{\rho(1+\rho-\rho^2)}, 0, \frac{1}{\rho(1+\rho-\rho^2)}, \frac{1}{\rho^2}, \frac{1}{\rho^2}\right).$$

The sojourn time variance vector of $\overline{K}$:

$$\widetilde{VAR} = \left(\frac{1-\rho^3}{\rho^6}, \frac{(1-\rho)^2}{\rho^2(2-\rho)^2}, 0, 0, \frac{(1-\rho)(1-\rho^2)}{\rho^2(1+\rho-\rho^2)^2}, 0, \frac{(1-\rho)(1-\rho^2)}{\rho^2(1+\rho-\rho^2)^2}, \frac{1-\rho^2}{\rho^4}, \frac{1-\rho^2}{\rho^4}\right).$$

The TPM for $EDTMC(\overline{K})$:

$$\widetilde{\mathbf{P}}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\rho}{2-\rho} & \frac{1-\rho}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC(\overline{K})$:

$$\tilde{\psi}^* = \frac{1}{2(6+3\rho-9\rho^2+2\rho^3)}(0, 2\rho(1-\rho)(2-\rho), (2-\rho)(1+\rho-\rho^2), (2-\rho)(1+\rho-\rho^2),$$
$$(2-\rho)(1+\rho-\rho^2), 2\rho^2(1-\rho), (2-\rho)(1+\rho-\rho^2), (2+\rho)(1-\rho), (2+\rho)(1-\rho)).$$

The steady-state PMF $\tilde{\psi}^*$ weighted by $\widetilde{SJ}$:

$$\frac{1}{2\rho^2(6+3\rho-9\rho^2+2\rho^3)}(0, 2\rho^2(1-\rho), 0, 0, \rho(2-\rho), 0, \rho(2-\rho), (2+\rho)(1-\rho), (2+\rho)(1-\rho)).$$

We normalize the steady-state weighted PMF dividing it by the sum of its components

$$\tilde{\psi}^* \widetilde{SJ}^T = \frac{2+\rho-\rho^2-\rho^3}{\rho^2(6+3\rho-9\rho^2+2\rho^3)}.$$

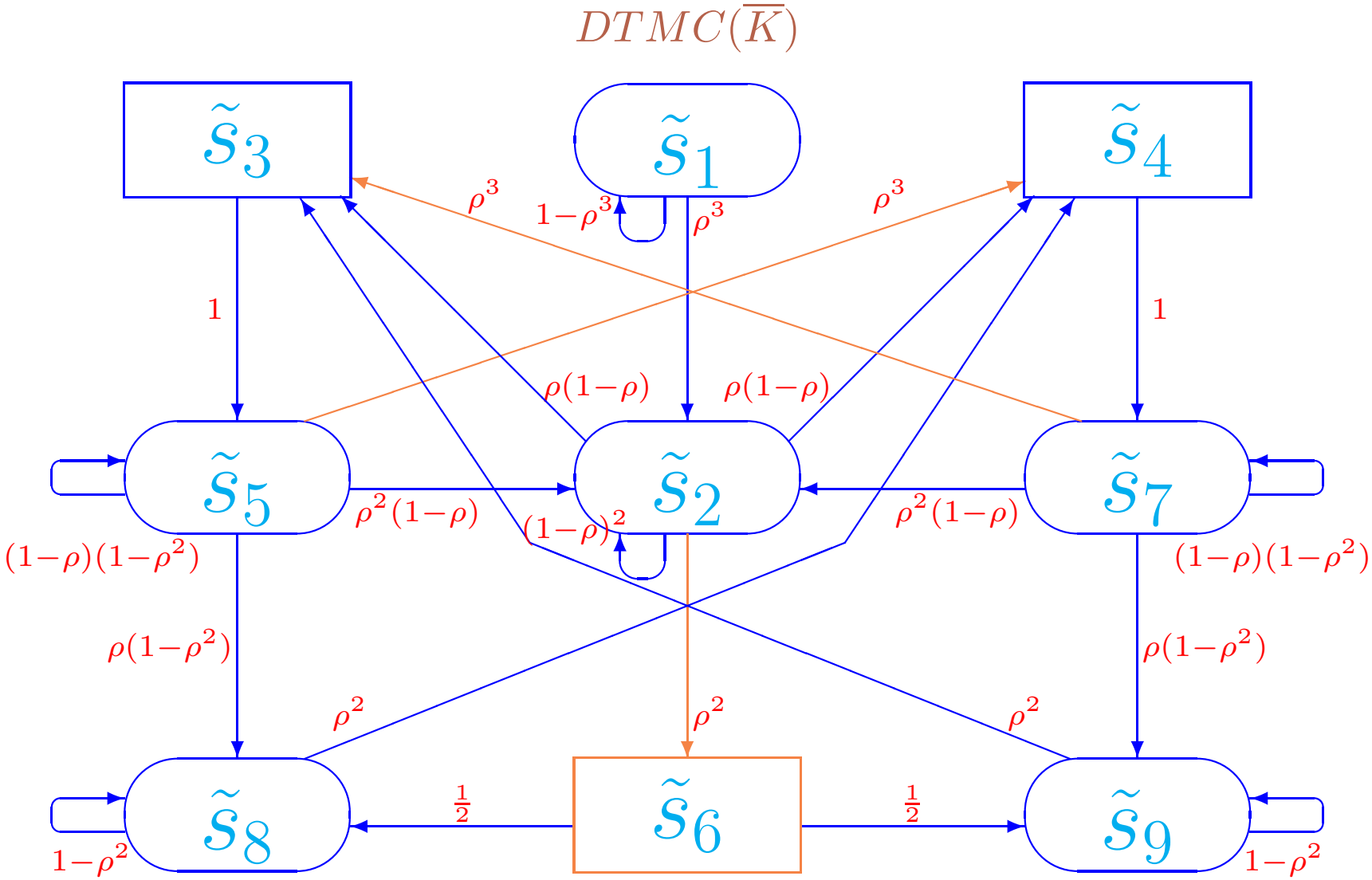
The steady-state PMF for $SMC(\overline{K})$:

$$\tilde{\varphi} = \frac{1}{2(2+\rho-\rho^2-\rho^3)}(0, 2\rho^2(1-\rho), 0, 0, \rho(2-\rho), 0, \rho(2-\rho), (2+\rho)(1-\rho), (2+\rho)(1-\rho)).$$

Otherwise, from $TS(\overline{K})$, we can construct $DTMC(\overline{K})$ and calculate $\tilde{\varphi}$ using it.

The TPM for $DTMC(\overline{K})$:

$$\widetilde{\mathbf{P}} = \begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\rho)^2 & \rho(1-\rho) & \rho(1-\rho) & 0 & \rho^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \rho^2(1-\rho) & 0 & \rho^3 & (1-\rho)(1-\rho^2) & 0 & 0 & \rho(1-\rho^2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \rho^2(1-\rho) & \rho^3 & 0 & 0 & 0 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & 0 & 0 & \rho^2 & 0 & 0 & 0 & 1-\rho^2 & 0 \\ 0 & 0 & \rho^2 & 0 & 0 & 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$



SHMGDTMC: The DTMC of the generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The steady-state PMF for $DTMC(\overline{K})$:

$$\tilde{\psi} = \frac{1}{2(1+\rho)(2-\rho+2\rho^2-2\rho^3)}(0, 2\rho^2(1-\rho), \rho^2(2-\rho)(1+\rho-\rho^2), \rho^2(2-\rho)(1+\rho-\rho^2),$$
$$\rho(2-\rho), 2\rho^4(1-\rho), \rho(2-\rho), (2+\rho)(1-\rho), (2+\rho)(1-\rho)).$$

Remember that $DR_T(\overline{K}) = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9\}$ and $DR_V(\overline{K}) = \{\tilde{s}_3, \tilde{s}_4, \tilde{s}_6\}$. Hence,

$$\sum_{\tilde{s}\in DR_T(\overline{K})} \tilde{\psi}(\tilde{s}) = \tilde{\psi}(\tilde{s}_1)+\tilde{\psi}(\tilde{s}_2)+\tilde{\psi}(\tilde{s}_5)+\tilde{\psi}(\tilde{s}_7)+\tilde{\psi}(\tilde{s}_8)+\tilde{\psi}(\tilde{s}_9) = \frac{2+\rho-\rho^2-\rho^3}{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}.$$

By Proposition PMFSMC:

$$\tilde{\varphi}(\tilde{s}_1) = 0 \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = 0,$$

$$\tilde{\varphi}(\tilde{s}_2) = \frac{\rho^2(1-\rho)}{(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3},$$

$$\tilde{\varphi}(\tilde{s}_3) = 0,$$

$$\tilde{\varphi}(\tilde{s}_4) = 0,$$

$$\tilde{\varphi}(\tilde{s}_5) = \frac{\rho(2-\rho)}{2(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)},$$

$$\tilde{\varphi}(\tilde{s}_6) = 0,$$

$$\tilde{\varphi}(\tilde{s}_7) = \frac{\rho(2-\rho)}{2(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)},$$

$$\tilde{\varphi}(\tilde{s}_8) = \frac{(2+\rho)(1-\rho)}{2(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{(2+\rho)(1-\rho)}{2(2+\rho-\rho^2-\rho^3)},$$

$$\tilde{\varphi}(\tilde{s}_9) = \frac{(2+\rho)(1-\rho)}{2(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{(2+\rho)(1-\rho)}{2(2+\rho-\rho^2-\rho^3)}.$$

The steady-state PMF for $SMC(\overline{K})$:

$$\tilde{\varphi}=\frac{1}{2(2+\rho-\rho^2-\rho^3)}(0,2\rho^2(1-\rho),0,0,\rho(2-\rho),0,\rho(2-\rho),(2+\rho)(1-\rho),(2+\rho)(1-\rho)).$$

This coincides with the result obtained with the use of $\tilde{\psi}^*$ and $\widetilde{SJ}$.

Alternatively, from $TS(\overline{K})$, we can construct $RDTMC(\overline{K})$,

and calculate $\tilde{\varphi}$ using it.

$DR_T(\overline{K}) = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_7, \tilde{s}_8, \tilde{s}_9\}$ and $DR_V(\overline{K}) = \{\tilde{s}_3, \tilde{s}_4, \tilde{s}_6\}$.

We reorder the elements of $DR(\overline{K})$ by

moving vanishing states to the first positions: $\tilde{s}_3, \tilde{s}_4, \tilde{s}_6, \tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_7, \tilde{s}_8, \tilde{s}_9$.

