CARL VON OSSIETZKY universität OLDENBURG

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AUS DEM DEPARTMENT FÜR INFORMATIK
der Fakultät II - Informatik, Wirtschafts- und Rechtswissenschaften

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Investigating equivalence relations in *dts*PBC

Dr. Igor V. Tarasyuk

Bericht

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Igor V. Tarasyuk

A.P. Ershov Institute of Informatics Systems
Siberian Branch of the Russian Academy of Sciences
6, Acad. Lavrentiev ave., 630090 Novosibirsk, Russia
itar@iis.nsk.su

Abstract

In the last decades, a number of stochastic enrichments of process algebras was constructed to specify stochastic processes within the well-developed framework of algebraic calculi. In 2003, a continuous time stochastic extension $sPBC$ of finite Petri box calculus (PBC) was enriched with iteration operator by H.S. Macià, V.R. Valero, D.L. Cazorla and F.G. Cuartero. In 2006, the author added iteration to the discrete time stochastic extension $dtsPBC$ of finite PBC. In this paper, in the framework of the $dtsPBC$ with iteration, we define a variety of stochastic equivalences. They allow one to identify stochastic processes with similar behaviour that are differentiated by too strict notion of the semantic equivalence. The interrelations of all the introduced equivalences are investigated. A logical characterization of the equivalences is presented via formulas of the new probabilistic modal logics. We demonstrate how to apply the equivalences to compare stationary behaviour. A problem of preservation of the equivalences by algebraic operations is discussed. As a result, we define an equivalence that is a congruence relation. At last, two case studies of performance evaluation in the algebra are presented.

Keywords: stochastic Petri net, stochastic process algebra, Petri box calculus, iteration, discrete time, transition system, operational semantics, dts-box, denotational semantics, empty loop, stochastic equivalence, modal logic, stationary behaviour, congruence relation, performance evaluation.

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1 Introduction

Stochastic Petri nets (SPNs) are a well-known model for quantitative analysis of discrete dynamic event systems proposed initially in [47]. Essentially, SPNs are a high level language for specification and performance analysis of concurrent systems. A stochastic process corresponding to this formal model is a Markov chain generated and analyzed by well-developed algorithms and methods. Firing probabilities distributed along continuous or discrete time scale are associated with transitions of an SPN. Thus, there exist SPNs with continuous [47, 27] and discrete [48] time. Markov chains of the corresponding types are associated with the SPNs. As a rule, for SPNs with continuous time (CTSPNs), exponential or phase distributions of transition probabilities are used. For SPNs with discrete time (DTSPNs), geometric or combinations of geometric distributions are usually used. Transitions of CTSPNs fire one by one at continuous time moments. Hence, the semantics of this model is an interleaving one. In this semantics, parallel computations are modeled by all possible execution sequences of their components. Transitions of DTSPNs fire concurrently in steps at discrete time moments. Hence, this model has a step semantics. In this semantics, parallel computations are modeled by sequences of concurrent occurrences (steps) of their components. In [19, 20], a labeling for transitions of CTSPNs with action names was proposed. The labeling allows SPNs to model processes with functionally similar components: the transitions corresponding to the similar components are labeled by the same action. Moreover, one can compare labeled SPNs by different behavioural equivalences, and this makes possible to check stochastic processes specified by labeled SPNs for functional similarity. Therefore, one can compare both functional and performance properties, and labeled SPNs turn into a formalism for quantitative and qualitative analysis.

Algebraic calculi occupy a special place among formal models for specification of concurrent systems and analysis of their behavioral properties. In such process algebras (PAs), a system or a process is specified by an algebraic formula. Verification of the properties is accomplished at a syntactic level by means of well-developed systems of equivalences, axioms and inference rules. One of the first PAs was CCS (Calculus of Communicating Systems) [46]. Process algebras have been acknowledged to be very suitable formalism to operate with real time and stochastic systems as well. In the last years, stochastic extensions of PAs called stochastic process algebras (SPAs) became very popular as a modeling framework. SPAs do not just specify actions which can happen (qualitative features) as usual process algebras, but they associate some quantitative parameters with actions (quantitative characteristics). The most popular SPAs proposed so far are $PEPA$ [30], $TIPP$ [32] and $EMPA$ [10]. An extension of CCS with probabilities and time called $TPCCS$ was defined in [29]. An enrichment of BPA with probabilistic choice, $prBPA$, as well as an extension of $prBPA$ with parallel composition operator named ACP_π^+ have been proposed in [1]. A stochastic process calculus PPA was constructed in [61, 63]. The papers [18, 24, 65, 11] propose a variety of other SPAs. A standard way for probabilistic extension of process algebras into the calculi of probabilistic transition systems was described in [33].

Process algebras allow one to specify processes in a compositional way via an expressive formal syntax. On the other hand, Petri nets provide one with an ability for visual representation of a process structure and execution. Hence, the relationship between SPNs and SPAs is of particular interest. To combine advantages of both models, a semantics of algebraic formulas in terms of Petri nets is usually defined. In the stochastic case, the Markov chain of the stochastic process specified by an SPA formula is built based on the state transition graph of the corresponding SPN.

As a rule, stochastic process calculi proposed in the literature are based on interleaving. As a semantic domain, the interleaving formalism of transition systems is often used. Therefore, investigation of a stochastic extension for more expressive and powerful algebraic calculi is an important issue. At present, the development of step or "true concurrency" (such that parallelism is considered as a causal independence) SPAs is in the very beginning. At the same time, there does not yet exist an algebra of infinite concurrent stochastic processes.