The reordered TPM for $DTMC(\overline{K})$:

$$\widetilde{\mathbf{P}}_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\ \rho(1-\rho) & \rho(1-\rho) & \rho^2 & 0 & (1-\rho)^2 & 0 & 0 & 0 & 0 \\ 0 & \rho^3 & 0 & 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) & 0 \\ \rho^3 & 0 & 0 & 0 & \rho^2(1-\rho) & 0 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & \rho^2 & 0 & 0 & 0 & 0 & 0 & 1-\rho^2 & 0 \\ \rho^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

The result of the decomposing $\widetilde{\mathbf{P}}_r$:

$$\widetilde{\mathbf{C}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \widetilde{\mathbf{D}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \widetilde{\mathbf{E}} = \begin{pmatrix} 0 & 0 & 0 \\ \rho(1-\rho) & \rho(1-\rho) & \rho^2 \\ 0 & \rho^3 & 0 \\ \rho^3 & 0 & 0 \\ 0 & \rho^2 & 0 \\ \rho^2 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathbf{F}} = \begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\ 0 & (1-\rho)^2 & 0 & 0 & 0 & 0 \\ 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) & 0 \\ 0 & \rho^2(1-\rho) & 0 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & 0 & 0 & 0 & 1-\rho^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

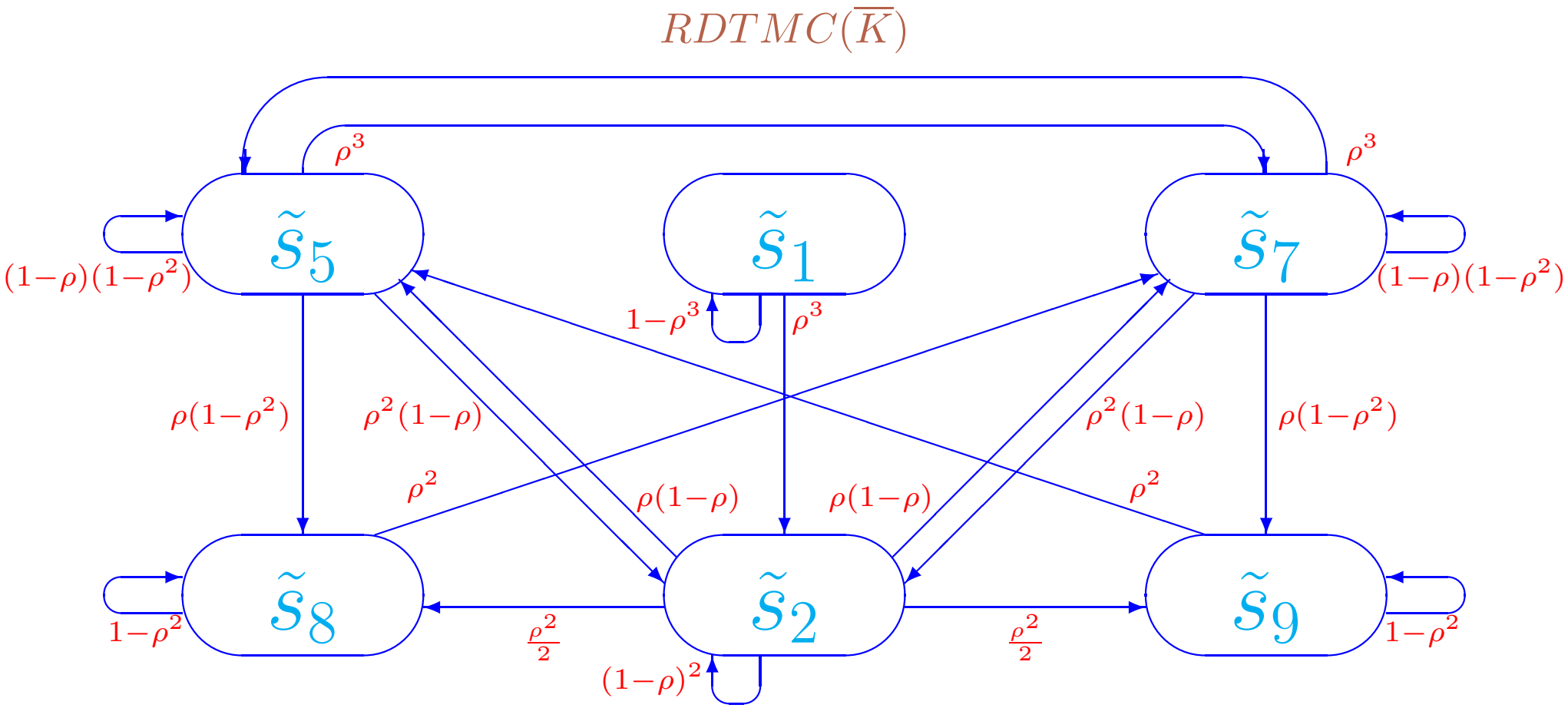
Since $\widetilde{\mathbf{C}}^1 = \mathbf{0}$, we have $\forall k > 0,\ \widetilde{\mathbf{C}}^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\widetilde{\mathbf{G}} = \sum_{k=0}^{l} \widetilde{\mathbf{C}}^k = \widetilde{\mathbf{C}}^0 = \mathbf{I}.$$

The TPM for $RDTMC(\overline{K})$:

$$\widetilde{\mathbf{P}}^{\diamond} = \widetilde{\mathbf{F}} + \widetilde{\mathbf{E}}\widetilde{\mathbf{G}}\widetilde{\mathbf{D}} = \widetilde{\mathbf{F}} + \widetilde{\mathbf{E}}\mathbf{I}\widetilde{\mathbf{D}} = \widetilde{\mathbf{F}} + \widetilde{\mathbf{E}}\widetilde{\mathbf{D}} =$$

$$\begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\ 0 & (1-\rho)^2 & \rho(1-\rho) & \rho(1-\rho) & \frac{\rho^2}{2} & \frac{\rho^2}{2} \\ 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & \rho^3 & \rho(1-\rho^2) & 0 \\ 0 & \rho^2(1-\rho) & \rho^3 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & 0 & 0 & \rho^2 & 1-\rho^2 & 0 \\ 0 & 0 & \rho^2 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$



SHMGRDTMC: The reduced DTMC of the generalized shared memory system

The steady-state PMF for $RDTMC(\overline{K})$:

$$\tilde{\psi}^{\diamond} = \frac{1}{2(2+\rho-\rho^2-\rho^3)}(0, 2\rho^2(1-\rho), \rho(2-\rho), \rho(2-\rho), (2+\rho)(1-\rho), (2+\rho)(1-\rho)).$$

Note that $\tilde{\psi}^{\diamond} = (\tilde{\psi}^{\diamond}(\tilde{s}_1), \tilde{\psi}^{\diamond}(\tilde{s}_2), \tilde{\psi}^{\diamond}(\tilde{s}_5), \tilde{\psi}^{\diamond}(\tilde{s}_7), \tilde{\psi}^{\diamond}(\tilde{s}_8), \tilde{\psi}^{\diamond}(\tilde{s}_9))$.

By Proposition PMFSMCT:

$$\tilde{\varphi}(\tilde{s}_1) = 0, \qquad \tilde{\varphi}(\tilde{s}_2) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}, \qquad \tilde{\varphi}(\tilde{s}_3) = 0,$$

$$\tilde{\varphi}(\tilde{s}_4) = 0, \qquad \tilde{\varphi}(\tilde{s}_5) = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}, \qquad \tilde{\varphi}(\tilde{s}_6) = 0,$$

$$\tilde{\varphi}(\tilde{s}_7) = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}, \qquad \tilde{\varphi}(\tilde{s}_8) = \frac{(2+\rho)(1-\rho)}{2(2+\rho-\rho^2-\rho^3)}, \qquad \tilde{\varphi}(\tilde{s}_9) = \frac{(2+\rho)(1-\rho)}{2(2+\rho-\rho^2-\rho^3)}.$$

The steady-state PMF for $SMC(\overline{K})$:

$$\tilde{\varphi} = \frac{1}{2(2+\rho-\rho^2-\rho^3)}(0, 2\rho^2(1-\rho), 0, 0, \rho(2-\rho), 0, \rho(2-\rho), (2+\rho)(1-\rho), (2+\rho)(1-\rho)).$$

This coincides with the result obtained with the use of $\tilde{\psi}^*$ and $\widetilde{SJ}$.

Performance indices

- The average recurrence time in the state $\tilde{s}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\tilde{\varphi}_2} = \frac{2+\rho-\rho^2-\rho^3}{\rho^2(1-\rho)}$.

- The common memory is available only in the states $\tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_6$.

  The steady-state probability that the memory is available is
  $\tilde{\varphi}_2 + \tilde{\varphi}_3 + \tilde{\varphi}_4 + \tilde{\varphi}_6 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} + 0 + 0 + 0 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} = \frac{2+\rho-2\rho^2}{2+\rho-\rho^2-\rho^3}$.

- After activation of the system, we leave the state $\tilde{s}_1$ for all, and the common memory is either requested or allocated in every remaining state, with exception of $\tilde{s}_2$.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving $\tilde{s}_2$, calculated as $\frac{\tilde{\varphi}_2}{\widetilde{SJ}_2} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} \cdot \frac{\rho(2-\rho)}{1} = \frac{\rho^3(1-\rho)(2-\rho)}{2+\rho-\rho^2-\rho^3}$.

- The parallel common memory request of two processors $\{(\{r_1\},\rho),(\{r_2\},\rho)\}$ is only possible from the state $\tilde{s}_2$.

  The request probability in this state is the sum of the execution probabilities for all multisets of activities containing both $(\{r_1\},\rho)$ and $(\{r_2\},\rho)$.

  The *steady-state probability of the shared memory request from two processors* is
  $\tilde{\varphi}_2 \sum_{\{\Upsilon|(\{(\{r_1\},\rho),(\{r_2\},\rho)\}\subseteq\Upsilon\}} PT(\Upsilon,\tilde{s}_2) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}\rho^2 = \frac{\rho^4(1-\rho)}{2+\rho-\rho^2-\rho^3}$.

- The common memory request of the first processor $(\{r_1\},\rho)$ is only possible from the states $\tilde{s}_2,\tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r_1\},\rho)$.

  The *steady-state probability of the shared memory request from the first processor* is
  $\tilde{\varphi}_2 \sum_{\{\Upsilon|(\{r_1\},\rho)\in\Upsilon\}} PT(\Upsilon,\tilde{s}_2) + \tilde{\varphi}_7 \sum_{\{\Upsilon|(\{r_1\},\rho)\in\Upsilon\}} PT(\Upsilon,\tilde{s}_7) =$
  $\frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}(\rho(1-\rho)+\rho^2) + \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}(\rho(1-\rho^2)+\rho^3) = \frac{\rho^2(2+\rho-2\rho^2)}{2(2+\rho-\rho^2-\rho^3)}$.

The abstract generalized system and its reduction

The static expression of the first processor is

$L_1 = [(\{x_1\}, \rho) * ((\{r\}, \rho); (\{d, y_1\}, \natural_l); (\{m, z_1\}, \rho)) * \mathsf{Stop}]$.

The static expression of the second processor is

$L_2 = [(\{x_2\}, \rho) * ((\{r\}, \rho); (\{d, y_2\}, \natural_l); (\{m, z_2\}, \rho)) * \mathsf{Stop}]$.

The static expression of the shared memory is

$L_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \rho) * ((((\{\widehat{y_1}\}, \natural_l); (\{\widehat{z_1}\}, \rho))[]((\{\widehat{y_2}\}, \natural_l); (\{\widehat{z_2}\}, \rho))) * \mathsf{Stop}]$.

The static expression of the abstract generalized shared memory system with two processors is

$L = (L_1 \| L_2 \| L_3)\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2$.

$DR(\overline{L})$ resembles $DR(\overline{K})$, and $TS(\overline{L})$ is similar to $TS(\overline{K})$.

$SMC(\overline{L}) \simeq SMC(\overline{K})$, thus, the average sojourn time vectors of $\overline{L}$ and $\overline{K}$,
the TPMs and the steady-state PMFs for $EDTMC(\overline{L})$ and $EDTMC(\overline{K})$ coincide.

Performance indices

The first, second, third and fourth performance indices are the same for the generalized system and its abstract modification.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \rho)$ is only possible from the states $\tilde{s}_2, \tilde{s}_5, \tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r\}, \rho)$.

  The *steady-state probability of the shared memory request from a processor* is

  $$\tilde{\varphi}_2 \sum_{\{\Upsilon|(\{r\},\rho)\in\Upsilon\}} PT(\Upsilon, \tilde{s}_2) + \tilde{\varphi}_5 \sum_{\{\Upsilon|(\{r\},\rho)\in\Upsilon\}} PT(\Upsilon, \tilde{s}_5) + \tilde{\varphi}_7 \sum_{\{\Upsilon|(\{r\},\rho)\in\Upsilon\}} PT(\Upsilon, \tilde{s}_7) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}(\rho(1-\rho)+\rho(1-\rho)+\rho^2) + \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}(\rho(1-\rho^2)+\rho^3) + \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}(\rho(1-\rho^2)+\rho^3) = \frac{\rho^2(2-\rho)(1+\rho-\rho^2)}{2+\rho-\rho^2-\rho^3}.$$

The quotient of the abstract generalized system

$DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_5, \widetilde{\mathcal{K}}_6\}$, where

$\widetilde{\mathcal{K}}_1 = \{\tilde{s}_1\}$ (the initial state),

$\widetilde{\mathcal{K}}_2 = \{\tilde{s}_2\}$ (the system is activated and the memory is not requested),

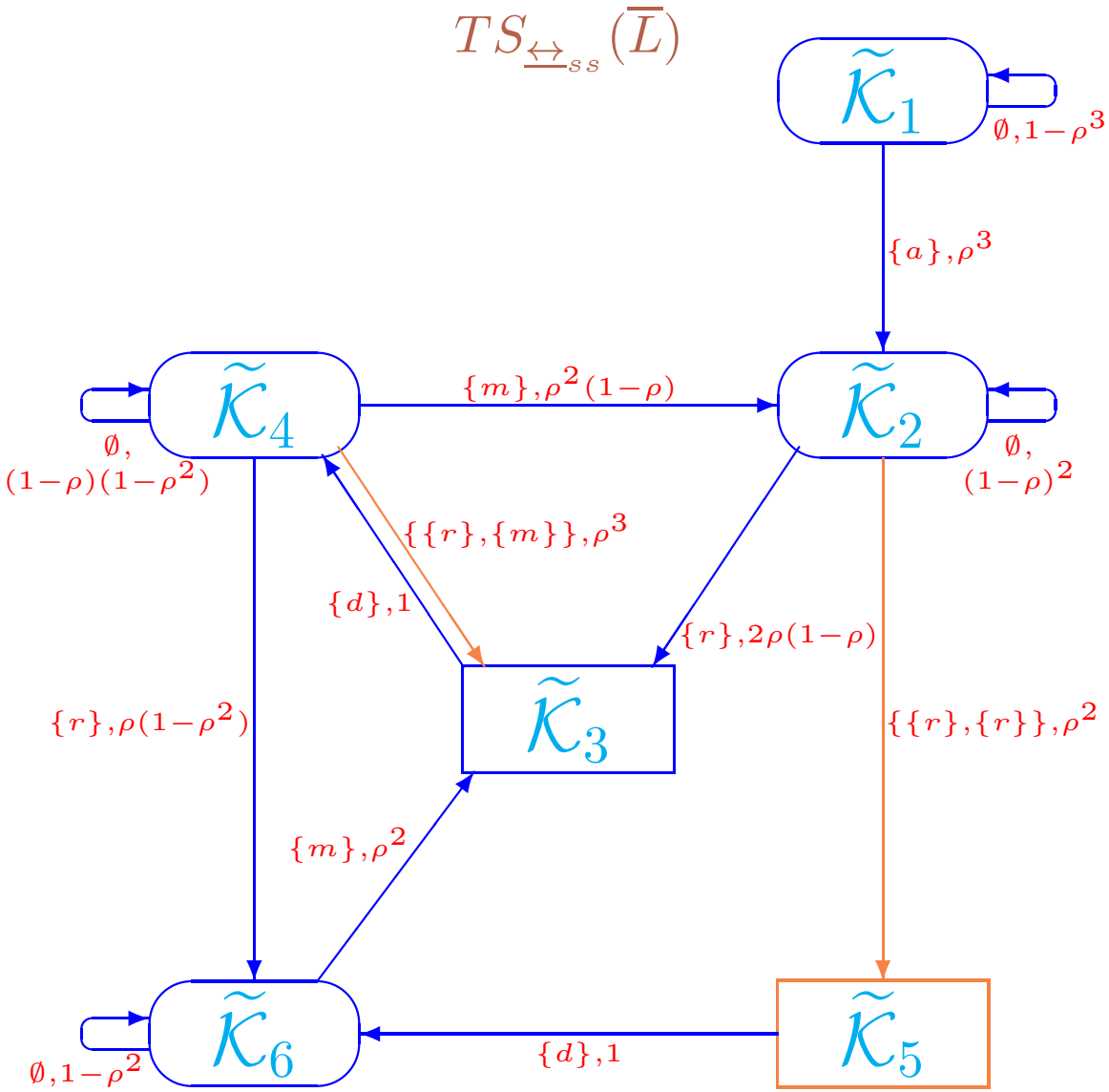
$\widetilde{\mathcal{K}}_3 = \{\tilde{s}_3, \tilde{s}_4\}$ (the memory is requested by one processor),

$\widetilde{\mathcal{K}}_4 = \{\tilde{s}_5, \tilde{s}_7\}$ (the memory is allocated to a processor),

$\widetilde{\mathcal{K}}_5 = \{\tilde{s}_6\}$ (the memory is requested by two processors),

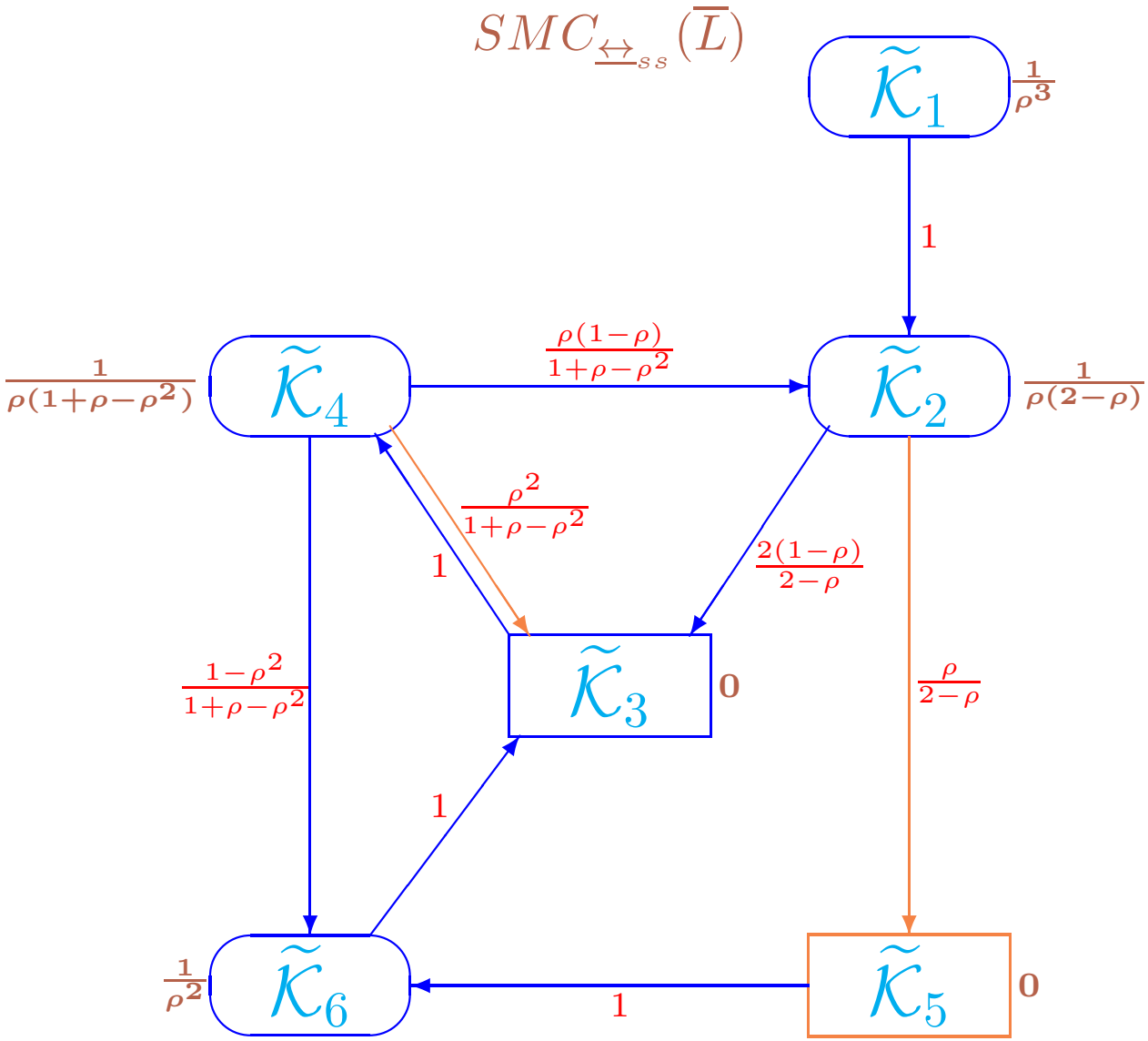
$\widetilde{\mathcal{K}}_6 = \{\tilde{s}_8, \tilde{s}_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).

$DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6\}$ and $DR_V(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5\}$.



SHMGQTS: The quotient transition system of the abstract generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)



SHMGQSMC: The quotient underlying SMC of the abstract generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The quotient average sojourn time vector of $\overline{F}$:

$$\widetilde{SJ}' = \left(\frac{1}{\rho^3}, \frac{1}{\rho(2-\rho)}, 0, \frac{1}{\rho(1+\rho-\rho^2)}, 0, \frac{1}{\rho^2}\right).$$

The quotient sojourn time variance vector of $\overline{F}$:

$$\widetilde{VAR}' = \left(\frac{1-\rho^3}{\rho^6}, \frac{(1-\rho)^2}{\rho^2(2-\rho)^2}, 0, \frac{(1-\rho)(1-\rho^2)}{\rho^2(1+\rho-\rho^2)^2}, 0, \frac{1-\rho^2}{\rho^4}\right).$$

The TPM for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}'^{*} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2(1-\rho)}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\psi}'^{*} = \frac{1}{6+3\rho-9\rho^2+2\rho^3}(0, \rho(1-\rho)(2-\rho), (2-\rho)(1+\rho-\rho^2), \\ (2-\rho)(1+\rho-\rho^2), \rho^2(1-\rho), (2+\rho)(1-\rho)).$$

The steady-state PMF $\tilde{\psi}'^{*}$ weighted by $\widetilde{SJ}'$:

$$\frac{1}{\rho^2(6+3\rho-9\rho^2+2\rho^3)}(0,\rho^2(1-\rho),0,\rho(2-\rho),0,(2+\rho)(1-\rho)).$$