Petri box calculus (PBC) is a flexible and expressive process algebra based on calculi CCS [46] and AFP_0 [37]. PBC was proposed fifteen years ago [3], and it was well explored since that time [2, 14, 21, 36, 38, 12, 13, 22, 23, 25, 31, 4, 5, 34, 6, 7, 8, 9]. Its goal was to propose a compositional semantics for high level constructs of concurrent programming languages in terms of elementary Petri nets. Thus, PBC serves as a bridge between theory and applications. Formulas of PBC are combined not from single actions (including the invisible one) and variables only, as in CCS, but from multisets of actions called multiactions (*basic formulas*) as well. In contrast to CCS, concurrency and synchronization are different operations (*concurrent constructs*). Synchronization is defined as a unary multi-way stepwise operation based on communication of actions and their conjugates. The other fundamental operations are sequence and choice (*sequential constructs*). The calculus includes also restriction and relabeling (*abstraction constructs*). To specify infinite processes, refinement, recursion and iteration operations were added (*hierarchical constructs*). Thus, unlike CCS, the algebra PBC has an additional iteration construction to specify infiniteness in the cases when finite Petri nets can be used as the semantic interpretation. For PBC, a denotational semantics was proposed in terms of a subclass of Petri nets equipped with interface and considered up to isomorphism. This subclass is called Petri boxes. The calculus PBC has a step operational semantics in terms of labeled transition systems based on the structural operational semantics (SOS) rules. Pomset operational semantics of PBC was defined in [38] such that the partial order information was extracted from "decorated" step traces. In these step sequences, multiactions were annotated with an information on the relative position of the expression part they were derived from. Last years, several extensions of PBC were presented.

A time extension of PBC called time Petri box calculus ($tPBC$) was proposed in [39]. In $tPBC$, timing information is added by combining instantaneous multiactions and time delays. A denotational semantics was defined in terms of a subclass of labeled time Petri nets (tPNs) called time Petri boxes (ct-boxes). $tPBC$ has interleaving time operational semantics in terms of labeled transition systems. Another time enrichment of PBC called Timed Petri box calculus ($TPBC$) was defined in [44, 45]. In contrast to $tPBC$, multiactions of $TPBC$ are not instantaneous but have time durations. Additionally, in $TPBC$ there exist no "illegal" multiaction occurrences unlike $tPBC$. The complexity of "illegal" occurrences mechanism was one of the main intentions to construct $TPBC$ though the calculus appeared to be more complicated than $tPBC$. A denotational semantics was defined in terms of a subclass of labeled Timed Petri nets (TPNs) called Timed Petri boxes (T-boxes). The algebra $tPBC$ has a step timed operational semantics in terms of labeled transition systems. Note that $tPBC$ and $TPBC$ differ in ways they capture time information, and they are not in competition but complement each other. The third time extension of PBC called arc time Petri box calculus ($atPBC$) was constructed in [62]. In $atPBC$, multiactions are associated with time delay intervals. A denotational semantics was defined on a subclass of arc time Petri nets (atPNs) called arc time Petri boxes (at-boxes). $atPBC$ possesses a step operational semantics in terms of labeled transition systems.

A stochastic extension of PBC called stochastic Petri box calculus ($sPBC$) was proposed in [58, 59, 60, 49, 54, 55, 56, 42]. In $sPBC$, multiactions have stochastic durations that follow negative exponential distribution. Each multiaction is instantaneous and equipped with a rate that is a parameter of the corresponding exponential distribution. The execution of a multiaction is possible only after the corresponding stochastic time delay. Just a finite part of PBC was used for the stochastic enrichment. This means that $sPBC$ has neither refinement nor recursion nor iteration operations. A denotational semantics was defined in terms of a subclass of labeled CTSPNs called stochastic Petri boxes (s-boxes). Calculus $sPBC$ has interleaving operational semantics in terms of labeled transition systems. Note that we have interleaving behaviour here because of the fact that a simultaneous firing of any two transitions has zero probability in accordance with the properties of continuous time distributions. Current research in this branch has an aim to extend the specification abilities of $sPBC$ and to define an appropriate congruence relation over algebraic formulas. The results on constructing the iteration for $sPBC$ were reported in [51, 52]. In the papers [50, 53], a number of new equivalence relations were proposed for regular terms of $sPBC$ to choose later a suitable candidate for a congruence. In [57], the special multiactions with zero time delay were added to $sPBC$. A denotational semantics of such an $sPBC$ extension was defined via a subclass of labeled generalized SPNs (GSPNs). The subclass is called generalized stochastic Petri boxes (gs-boxes).

An ambient extension of PBC called Ambient Petri box calculus ($APBC$) was proposed in [26]. Ambient calculus is used to model behaviour of mobile systems. Ambient is a named environment delimited by a boundary. The ambients can be moved to a new location thus modeling mobility. The algebra $APBC$ includes ambients and mobility capabilities. Hence, it could be interpreted as an extension of the Ambient Calculus with the operations of PBC. Basic actions of $APBC$ are capabilities defined over ambient names and standard multiactions of PBC. Only finite part of PBC was taken for the ambient enrichment. Moreover, just concurrency and sequence were transferred into $APBC$ from the set of PBC operations in [26]. This reduced algebra was called Simple Ambient Petri box calculus ($SAPBC$). A denotational semantics was defined in terms of Elementary Object Systems (EOSs) that are two-level net systems composed from a system net and object nets. Object nets could be interpreted as high-level tokens of the system net modeling the execution of mobilie processes. The calculus $SAPBC$ has a step operational semantics in terms of labeled transition systems.

Nevertheless, there were no stochastic extension of PBC with step semantics until recent times. It can be done with the use of labeled DTSPNs as a semantic area, since discrete time models allow for concurrent action occurrences. The enrichment based of DTSPNs is natural because PBC has a step operational semantics.