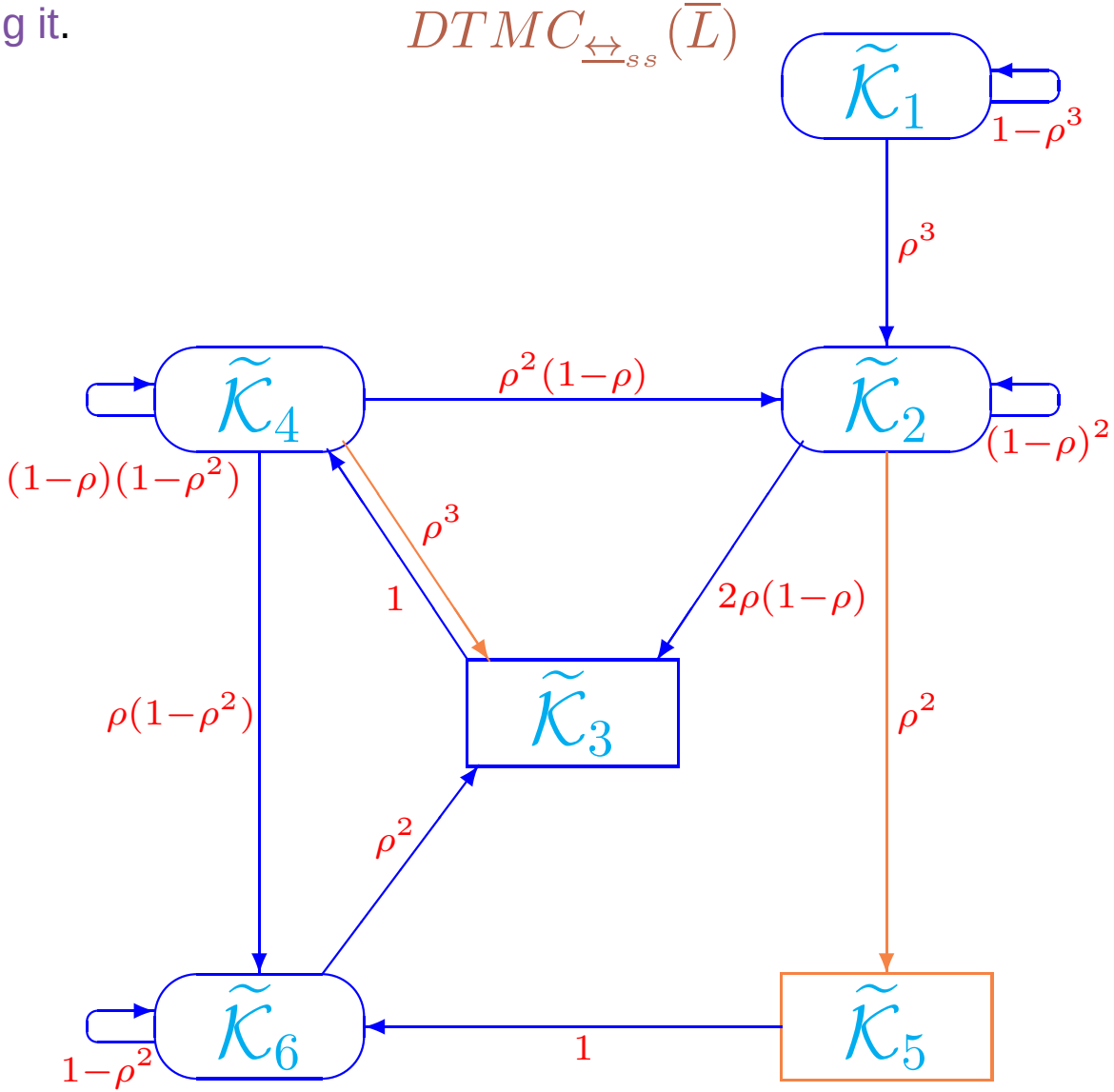
We normalize the steady-state weighted PMF dividing it by the sum of its components

$$\tilde{\psi}'^{*}\widetilde{SJ}'^{T}=\frac{2+\rho-\rho^2-\rho^3}{\rho^2(6+3\rho-9\rho^2+2\rho^3)}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\varphi}'=\frac{1}{2+\rho-\rho^2-\rho^3}(0,\rho^2(1-\rho),0,\rho(2-\rho),0,(2+\rho)(1-\rho)).$$

Otherwise, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{L})$, we can construct the quotient DTMC of $\overline{L}$, $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$, and calculate $\tilde{\varphi}'$ using it.



SHMGQDTMC: The quotient DTMC of the abstract generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}' = \begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\ 0 & (1-\rho)^2 & 2\rho(1-\rho) & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \rho^2(1-\rho) & \rho^3 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \rho^2 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

The steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\psi}' = \frac{1}{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}(0, \rho^2(1-\rho), \rho^2(2-\rho)(1+\rho-\rho^2), \rho(2-\rho), \rho^4(1-\rho), (2+\rho)(1-\rho)).$$

$DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6\}$ and $DR_V(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5\}$. Hence,

$$\sum_{\widetilde{\mathcal{K}} \in DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})}} \tilde{\psi}'(\widetilde{\mathcal{K}}) = \tilde{\psi}'(\widetilde{\mathcal{K}}_1) + \tilde{\psi}'(\widetilde{\mathcal{K}}_2) + \tilde{\psi}'(\widetilde{\mathcal{K}}_4) + \tilde{\psi}'(\widetilde{\mathcal{K}}_6) = \frac{2+\rho-\rho^2-\rho^3}{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}.$$

By the "quotient" analogue of Proposition PMFSMC:

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_1) = 0 \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = 0,$$

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_2) = \frac{\rho^2(1-\rho)}{(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3},$$

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_3) = 0,$$

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_4) = \frac{\rho(2-\rho)}{(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3},$$

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_5) = 0,$$

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_6) = \frac{(2+\rho)(1-\rho)}{(1+\rho)(2-\rho+2\rho^2-2\rho^3)} \cdot \frac{(1+\rho)(2-\rho+2\rho^2-2\rho^3)}{2+\rho-\rho^2-\rho^3} = \frac{(2+\rho)(1-\rho)}{2+\rho-\rho^2-\rho^3}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\varphi}' = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), 0, \rho(2-\rho), 0, (2+\rho)(1-\rho)).$$

This coincides with the result obtained with the use of $\tilde{\psi}'^*$ and $\widetilde{SJ}'$.

Alternatively, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{L})$, we can construct $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ and calculate $\tilde{\varphi}'$ using it.

$DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6\}$ and $DR_V(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5\}$.

We reorder the elements of $DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})}$ by moving the equivalence classes of vanishing states to the first positions: $\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5, \widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6$.

The reordered TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}'_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-\rho^3 & \rho^3 & 0 & 0 \\ 2\rho(1-\rho) & \rho^2 & 0 & (1-\rho)^2 & 0 & 0 \\ \rho^3 & 0 & 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & \rho(1-\rho^2) \\ \rho^2 & 0 & 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

The result of the decomposing $\widetilde{\mathbf{P}}'_r$:

$$\widetilde{\mathbf{C}}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \widetilde{\mathbf{D}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widetilde{\mathbf{E}}' = \begin{pmatrix} 0 & 0 \\ 2\rho(1-\rho) & \rho^2 \\ \rho^3 & 0 \\ \rho^2 & 0 \end{pmatrix},$$

$$\widetilde{\mathbf{F}}' = \begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 \\ 0 & (1-\rho)^2 & 0 & 0 \\ 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & \rho(1-\rho^2) \\ 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

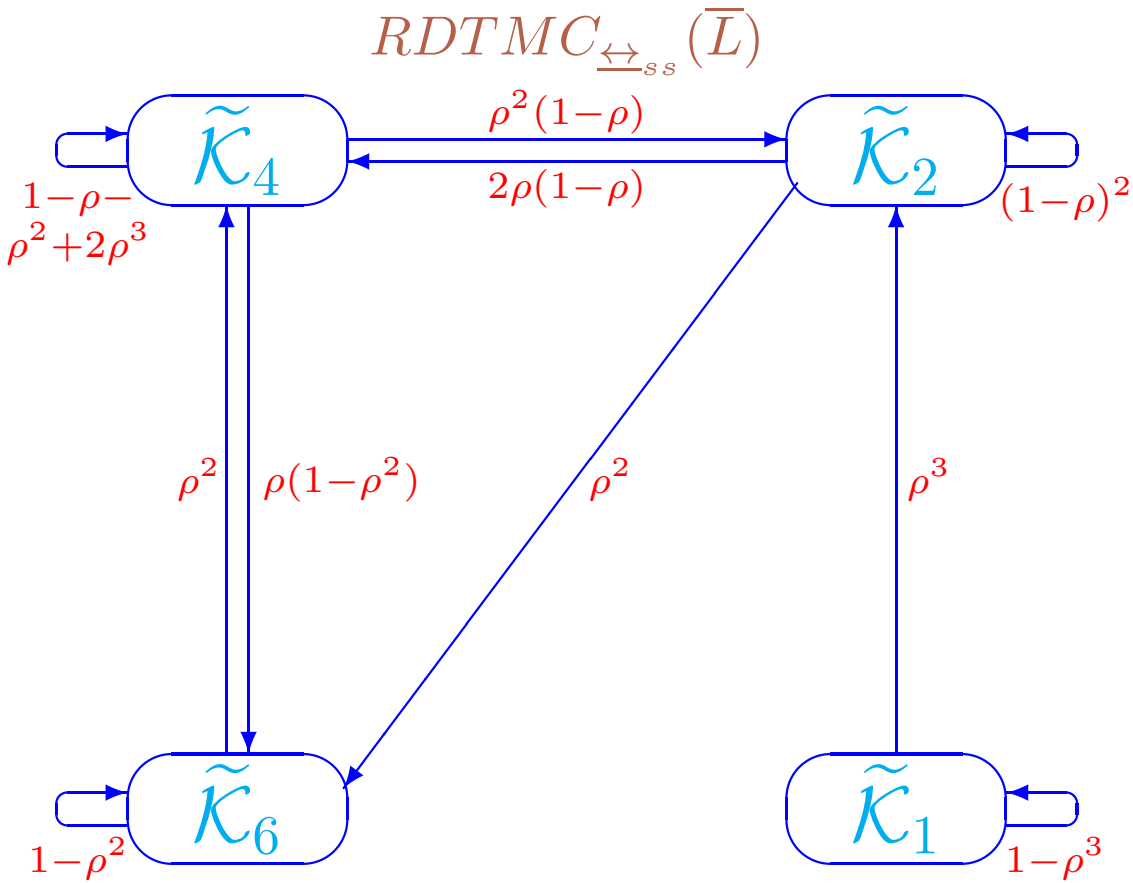
Since $\widetilde{\mathbf{C}}'^1 = \mathbf{0}$, we have $\forall k > 0,\ \widetilde{\mathbf{C}}'^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\widetilde{\mathbf{G}}' = \sum_{k=0}^{l} \widetilde{\mathbf{C}}'^l = \widetilde{\mathbf{C}}'^0 = \mathbf{I}.$$

The TPM for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}'^{\diamond} = \widetilde{\mathbf{F}}' + \widetilde{\mathbf{E}}'\widetilde{\mathbf{G}}'\widetilde{\mathbf{D}}' = \widetilde{\mathbf{F}}' + \widetilde{\mathbf{E}}'\mathbf{I}\widetilde{\mathbf{D}}' = \widetilde{\mathbf{F}}' + \widetilde{\mathbf{E}}'\widetilde{\mathbf{D}}' =$$

$$\begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 \\ 0 & (1-\rho)^2 & 2\rho(1-\rho) & \rho^2 \\ 0 & \rho^2(1-\rho) & 1-\rho-\rho^2+2\rho^3 & \rho(1-\rho^2) \\ 0 & 0 & \rho^2 & 1-\rho^2 \end{pmatrix}.$$



SHMGQRDTMC: The reduced quotient DTMC of the abstract generalized shared memory system

The steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\psi}'^{\diamond} = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), \rho(2-\rho), (2+\rho)(1-\rho)).$$

Note that $\tilde{\psi}'^{\diamond} = (\tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_1), \tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_2), \tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_4), \tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_6))$.