A notion of equivalence is very important in formal theory of computing processes and systems. Behavioural equivalences are applied during verification stage both to compare behaviour of systems and reduce their structure. At present time, there exists a great diversity of different equivalence notions for concurrent systems, and their interrelations were well explored in the literature. The most popular and widely used one is bisimulation. Unfortunately, the mentioned behavioural equivalences take into account only functional (qualitative) but not performance (quantitative) aspects of system behaviour. Additionally, the equivalences are often interleaving ones, and they do not respect concurrency. SPAs inherited from untimed PAs a possibility to apply equivalences for comparison of specified processes. Like equivalences for other stochastic models, the relations for SPAs have special requirements due to the summation of probabilities. The states from which similar future behaviours start have to be grouped into equivalence classes. The classes form elements of the aggregated state space, and they are defined a posteriori while searching for equivalences on state space of a model. In [11], a notion of interleaving stochastic bisimulation equivalence for process terms was introduced. The authors proved that the equivalence is preserved by formula composition within SPAs considered in the paper, i.e., the relation is a congruence. At the same time, no appropriate equivalence notion was defined for concurrent SPAs so far. Thus, it is desirable to propose an equivalence relation for parallel SPAs that relates formulas specifying processes with similar behavior and differentiates those having non-similar one from a certain viewpoint. It would be fine to find a relation that is a congruence with respect to the algebraic operations. In this case, the formulas combined by algebraic operations from equivalent subformulas will be equivalent as well. This is very significant property while bottom-up design of processes.

We did some work on the development of concurrent discrete time SPNs and SPAs as well as on defining a variety of concurrent probabilistic equivalences. In [15], labeled weighted DTSPNs (LWDTSPNs) were proposed that is a modification of DTSPNs by transition labeling and weights. In [17, 72], labeled DTSPNs (LDTSPNs) were introduced. Transitions of LWDTSPNs and LDTSPNs are labeled by actions which represent elementary activities and can be visible or invisible to an external observer. For these two net classes, a number of new probabilistic τ-trace and τ-bisimulation equivalences were defined that abstract from invisible actions (denoted by τ) and respect concurrency in different degrees (interleaving and step relations). In addition, probabilistic relations that require back or back-forth simulation were introduced. An application of the probabilistic back-forth τ-bisimulation equivalences to compare stationary behaviour of the LWDTSPNs or LDTSPNs was demonstrated. In [67, 17], a logical characterization was presented for interleaving and step probabilistic τ-bisimulation equivalences via formulas of the new probabilistic modal logics. The characterization means that two LWDTSPNs or LDTSPNs are (interleaving or step) probabilistic τ-bisimulation equivalent if they satisfy the same formulas of the corresponding probabilistic modal logic. Thus, instead of comparing nets operationally, one have to check the corresponding satisfaction relation only applying standard verification techniques. The new interleaving and step logics are modifications of that called PML proposed in [40] on probabilistic transition systems with visible actions. In [16, 17, 72], a stochastic algebra of finite nondeterministic processes $StAFP_0$ was proposed with semantics in terms of a subclass of LWDTSPNs and LDTSPNs called stochastic acyclic nets (SANs). The calculus defined is a stochastic extension of the algebra AFP_0 introduced in [35]. $StAFP_0$ specifies concurrent stochastic processes. Another feature of the algebra is a net semantics allowing one to preserve the level of parallelism, since Petri nets is a classical "true concurrency" model. Usually, transition systems are used for this purpose, but they are not able to respect concurrency completely. An axiomatization for the semantic equivalence of $StAFP_0$ was proposed. It was proved that any algebraic formula could be reduced to the "fully stratified" one with the use of the axiom system. This simplifies semantic comparison of formulas. In [68, 72], we considered different classes of stochastic Petri nets. We explored how transition labeling could be defined to compare SPNs by equivalences. An suitability of the SPN classes for modeling and analysis of different kinds of dynamic systems was investigated. In [69, 71], a discrete time stochastic extension $dtsPBC$ of finite PBC was constructed. A step operational and a net denotational semantics of $dtsPBC$ were defined, and their consistency was demonstrated. In addition, a variety of probabilistic equivalences were proposed to identify stochastic processes with similar behaviour which are differentiated by the semantic equivalence. The interrelations of all the introduced equivalences were studied. In [70], we constructed an enrichment of $dtsPBC$

with the iteration operator used to specify infinite processes.

In this paper, we investigate equivalence notions for *dtsPBC* with iteration. First, we present the syntax of the extended *dtsPBC*. Each multiaction of the initial calculus *PBC* is associated with a probability. Such a pair is called stochastic multiaction or activity. Second, we propose semantics of *dtsPBC*. A step operational semantics is constructed in terms of labeled transition systems based on action and inaction rules. A denotational semantics is defined in terms of a subclass of LDTSPNs called discrete time stochastic Petri boxes (dts-boxes). Consistency of operational and denotational semantics is proved. Further, we define a number of stochastic equivalences in the algebraic setting based of transition systems without empty behaviour. These relations are weaker than the semantic equivalence of *dtsPBC*. They are used to identify stochastic processes with similar behaviour which are differentiated by the semantic equivalence that is too strict in many cases. The interrelations diagram of all the introduced equivalences is built. We present a characterization of the stochastic bisimulation equivalences via two new probabilistic modal logics based on *PML*. It is demonstrated how to compare stochastic processes in their steady states with the use of the relations. Moreover, a problem of preservation of the equivalence notions by algebraic operations is discussed. The proposed equivalences are used to construct a congruence relation. At the end, we present two case studies explaining how to analyze performance of systems within the calculus. We consider algebraic models of shared memory system and dining philosophers one.

The paper is organized as follows. In the next Section 2, the syntax of the extended calculus *dtsPBC* is presented. Then, in Section 3, we construct operational semantics of the algebra in terms of labeled transition systems. In Section 4, we propose denotational semantics based on a subclass of LDTSPNs. Section 5 is devoted to the construction and the interrelations of stochastic algebraic equivalences based on transition systems without empty loops. A logical characterization of the equivalences is presented in Section 6. In Section 7, an application of the relations to comparison of stationary behaviour is investigated. A preservation of the equivalences by the algebraic operations, i.e., a congruence problem is discussed in Section 8. Section 9 contains two examples of performance evaluation for systems specified by the algebraic expressions. The concluding Section 10 summarizes the results obtained and outlines research perspectives in this area.