By the "quotient" analogue of Proposition PMFSMCT:

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_1) = 0, \qquad \tilde{\varphi}'(\widetilde{\mathcal{K}}_2) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}, \qquad \tilde{\varphi}'(\widetilde{\mathcal{K}}_3) = 0,$$
$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_4) = \frac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3}, \qquad \tilde{\varphi}'(\widetilde{\mathcal{K}}_5) = 0, \qquad \tilde{\varphi}'(\widetilde{\mathcal{K}}_6) = \frac{(2+\rho)(1-\rho)}{2+\rho-\rho^2-\rho^3}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\varphi}' = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), 0, \rho(2-\rho), 0, (2+\rho)(1-\rho)).$$

This coincides with the result obtained with the use of $\tilde{\psi}'^{*}$ and $\widetilde{SJ}'$.

Performance indices

- The average recurrence time in the state $\widetilde{\mathcal{K}}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\tilde{\varphi}'_2} = \frac{2+\rho-\rho^2-\rho^3}{\rho^2(1-\rho)}$.

- The common memory is available only in the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5$.

  The steady-state probability that the memory is available is
  $\tilde{\varphi}'_2 + \tilde{\varphi}'_3 + \tilde{\varphi}'_5 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} + 0 + 0 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} = \frac{2+\rho-2\rho^2}{2+\rho-\rho^2-\rho^3}$.

- After activation of the system, we leave the state $\widetilde{\mathcal{K}}_1$ for all, and the common memory is either requested or allocated in every remaining state, with exception of $\widetilde{\mathcal{K}}_2$.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving $\widetilde{\mathcal{K}}_2$, calculated as $\frac{\tilde{\varphi}'_2}{\widetilde{SJ}'_2} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} \cdot \frac{\rho(2-\rho)}{1} = \frac{\rho^3(1-\rho)(2-\rho)}{2+\rho-\rho^2-\rho^3}$.

- The parallel common memory request of two processors $\{\{r\},\{r\}\}$ is only possible from the state $\widetilde{\mathcal{K}}_2$.

  The request probability in this state is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$ twice.

  The *steady-state probability of the shared memory request from two processors* is
  $\tilde{\varphi}'_2 \sum_{\{A,\widetilde{\mathcal{K}}|\{\{r\},\{r\}\}\subseteq A,\ \widetilde{\mathcal{K}}_2 \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{K}}) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}\rho^2 = \frac{\rho^4(1-\rho)}{2+\rho-\rho^2-\rho^3}$.

- The common memory request of a processor $\{r\}$ is only possible from the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is
  $\tilde{\varphi}'_2 \sum_{\{A,\widetilde{\mathcal{K}}|\{r\}\in A,\ \widetilde{\mathcal{K}}_2 \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{K}}) + \tilde{\varphi}'_4 \sum_{\{A,\widetilde{\mathcal{K}}|\{r\}\in A,\ \widetilde{\mathcal{K}}_4 \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_4,\widetilde{\mathcal{K}}) =$
  $\frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}(2\rho(1-\rho)+\rho^2) + \frac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3}(\rho(1-\rho^2)+\rho^3) = \frac{\rho^2(2-\rho)(1+\rho-\rho^2)}{2+\rho-\rho^2-\rho^3}$.

The performance indices are the same for the complete and quotient abstract generalized shared memory systems.

The coincidence of the first and second performance indices illustrates Proposition STPROB.

The coincidence of the third performance index illustrates Proposition STPROB and Proposition SJAVVA.

The coincidence of the fourth performance index is by Theorem STTRAC:

one should apply its result to the derived step trace $\{\{r\},\{r\}\}$ of $\overline{L}$ and itself.

The coincidence of the fifth performance index is by Theorem STTRAC:

one should apply its result to the derived step traces $\{\{r\}\},\ \{\{r\},\{r\}\},\ \{\{r\},\{m\}\}$ of $\overline{L}$ and itself,

and sum the left and right parts of the three resulting equalities.

Effect of quantitative changes of ρ to performance of the quotient abstract generalized shared memory system in its steady state

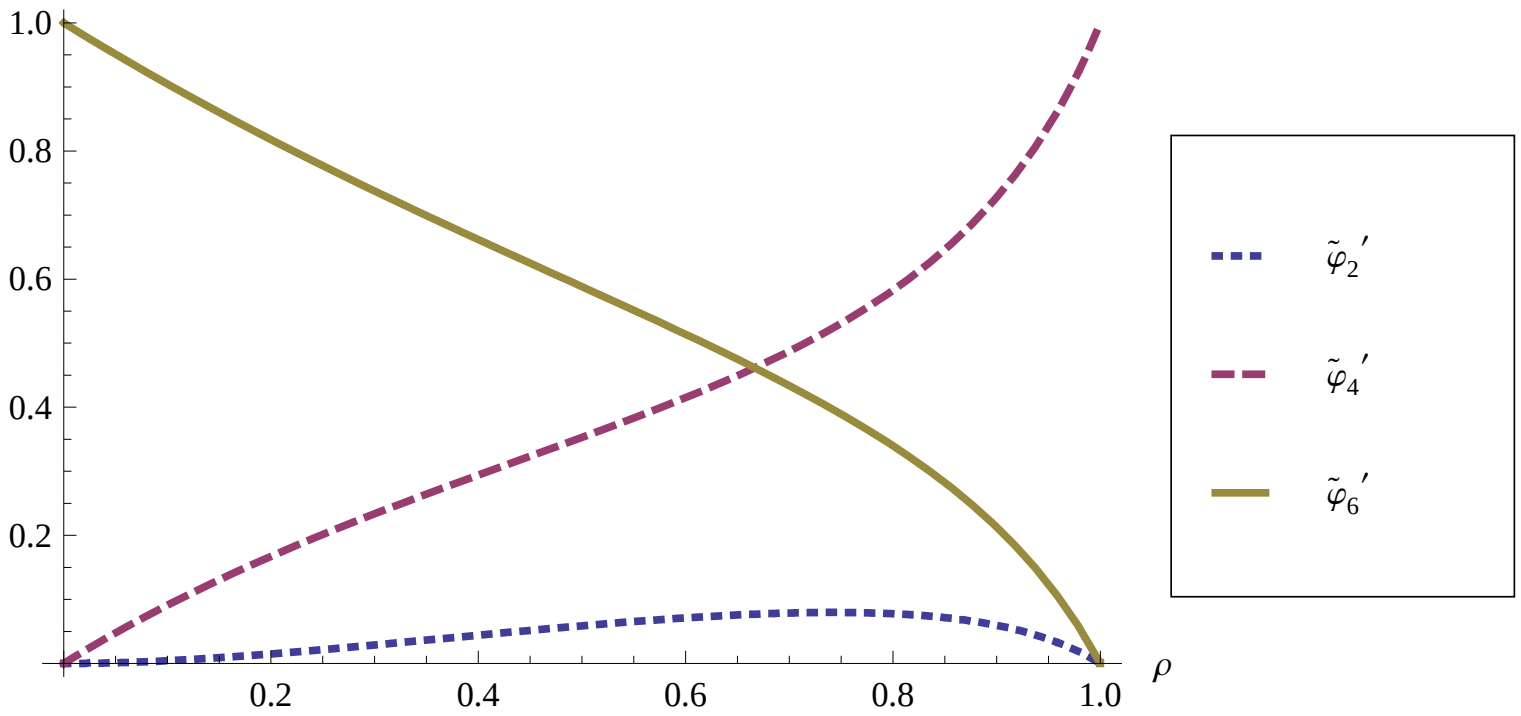
$\rho \in (0;1)$ is the probability of every multiaction of the system.

The closer is ρ to 0, the less is the probability to execute some activities at every discrete time step: the system will most probably *stand idle*.

The closer is ρ to 1, the greater is the probability to execute some activities at every discrete time step: the system will most probably *operate*.

$\tilde{\varphi}'_1 = \tilde{\varphi}'_3 = \tilde{\varphi}'_5 = 0$ are constants, and they do not depend on ρ.

$\tilde{\varphi}'_2 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}$, $\tilde{\varphi}'_4 = \frac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3}$, $\tilde{\varphi}'_6 = \frac{(2+\rho)(1-\rho)}{2+\rho-\rho^2-\rho^3}$ depend on ρ.



SHMGQSSP: Steady-state probabilities $\tilde{\varphi}'_2,\ \tilde{\varphi}'_4,\ \tilde{\varphi}'_6$ as functions of the parameter ρ

$\tilde{\varphi}'_2,\ \tilde{\varphi}'_4$ tend to 0 and $\tilde{\varphi}'_6$ tends to 1 when ρ approaches 0.

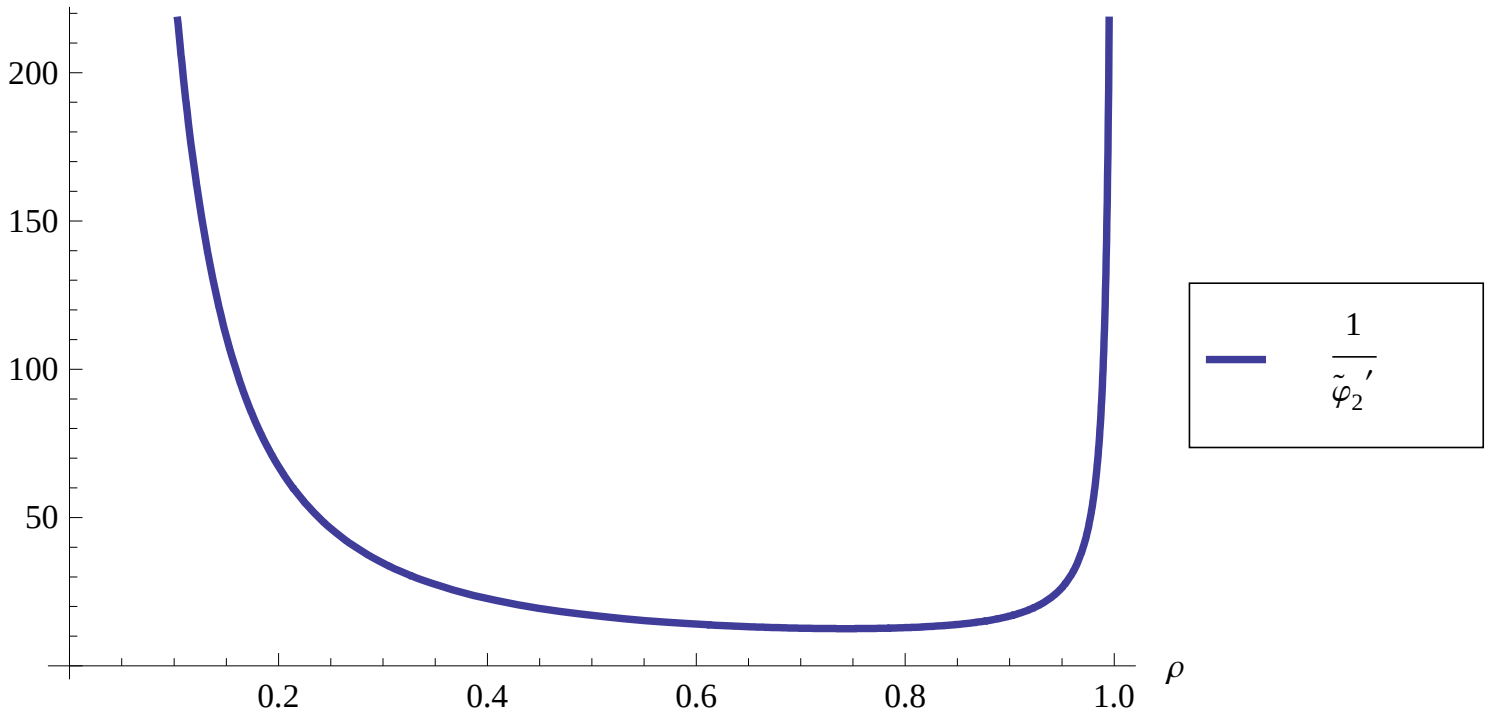
When ρ is closer to 0, the probability that the memory is allocated to a processor and the memory is requested by another processor increases: *more unsatisfied memory requests*.