2 Syntax

In this section, we propose the syntax of discrete time stochastic extension of finite *PBC* enriched with iteration called *discrete time stochastic Petri box calculus* (*dtsPBC*).

First, we recall a definition of multiset that is an extension of the set notion by allowing several identical elements.

Definition 2.1 *Let X be a set. A finite* multiset (bag) *M over X is a mapping $M : X \to I\!N$ such that $|\{x \in X \mid M(x) > 0\}| < \infty$, i.e., it can contain finite number of elements only.*

We denote the *set of all finite multisets* over X by $I\!N_f^X$. When $\forall x \in X\ M(x) \leq 1$, M is a proper set. The *cardinality* of a multiset M is defined as $|M| = \sum_{x\in X} M(x)$. We write $x \in M$ if $M(x) > 0$ and $M \subseteq M'$ if $\forall x \in X\ M(x) \leq M'(x)$. We define $(M + M')(x) = M(x) + M'(x)$ and $(M - M')(x) = \max\{0, M(x) - M'(x)\}$.

Let $Act = \{a, b, \ldots\}$ be the set of *elementary actions*. Then $\widehat{Act} = \{\hat{a}, \hat{b}, \ldots\}$ is the set of *conjugated actions (conjugates)* such that $a \neq \hat{a}$ and $\hat{\hat{a}} = a$. Let $\mathcal{A} = Act \cup \widehat{Act}$ be the set of *all actions*, and $\mathcal{L} = I\!N_f^{\mathcal{A}}$ be the set of *all multiactions*. Note that $\emptyset \in \mathcal{L}$, this corresponds to an internal activity, i.e., the execution of a multiaction that contains no visible action names. The *alphabet* of $\alpha \in \mathcal{L}$ is defined as $\mathcal{A}(\alpha) = \{x \in \mathcal{A} \mid \alpha(x) > 0\}$.

An *activity (stochastic multiaction)* is a pair (α, ρ), where $\alpha \in \mathcal{L}$ and $\rho \in (0; 1)$ is the probability of the multiaction α. The multiaction probabilities are used to calculate probabilities of state changes (steps) at discrete time moments. The multiaction probabilities are required not to be equal to 1, since otherwise, the multiactions with probability 1 always happen in a step, and all other with the less probabilities do not. In this case, technical difficulties appear with conflicts resolving, see [48]. Let $\mathcal{SL}$ be the set of *all activities*. Let us note that the same multiaction $\alpha \in \mathcal{L}$ may have different probabilities in the same specification. The *alphabet* of $(\alpha, \rho) \in \mathcal{SL}$ is defined as $\mathcal{A}(\alpha, \rho) = \mathcal{A}(\alpha)$. For $(\alpha, \rho) \in \mathcal{SL}$, we define its *multiaction part* as $\mathcal{L}(\alpha, \rho) = \alpha$ and its *probability part* as $\Omega(\alpha, \rho) = \rho$.

Activities are combined into formulas by the following operations: *sequential execution* ;, *choice* [], *parallelism* $\|$, *relabeling* $[f]$, *restriction* rs, *synchronization* sy and *iteration* $[**]$.

Relabeling functions $f : \mathcal{A} \to \mathcal{A}$ are bijections preserving conjugates, i.e., $\forall x \in \mathcal{A}\ f(\hat{x}) = \widehat{f(x)}$. Let $\alpha, \beta \in \mathcal{L}$ be two multiactions such that for some action $a \in Act$ we have $a \in \alpha$ and $\hat{a} \in \beta$ or $\hat{a} \in \alpha$ and $a \in \beta$. Then synchronization of α and β by a is defined as $\alpha \oplus_a \beta = \gamma$, where

$$\gamma(x) = \begin{cases} \alpha(x) + \beta(x) - 1, & x = a \text{ or } x = \hat{a}; \\ \alpha(x) + \beta(x), & \text{otherwise.} \end{cases}$$

Static expressions specify the structure of a system. As we shall see, they correspond to unmarked SPNs.

Definition 2.2 *Let $(\alpha, \rho) \in \mathcal{SL}$ and $a \in Act$. A* static expression *of dtsPBC is defined as*

$$E ::= \ (\alpha, \rho) \mid E; E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E * E * E].$$

Let $StatExpr$ denote the set of *all static expressions* of $dtsPBC$.

To avoid inconsistency of the iteration operator, we should not allow any concurrency in the highest level of the second argument of iteration. This is not a severe restriction though, since we can always prefix parallel expressions by an activity with the empty multiaction and an appropriate probability.

Definition 2.3 *Let $(\alpha, \rho) \in \mathcal{SL}$ and $a \in Act$. A* regular static expression *of dtsPBC is defined as*

$$\begin{aligned} D &::= \ (\alpha, \rho) \mid D; E \mid D[]D \mid D[f] \mid D \text{ rs } a \mid D \text{ sy } a \mid [D * D * E], \\ E &::= \ (\alpha, \rho) \mid E; E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E * D * E]. \end{aligned}$$

Let $RegStatExpr$ denote the set of *all regular static expressions* of $dtsPBC$.

Dynamic expressions specify the states of a system. As we shall see, they correspond to marked SPNs. Note that if an underlying static expression of a dynamic one is not regular, the corresponding marked SPN can be unsafe (though, it is 2-bounded in the worst case, see [6]).