$\tilde{\varphi}'_2,\ \tilde{\varphi}'_6$ tend to 0 and $\tilde{\varphi}'_4$ tends to 1 when ρ approaches 1.

When ρ is closer to 1, the probability that the memory is allocated to a processor (and not requested by another one) increases: *less unsatisfied memory requests*.

The maximal value 0.0797 of $\tilde{\varphi}'_2$ is reached when $\rho \approx 0.7433$.

In this case, the probability that the system is activated and the memory is not requested is maximal: *maximal shared memory availability* is about 8%.



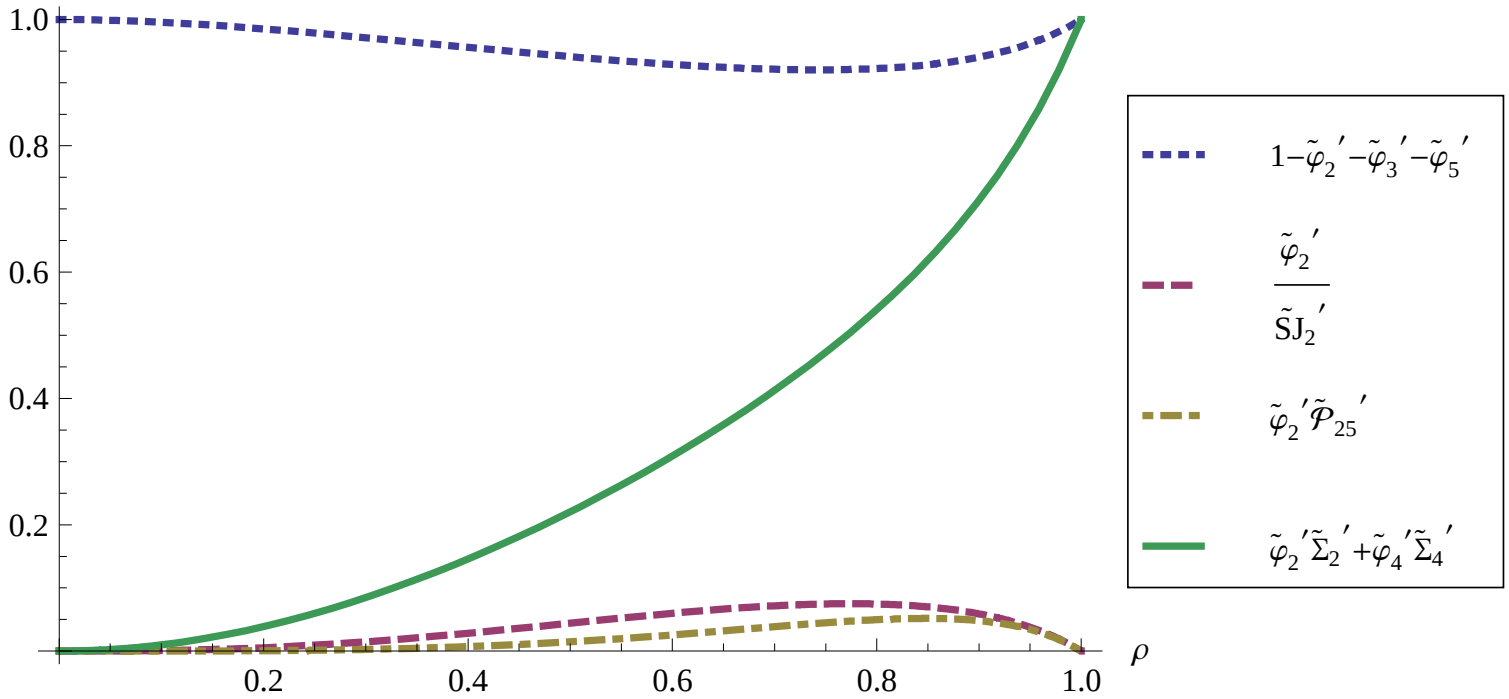
SHMGQART: Average system run-through $\frac{1}{\tilde{\varphi}_2'}$ as a function of the parameter ρ

The average system run-through is $\frac{1}{\tilde{\varphi}_2'}$.

It tends to ∞ when ρ approaches 0 or 1.

The minimal value 12.5516 of $\frac{1}{\tilde{\varphi}_2'}$ is reached when $\rho \approx 0.7433$.

To speed up the system's operation: take the parameter ρ closer to 0.7433.



SHMGQIND: Some performance indices as functions of the parameter ρ

The shared memory utilization is $1 - \tilde{\varphi}_2' - \tilde{\varphi}_3' - \tilde{\varphi}_5'$.

It tends to 1 when ρ approaches 0 and when ρ approaches 1.

The minimal value 0.9203 of the utilization is reached when $\rho \approx 0.7433$.

The *minimal shared memory utilization* is about 92%.

To increase the utilization: take the parameter ρ closer to 0 or 1.

The rate with which the necessity of shared memory emerges is $\frac{\tilde{\varphi}_2'}{\widetilde{SJ}_2'}$.

It tends to 0 when ρ approaches 0 and when ρ approaches 1.

The maximal value 0.0751 of the rate is reached when $\rho \approx 0.7743$.

The *maximal rate with which the necessity of shared memory emerges* is about $\frac{1}{13}$.

To decrease the rate: take the parameter ρ closer to 0 or 1.

The steady-state probability of the shared memory request from two processors is $\tilde{\varphi}'_2\widetilde{\mathcal{P}}'_{25}$, where $\widetilde{\mathcal{P}}'_{25} = \sum_{\{A,\widetilde{\mathcal{K}}|\{\{r\},\{r\}\}\subseteq A,\ \widetilde{\mathcal{K}}_2 \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}) = PM(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_5)$.

It tends to 0 when ρ approaches 0 and when ρ approaches 1.

The maximal value 0.0517 of the rate is reached when $\rho \approx 0.8484$.

To decrease the probability: take the parameter ρ closer to 0 or 1.

The steady-state probability of the shared memory request from a processor is $\tilde{\varphi}'_2\widetilde{\Sigma}'_2 + \tilde{\varphi}'_4\widetilde{\Sigma}'_4$, where $\widetilde{\Sigma}'_i = \sum_{\{A,\widetilde{\mathcal{K}}|\{r\}\in A,\ \widetilde{\mathcal{K}}_i \stackrel{A}{\to} \widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_i, \widetilde{\mathcal{K}}),\ i \in \{2, 4\}$.

It tends to 0 when ρ approaches 0 and it tends to 1 when ρ approaches 1.

To increase the probability: take the parameter ρ closer to 1.

Overview and open questions

Concurrency interpretation

Interleaving transition relation

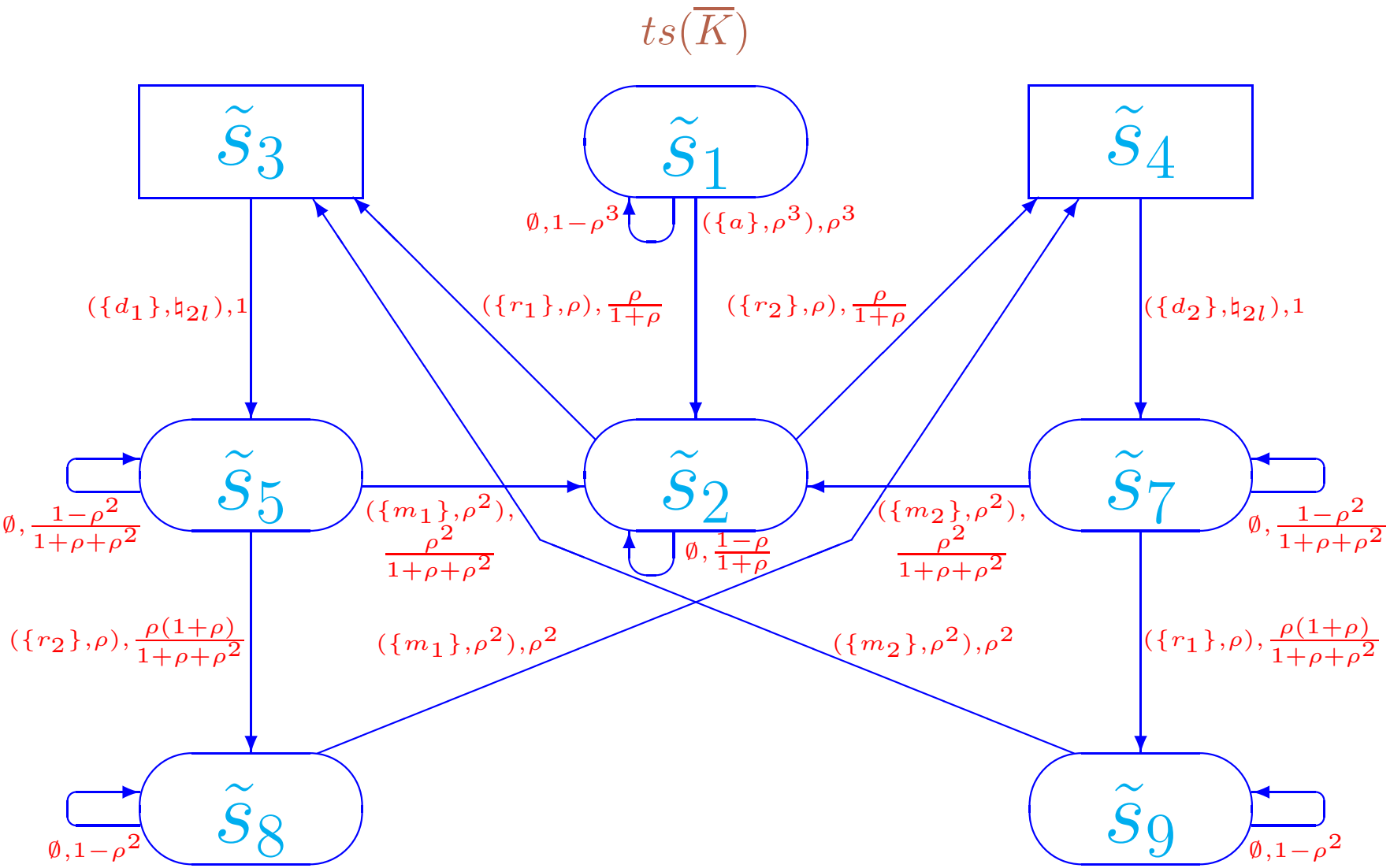
Let G be a dynamic expression, $s \in DR(G)$, $\Upsilon \in Exec(s)$ and $|\Upsilon| \leq 1$.

The *probability to execute the multiset of activities* Υ *in* s*, when only zero-element steps (i.e. empty loops) or one-element steps are allowed*:

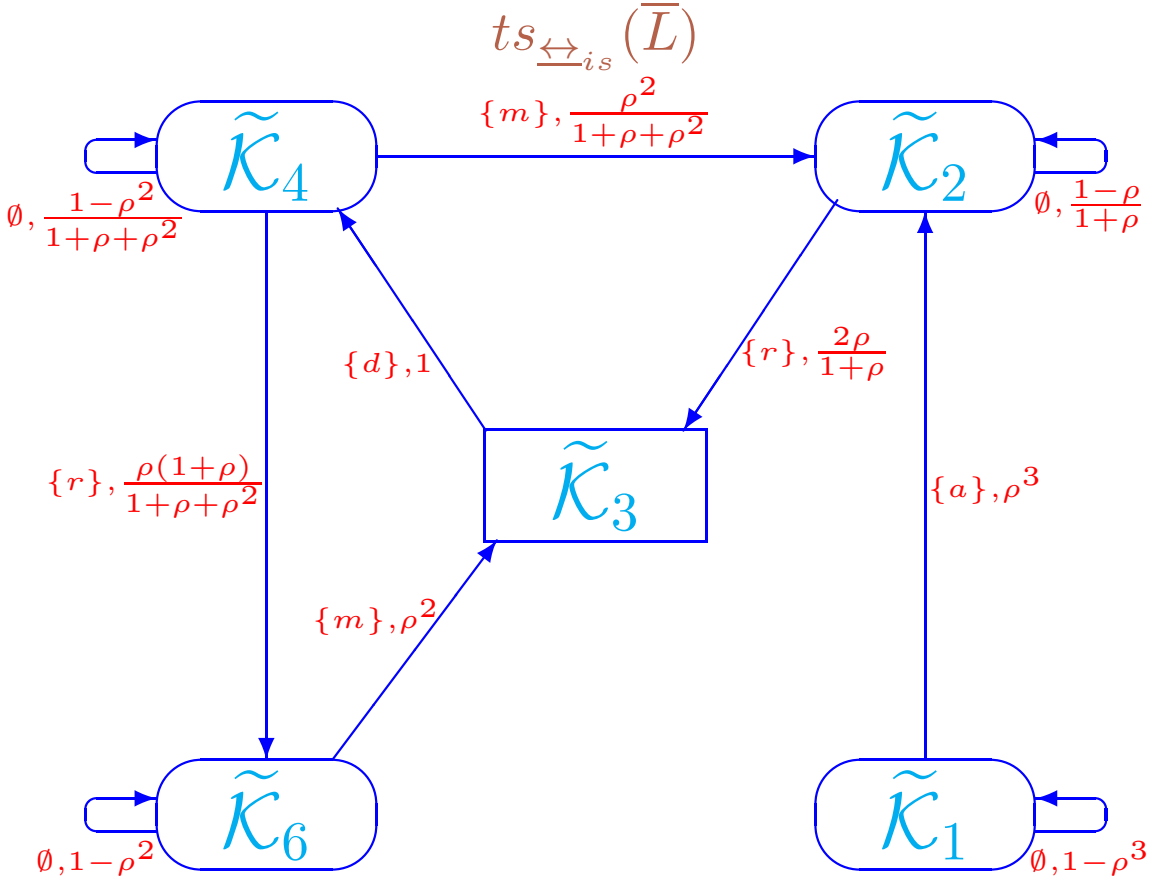
$$pt(\Upsilon, s) = \frac{PT(\Upsilon, s)}{\sum_{\{\Xi \mid |\Xi| \leq 1\}} PT(\Xi, s)}.$$

Overview and open questions

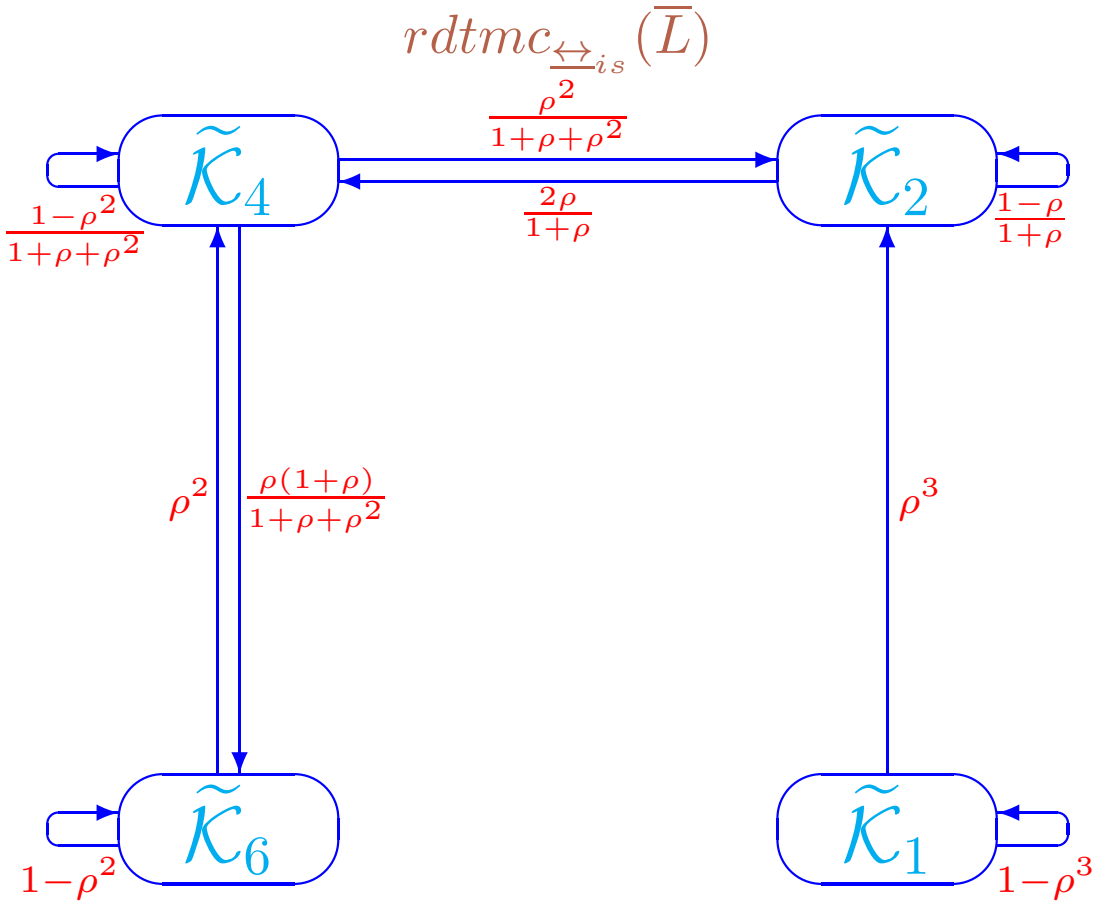
Concurrency interpretation



SHMGTSI: The interleaving transition system of the generalized shared memory system



SHMGQTSI: The interleaving quotient transition system of the abstract generalized shared memory system



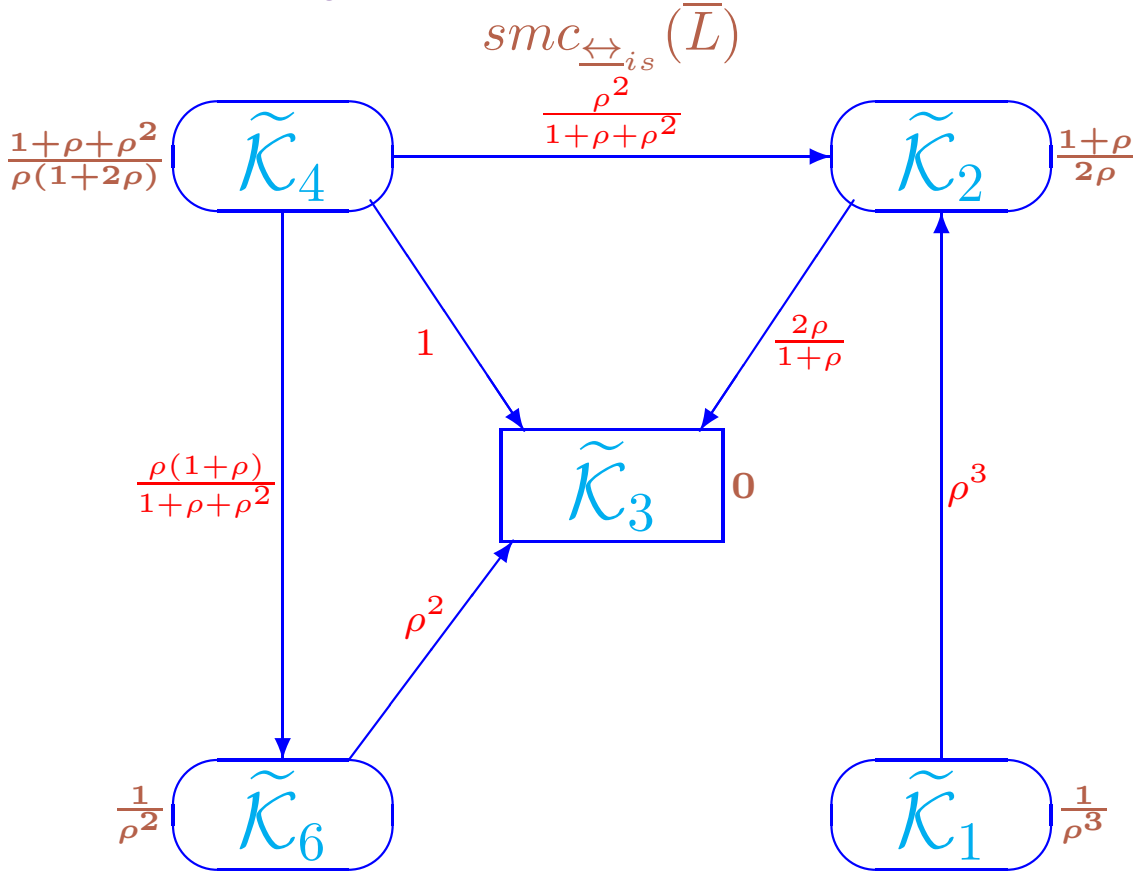
SHMGQRDTMCI: The interleaving reduced quotient DTMC of the abstract generalized shared memory system

The steady-state PMF for $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{L})$:

$\tilde{\phi}'^{\diamond} = \frac{1}{2+4\rho+3\rho^2+3\rho^3}(0, \rho^2(1+\rho), 2\rho(1+\rho+\rho^2), 2(1+\rho))$, whereas

the steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$\tilde{\psi}'^{\diamond} = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), \rho(2-\rho), (2+\rho)(1-\rho))$.