Definition 2.4 *Let $(\alpha, \rho) \in \mathcal{SL}$, $a \in Act$ and $E \in RegStatExpr$. A* regular dynamic expression *of dtsPBC is defined as*

$$\begin{aligned} G ::= \ & \overline{E} \mid \underline{E} \mid G; E \mid E; G \mid G[]E \mid E[]G \mid G\|G \mid G[f] \mid G \text{ rs } a \mid G \text{ sy } a \mid \\ & [G * E * E] \mid [E * G * E] \mid [E * E * G]. \end{aligned}$$

Let $RegDynExpr$ denote the set of *all regular dynamic expressions* of $dtsPBC$.

In the following, we shall consider regular expressions only, and we can omit the word "regular".

3 Operational semantics

In this section, we construct the step operational semantics in terms of labeled transition systems.

3.1 Inaction rules

First, we define inaction rules for overlined and underlined static expressions. Let $E, F, K \in RegStatExpr$ and $a \in Act$.

$$\begin{array}{llll}
\overline{E; F} \xrightarrow{\emptyset} \overline{E}; F & \underline{E}; F \xrightarrow{\emptyset} E; \overline{F} & E; \underline{F} \xrightarrow{\emptyset} \underline{E; F} & \overline{E[]F} \xrightarrow{\emptyset} \overline{E}[]F \\
\overline{E[]F} \xrightarrow{\emptyset} E[]\overline{F} & \underline{E}[]F \xrightarrow{\emptyset} \underline{E[]F} & E[]\underline{F} \xrightarrow{\emptyset} \underline{E[]F} & \overline{E\|F} \xrightarrow{\emptyset} \overline{E}\|\overline{F} \\
\underline{E}\|\underline{F} \xrightarrow{\emptyset} \underline{E\|F} & \overline{E[f]} \xrightarrow{\emptyset} \overline{E}[f] & \underline{E}[f] \xrightarrow{\emptyset} \underline{E[f]} & \overline{E \text{ rs } a} \xrightarrow{\emptyset} \overline{E} \text{ rs } a \\
\underline{E} \text{ rs } a \xrightarrow{\emptyset} \underline{E \text{ rs } a} & \overline{E \text{ sy } a} \xrightarrow{\emptyset} \overline{E} \text{ sy } a & \underline{E} \text{ sy } a \xrightarrow{\emptyset} \underline{E \text{ sy } a} & \overline{[E * F * K]} \xrightarrow{\emptyset} [\overline{E} * F * K] \\
[\underline{E} * F * K] \xrightarrow{\emptyset} [E * \overline{F} * K] & [E * \underline{F} * K] \xrightarrow{\emptyset} [E * \overline{F} * K] & [E * \underline{F} * K] \xrightarrow{\emptyset} [E * F * \overline{K}] & [E * F * \underline{K}] \xrightarrow{\emptyset} \underline{[E * F * K]}
\end{array}$$

Second, we propose inaction rules for dynamic expressions. Let $E, F \in RegStatExpr$, $G, H, \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

$$\begin{array}{ccccc}
G \xrightarrow{\emptyset} G & \dfrac{G \xrightarrow{\emptyset} \widetilde{G}, \ \circ \in \{;, []\}}{G \circ E \xrightarrow{\emptyset} \widetilde{G} \circ E} & \dfrac{G \xrightarrow{\emptyset} \widetilde{G}, \ \circ \in \{;, []\}}{E \circ G \xrightarrow{\emptyset} E \circ \widetilde{G}} & \dfrac{G \xrightarrow{\emptyset} \widetilde{G}}{G\|H \xrightarrow{\emptyset} \widetilde{G}\|H} & \dfrac{H \xrightarrow{\emptyset} \widetilde{H}}{G\|H \xrightarrow{\emptyset} G\|\widetilde{H}} \\
\dfrac{G \xrightarrow{\emptyset} \widetilde{G}}{G[f] \xrightarrow{\emptyset} \widetilde{G}[f]} & \dfrac{G \xrightarrow{\emptyset} \widetilde{G}, \ \circ \in \{\text{rs}, \text{sy}\}}{G \circ a \xrightarrow{\emptyset} \widetilde{G} \circ a} & \dfrac{G \xrightarrow{\emptyset} \widetilde{G}}{[G * E * F] \xrightarrow{\emptyset} [\widetilde{G} * E * F]} & \dfrac{G \xrightarrow{\emptyset} \widetilde{G}}{[E * G * F] \xrightarrow{\emptyset} [E * \widetilde{G} * F]} & \dfrac{G \xrightarrow{\emptyset} \widetilde{G}}{[E * F * G] \xrightarrow{\emptyset} [E * F * \widetilde{G}]}
\end{array}$$

Note that the rule $G \xrightarrow{\emptyset} G$ is intentionally included in the set of rules above. It reflects a non-zero probability to stay in a state at the next time moment that is an essential feature of discrete time stochastic processes. This ia a new rule that has no prototype among inaction rules of PBC.

A regular dynamic expression G is *operative* if no inaction rule can be applied to it, with the exception of $G \stackrel{\emptyset}{\rightarrow} G$. Note that any dynamic expression can be always transformed into a (not necessarily unique) operative one using inaction rules. Let $OpRegDynExpr$ denote the set of *all operative regular dynamic expressions* of $dtsPBC$.