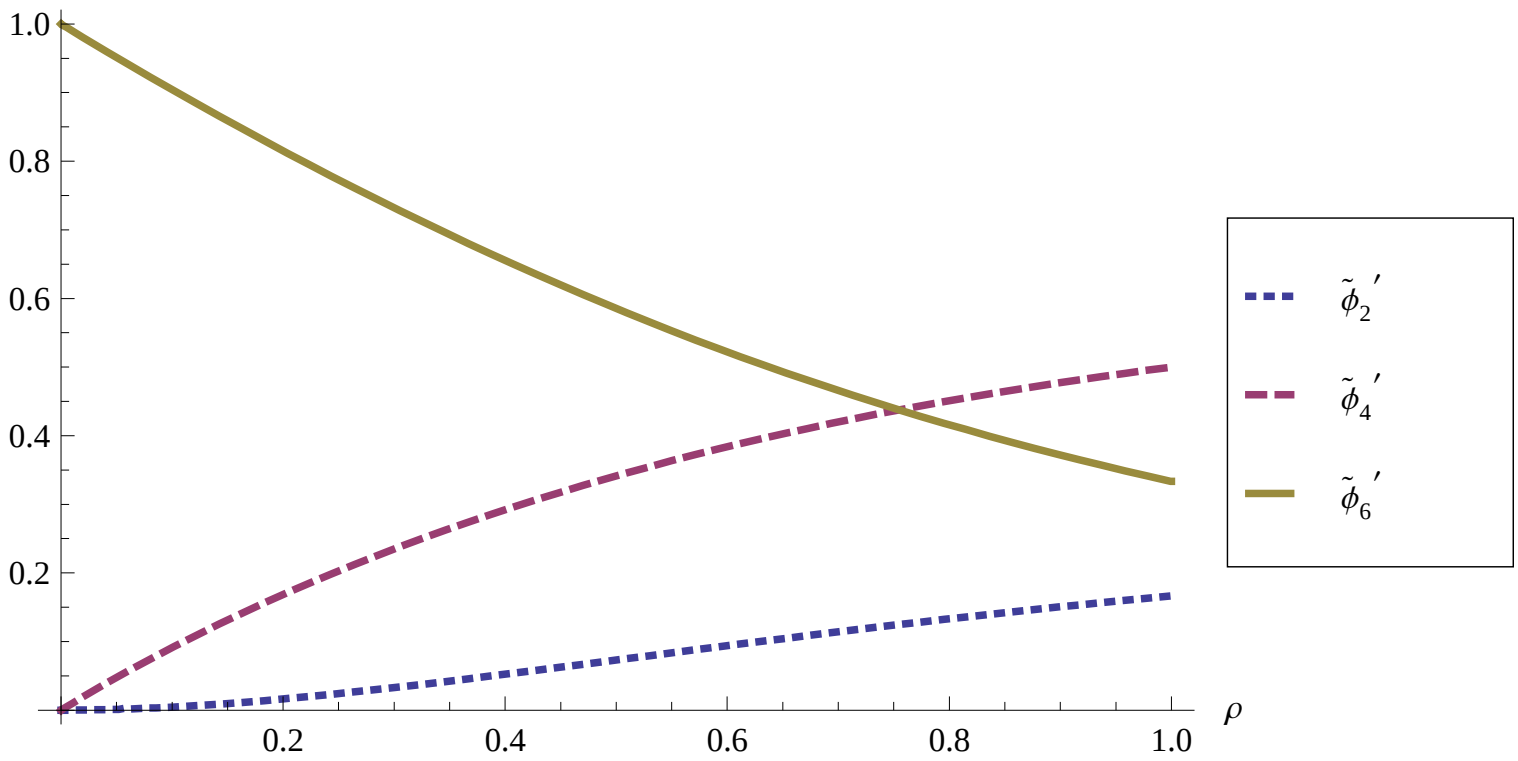
SHMGQSMCI: The interleaving quotient underlying SMC of the abstract generalized shared memory system

The steady-state PMF for $smc_{\underline{\leftrightarrow}_{is}}(\overline{L})$:

$\tilde{\phi}' = \frac{1}{2+4\rho+3\rho^2+3\rho^3}(0, \rho^2(1+\rho), 0, 2\rho(1+\rho+\rho^2), 0, 2(1+\rho))$, whereas

the steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$\tilde{\varphi}' = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), 0, \rho(2-\rho), 0, (2+\rho)(1-\rho))$.



SHMGQSSPI: Interleaving steady-state probabilities $\tilde{\phi}'_2,\ \tilde{\phi}'_4,\ \tilde{\phi}'_6$ as functions of the parameter ρ

The differences between Figures SHMGQSSP and SHMGQSSPI:

when ρ tends to 1, the increase of performance

(the time fraction when the memory is allocated to a processor and not required by another one)

is much more obvious in step semantics than in the interleaving one.

SHMQRTPI: Transient and steady-state probabilities for the interleaving reduced quotient DTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	∞
${\phi'_1}^{\diamond}[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	0
${\phi'_2}^{\diamond}[k]$	0	0.1499	0.1155	0.0950	0.0844	0.0789	0.0761	0.0747	0.0739	0.0736	0.0734	0.0732
${\phi'_3}^{\diamond}[k]$	0	0.1992	0.2722	0.3061	0.3233	0.3322	0.3367	0.3390	0.3402	0.3408	0.3411	0.3415
${\phi'_4}^{\diamond}[k]$	0	0.1379	0.3493	0.4640	0.5231	0.5534	0.5690	0.5770	0.5811	0.5832	0.5842	0.5854

Let $\rho = \frac{1}{2}$ and $l = 1$ in the above interleaving transition systems and DTMC.

The result: the interleaving transition system $ts(\overline{E})$, quotient transition system $ts_{\underline{\leftrightarrow}_{is}}(\overline{F})$, reduced quotient DTMC $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{F})$ of the concrete and abstract *standard* shared memory system.

The steady-state PMF for $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{F})$: $\phi'^{\diamond} = \left(0, \frac{3}{41}, \frac{14}{41}, \frac{24}{41}\right)$, whereas the steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$: $\psi'^{\diamond} = \left(0, \frac{1}{17}, \frac{6}{17}, \frac{10}{17}\right)$.

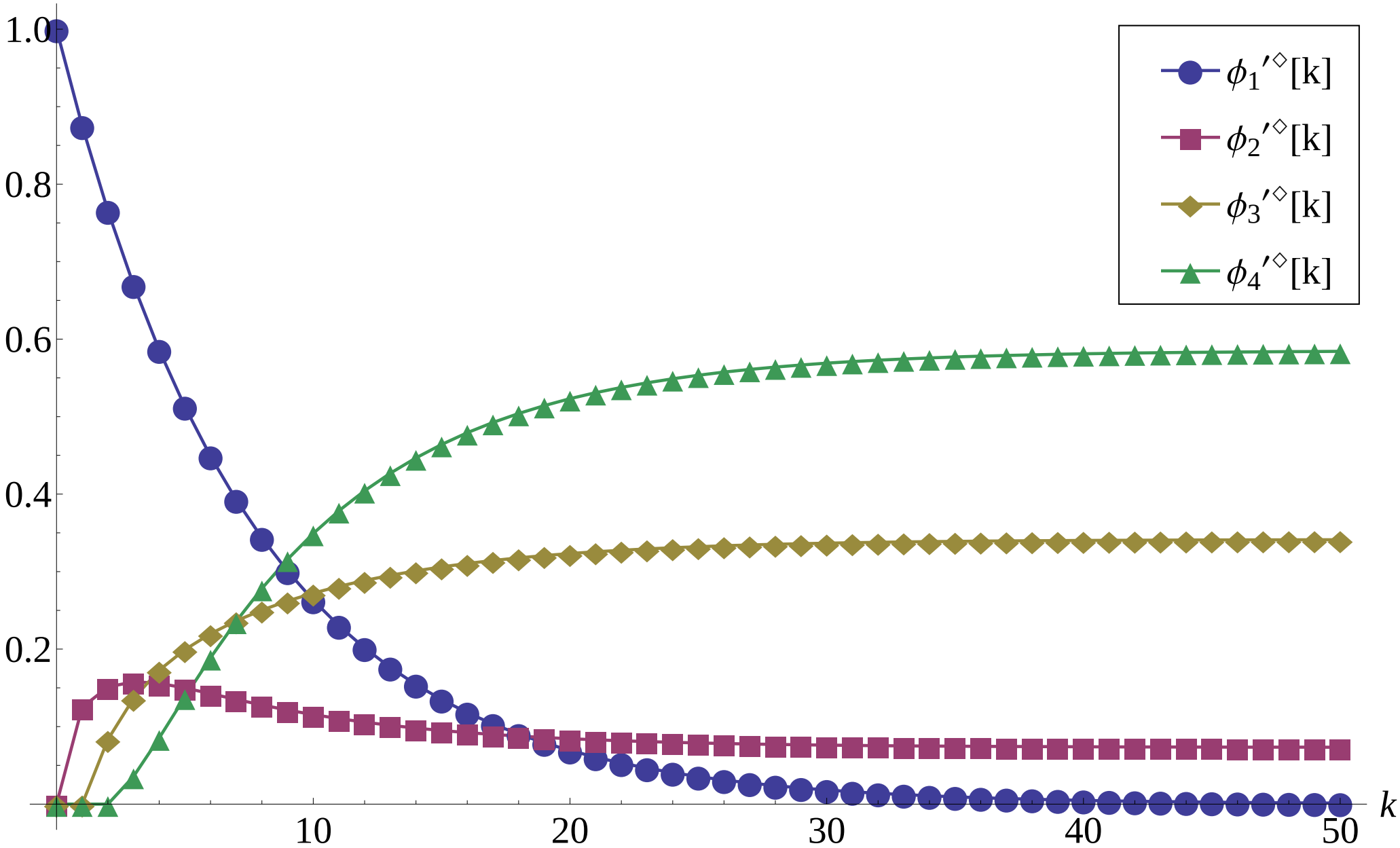
With k growing, ${\phi'_4}^{\diamond}[k] = \phi'^{\diamond}[k](\mathcal{K}_6)$ stabilizes slower than ${\psi'_4}^{\diamond}[k] = \psi'^{\diamond}[k](\mathcal{K}_6)$ from Table SHMQRTP and Figure SHMQRTP.

One reason: $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{F})$ has no transition from $\mathcal{K}_2$ to $\mathcal{K}_6$, unlike $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$.

The absolute relative differences for $k = 5$:

$\left|\frac{{\phi'_4}^{\diamond} - {\phi'_4}^{\diamond}[5]}{{\phi'_4}^{\diamond}}\right| = \left|\frac{0.5854 - 0.1379}{0.5854}\right| = \frac{0.4475}{0.5854} \approx 0.7644$ (76%),

$\left|\frac{{\psi'_4}^{\diamond} - {\psi'_4}^{\diamond}[5]}{{\psi'_4}^{\diamond}}\right| = \left|\frac{0.5882 - 0.1901}{0.5882}\right| = \frac{0.3981}{0.5882} \approx 0.6768$ (68%, i.e. 8% less).



SHMQRTPI: Transient probabilities alteration diagram for the interleaving reduced quotient DTMC of the abstract shared memory system

The results obtained

- A discrete time stochastic and immediate extension $dtsiPBC$ of finite PBC enriched with iteration.
- The step operational semantics based on labeled probabilistic transition systems.
- The denotational semantics in terms of a subclass of LDTSIPNs.
- The method of performance evaluation based on underlying SMCs.
- Step stochastic bisimulation equivalence of the expressions and dtsi-boxes.
- The transition systems and SMCs reduction modulo the equivalence.
- A comparison of stationary behaviour up to the equivalence.
- Performance analysis simplification with the equivalence.
- The case study: the shared memory system.

Further research

- Constructing a congruence relation: the equivalence that withstands application of the algebraic operations.
- Introducing the deterministically timed multiactions with fixed time delays (including the zero delay).
- Extending the syntax with recursion operator.

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The slides can be downloaded from Internet:

http://itar.iis.nsk.su/files/itar/pages/dtsipbcsemrw.pdf

Thank you for your attention!