Definition 3.1 *Let $\simeq = (\stackrel{\emptyset}{\rightarrow} \cup \stackrel{\emptyset}{\leftarrow})^*$ be dynamic expression isomorphism in $dtsPBC$. Thus, two dynamic expressions G and G' are* isomorphic, *denoted by $G \simeq G'$, if they can be reached from each other by applying inaction rules.*

3.2 Action rules

Now we propose action rules which describe expression transformations due to the execution of multisets of activities. Let $(\alpha, \rho), (\beta, \chi) \in \mathcal{SL},\ E, F \in RegStatExpr,\ G, H \in OpRegDynExpr,\ \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$. Moreover, let $\Gamma, \Delta \in I\!N_f^{\mathcal{SL}}$. The *alphabet* of $\Gamma \in I\!N_f^{\mathcal{SL}}$ is defined as $\mathcal{A}(\Gamma) = \cup_{(\alpha,\rho)\in\Gamma}\mathcal{A}(\alpha)$.

$$\overline{(\alpha,\rho)} \stackrel{\{(\alpha,\rho)\}}{\longrightarrow} \underline{(\alpha,\rho)} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{G;E \stackrel{\Gamma}{\rightarrow} \widetilde{G};E} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{E;G \stackrel{\Gamma}{\rightarrow} E;\widetilde{G}} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{G[]E \stackrel{\Gamma}{\rightarrow} \widetilde{G}[]E} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{E[]G \stackrel{\Gamma}{\rightarrow} E[]\widetilde{G}} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{G\|H \stackrel{\Gamma}{\rightarrow} \widetilde{G}\|H}$$

$$\frac{H \stackrel{\Gamma}{\rightarrow} \widetilde{H}}{G\|H \stackrel{\Gamma}{\rightarrow} G\|\widetilde{H}} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G},\ H \stackrel{\Delta}{\rightarrow} \widetilde{H}}{G\|H \stackrel{\Gamma+\Delta}{\longrightarrow} \widetilde{G}\|\widetilde{H}} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{G[f] \stackrel{f(\Gamma)}{\longrightarrow} \widetilde{G}[f]} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G},\ a,\hat{a} \notin \mathcal{A}(\Gamma)}{G \textsf{ rs } a \stackrel{\Gamma}{\rightarrow} \widetilde{G} \textsf{ rs } a} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{[G*E*F] \stackrel{\Gamma}{\rightarrow} [\widetilde{G}*E*F]} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{[E*G*F] \stackrel{\Gamma}{\rightarrow} [E*\widetilde{G}*F]}$$

$$\frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{[E*F*G] \stackrel{\Gamma}{\rightarrow} [E*F*\widetilde{G}]} \qquad \frac{G \stackrel{\Gamma}{\rightarrow} \widetilde{G}}{G \textsf{ sy } a \stackrel{\Gamma}{\rightarrow} \widetilde{G} \textsf{ sy } a} \qquad \frac{G \textsf{ sy } a \stackrel{\Gamma+\{(\alpha,\rho)\}+\{(\beta,\chi)\}}{\longrightarrow} \widetilde{G} \textsf{ sy } a,\ a \in \mathcal{A}(\alpha),\ \hat{a} \in \mathcal{A}(\beta)}{G \textsf{ sy } a \stackrel{\Gamma+\{(\alpha\oplus_a\beta,\rho\cdot\chi)\}}{\longrightarrow} \widetilde{G} \textsf{ sy } a}$$

Note that in the second rule for synchronization above we multiply the probabilities of synchronized multi-actions since this corresponds to the probability of event intersection. This is a new rule that has no analogous action rule in PBC.

3.3 Transition systems

Now we define labeled probabilistic transition systems associated with dynamic expressions.

Note that expressions of $dtsPBC$ can contain identical activities. To avoid technical difficulties, such as the proper calculation of the state change probabilities for multiple transitions, we can always enumerate coinciding activities from left to right in the syntax of expressions. In the following, we suppose that all identical activities are enumerated. The new activities resulted from synchronization will be annotated with the concatenation of the numbering of the activities they come from. Such new activities will be considered up to the permutation of their numbering resulting from the applications of the second rule for synchronization. After such an enumeration, the multisets of activities over arrows in the action rules will be the proper sets.

Let X be some set. We denote the cartesian product $X \times X$ by X^2. Let $\mathcal{E} \subseteq X^2$ be an equivalence relation on X. Then the *equivalence class* (with respect to $\mathcal{E}$) of an element $x \in X$ is defined by $[x]_{\mathcal{E}} = \{y \in X \mid (x, y) \in \mathcal{E}\}$. The equivalence $\mathcal{E}$ partitions X into the *set of equivalence classes* $X/_{\mathcal{E}} = \{[x]_{\mathcal{E}} \mid x \in X\}$.

Definition 3.2 *Let G be a dynamic expression. Then $[G]_\simeq = \{H \mid G \simeq H\}$ is the equivalence class of G with respect to isomorphism (the isomorphism class). The* derivation set *of a dynamic expression G, denoted by $DR(G)$, is the minimal set such that*

- *$[G]_\simeq \in DR(G)$;*
- *if $[H]_\simeq \in DR(G)$ and $\exists\Gamma\ H \stackrel{\Gamma}{\rightarrow} \widetilde{H}$ then $[\widetilde{H}]_\simeq \in DR(G)$.*

Let G be a dynamic expression and $s \in DR(G)$.

The set of *all multisets of activities executable in* s is defined as $Exec(s) = \{\Gamma \mid \exists H \in s\ \exists\widetilde{H}\ H \stackrel{\Gamma}{\rightarrow} \widetilde{H}\}$.

Let $\Gamma \in Exec(s)$. The probability that the activities from Γ *try to happen* in s is

$$PF(\Gamma, s) = \prod_{(\alpha,\rho)\in\Gamma} \rho \cdot \prod_{\{\{(\beta,\chi)\}\in Exec(s) \mid (\beta,\chi)\notin\Gamma\}} (1 - \chi).$$

In the case $\Gamma = \emptyset$ we define

$$PF(\emptyset, s) = \begin{cases} \prod_{\{(\beta,\chi)\}\in Exec(s)}(1-\chi), & Exec(s) \neq \{\emptyset\}; \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $PF(\Gamma, s)$ could be interpreted as a *joint* probability of independent events. Each such an event is interpreted as trying or not trying to occur of a particular activity from Γ. The multiplication in the definition is used because it reflects the probability of event intersection. When only empty multiset of activities can happen in s, we have $PF(\emptyset, s) = 1$, since we stay in s in this case.

The probability that the activities from Γ *happen* in s is

$$PT(\Gamma, s) = \frac{PF(\Gamma, s)}{\sum_{\Delta \in Exec(s)} PF(\Delta, s)}.$$

Thus, $PT(\Gamma, s)$ is the probability that the multiset of activities Γ tries to happen *normalized* by the probability to occur for *any* multiset executable in s. The denominator of the fraction above is a sum since it reflects the probability of the event union.

Note that the sum of outgoing probabilities for the expressions belonging to the derivations of G is equal to one. More formally, $\forall s \in DR(G)\ \sum_{\Gamma \in Exec(s)} PT(\Gamma, s) = 1$. This obviously follows from the definition of $PT(\Gamma, s)$ and guarantees that $PT(\Gamma, s)$ defines a probability distribution.

The probability that the execution of *any* activities changes s to $\tilde{s}$ is

$$PM(s, \tilde{s}) = \sum_{\{\Gamma \mid \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}} PT(\Gamma, s).$$

Since $PM(s, \tilde{s})$ is the probability for *any* multiset of activities to change s to $\tilde{s}$, we use summation in the definition. Note that $\forall s \in DR(G)\ \sum_{\{\tilde{s} \mid \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ \exists \Gamma\ H \xrightarrow{\Gamma} \widetilde{H}\}} PM(s, \tilde{s}) = \sum_{\{\tilde{s} \mid \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ \exists \Gamma\ H \xrightarrow{\Gamma} \widetilde{H}\}} \sum_{\{\Gamma \mid \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}} PT(\Gamma, s) = \sum_{\Gamma \in Exec(s)} PT(\Gamma, s) = 1$.

Definition 3.3 *Let G be a dynamic expression. The* (labeled probabilistic) transition system *of G is a quadruple $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, where*

- *the set of* states *is $S_G = DR(G)$;*
- *the set of* labels *is $L_G \subseteq \mathbb{N}_f^{\mathcal{SL}} \times (0; 1]$;*
- *the set of* transitions *is $\mathcal{T}_G = \{(s, (\Gamma, PT(\Gamma, s)), \tilde{s}) \mid s \in DR(G),\ \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}$;*
- *the* initial state *is $s_G = [G]_\simeq$.*

Thus, the transition system $TS(G)$ associated with a dynamic expression G describes all steps that happen at moments of discrete time with some (one-step) probability and consist of multisets of activities. These steps change states, and the states are the isomorphism classes of dynamic expressions obtained by application of action rules starting from the expressions belonging to $[G]_\simeq$. A transition $(s, (\Gamma, \mathcal{P}), \tilde{s}) \in \mathcal{T}_G$ will be written as $s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s}$. It is interpreted as follows: the probability to change the state s to $\tilde{s}$ as a result of executing Γ is $\mathcal{P}$. The step probabilities belong to the interval $(0; 1]$. The value 1 is the case when we cannot leave a state s, and hence there exists the only transition from s to itself $s \xrightarrow{\emptyset}_1 s$.

We write $s \xrightarrow{\Gamma} \tilde{s}$ if $\exists \mathcal{P}\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s}$ and $s \to \tilde{s}$ if $\exists \Gamma\ s \xrightarrow{\Gamma} \tilde{s}$. For one-element multiset $\Gamma = \{(\alpha, \rho)\}$ we write $s \xrightarrow{(\alpha,\rho)}_{\mathcal{P}} \tilde{s}$ and $s \xrightarrow{(\alpha,\rho)} \tilde{s}$.

Note that Γ could be the empty set, and its execution does not change isomorphism classes. This corresponds to the application of inaction rules to the expressions from the isomorphism classes. We have to keep track of such executions called *empty loops*, because they have nonzero probabilities. It follows from the definition of $PF(\emptyset, s)$ and the fact that multiaction probabilities cannot be equal to 1 as they belong to the interval $(0; 1)$.

Definition 3.4 *Let G, G' be dynamic expressions and $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, $TS(G') = (S_{G'}, L_{G'}, \mathcal{T}_{G'}, s_{G'})$ be their transition systems. A mapping $\beta : S_G \to S_{G'}$ is an* isomorphism *between $TS(G)$ and $TS(G')$, denoted by $\beta : TS(G) \simeq TS(G')$, if*

1. *β is a bijection such that $\beta(s_G) = s_{G'}$;*
2. *$\forall s, \tilde{s} \in S_G\ \forall \Gamma\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \xrightarrow{\Gamma}_{\mathcal{P}} \beta(\tilde{s})$.*

Two transition systems $TS(G)$ and $TS(G')$ are isomorphic*, denoted by $TS(G) \simeq TS(G')$, if $\exists \beta : TS(G) \simeq TS(G')$.*

Transition systems of static expressions can be defined as well. For $E \in RegStatExpr$ let $TS(E) = TS(\overline{E})$.

Definition 3.5 *Two dynamic expressions G and G' are* isomorphic with respect to transition systems*, denoted by $G =_{ts} G'$, if $TS(G) \simeq TS(G')$.*

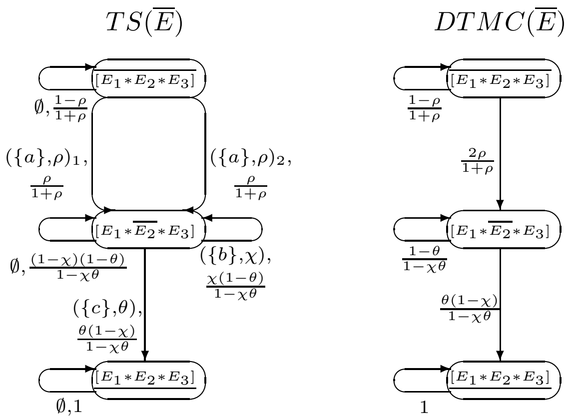

Figure 1: The transition system and the underlying DTMC of $\overline{E}$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2)*(\{b\},\chi)*(\{c\},\theta)]$

Definition 3.6 *Let G be a dynamic expression. The* underlying discrete time Markov chain (DTMC) *of G, denoted by $DTMC(G)$, has the state space $DR(G)$ and the transitions $s \rightarrow_{\mathcal{P}} \tilde{s}$, if $s \rightarrow \tilde{s}$ and $\mathcal{P} = PM(s,\tilde{s})$.*

Underlying DTMCs of static expressions can be defined as well. For $E \in RegStatExpr$, let $DTMC(E) = DTMC(\overline{E})$.

Example 3.1 *Let $E_1 = (\{a\},\rho)[](\{a\},\rho)$, $E_2 = (\{b\},\chi)$, $E_3 = (\{c\},\theta)$ and $E = [E_1 * E_2 * E_3]$. The identical activities of the composite static expression are enumerated as follows: $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$. In Figure 1 the transition system $TS(\overline{E})$ and the underlying DTMC $DTMC(\overline{E})$ are presented. Note that, for simplicity of the graphical representation, states are depicted by expressions belonging to the corresponding isomorphism classes, and singleton multisets of activities are written without braces. $DR(\overline{E})$ consists of isomorphism classes $s_1 = [[\overline{E_1 * E_2 * E_3}]]_{\simeq}$, $s_2 = [[E_1 * \overline{E_2} * E_3]]_{\simeq}$, $s_3 = [[\underline{E_1 * E_2 * E_3}]]_{\simeq}$.*

Let us demonstrate how the transition probabilities are calculated. For instance, we have $PF(\{(\{a\},\rho)_1\},s_1) = PF(\{(\{a\},\rho)_2\},s_1) = \rho(1-\rho)$ and $PF(\emptyset,s_1) = (1-\rho)^2$. Hence, $\sum_{\Delta\in Exec(s_1)} PF(\Delta,s_1) = 2\rho(1-\rho) + (1-\rho)^2 = 1-\rho^2$. Thus, $PT(\{(\{a\},\rho)_1\},s_1) = PT(\{(\{a\},\rho)_2\},s_1) = \frac{\rho(1-\rho)}{1-\rho^2} = \frac{\rho}{1+\rho}$ and $PT(\emptyset,s_1) = \frac{(1-\rho)^2}{1-\rho^2} = \frac{1-\rho}{1+\rho}$. The other probabilities are calculated in a similar way.

Every state of $DTMC(\overline{E})$ is associated with a discrete random value meaning a residence time in it. One can interpret staying in a state in the next discrete time moment as a failure and leaving it as a success of some trial series. It is easy to see that the random values are geometrically distributed, since the probability to stay in the state s_i $(1 \le i \le 3)$ for $k-1$ time moments and leave it at moment $k \ge 1$ is $PT(\emptyset,s_i)^{k-1}(1-PT(\emptyset,s_i))$ (the residence time is k in this case). The mean value formula for geometrical distribution allows us to calculate sojourn time in the states as $SJ(s_i) = \frac{1}{1-PT(\emptyset,s_i)}$. Thus, the sojourn time vector is

$$SJ = \left(\frac{1+\rho}{2\rho}, \frac{1}{\chi}, \infty\right).$$

4 Denotational semantics

In this section, we construct the denotational semantics in terms of a subclass of labeled DTSPNs called discrete time stochastic Petri boxes (dts-boxes).

4.1 Labeled DTSPNs

Now we introduce a class of labeled discrete time stochastic Petri nets.

Definition 4.1 *A* labeled DTSPN (LDTSPN) *is a tuple $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$, where*

- *P_N and T_N are finite sets of* places *and* transitions*, respectively, such that $P_N \cup T_N \neq \emptyset$ and $P_N \cap T_N = \emptyset$;*
- *$W_N : (P_N \times T_N) \cup (T_N \times P_N) \rightarrow I\!N$ is a function describing the* weights of arcs *between places and transitions;*

- $\Omega_N : T_N \to (0;1)$ *is the* transition probability *function associating transitions with probabilities;*
- $L_N : T_N \to Act_\tau$ *is the* transition labeling *function assigning labels from a finite set of visible actions Act or an invisible action τ to transitions (i.e., $Act_\tau = Act \cup \{\tau\}$);*
- $M_N \in I\!N_f^{P_N}$ *is the* initial marking*.*

A graphical representation of LDTSPNs is like that for standard labeled Petri nets but with probabilities written near the corresponding transitions. In the case the probabilities are not specified in the picture, they are considered to be of no importance in the corresponding examples, such as those used to describe stationary behaviour. The arc weights are depicted near them. The names of places and transitions are depicted near them when needed. If the names are omitted but used, it is supposed that the places and transitions are numbered from left to right and from top to down.

Let N be an LDTSPN and $t \in T_N$, $U \in I\!N_f^{T_N}$. The *precondition* ${}^\bullet t$ and the *postcondition* $t^\bullet$ of t are the multisets of places defined as $({}^\bullet t)(p) = W_N(p,t)$ and $(t^\bullet)(p) = W_N(t,p)$. The *precondition* ${}^\bullet U$ and the *postcondition* $U^\bullet$ of U are the multisets of places defined as ${}^\bullet U = \sum_{t\in U} {}^\bullet t$ and $U^\bullet = \sum_{t\in U} t^\bullet$.

A transition $t \in T_N$ is enabled in a marking $M \in I\!N_f^{P_N}$ of LDTSPN N if ${}^\bullet t \subseteq M$. Let $Ena(M)$ be the set of *all transitions such that each of them is enabled in a marking* M. A set of transitions $U \subseteq Ena(M)$ is enabled in a marking M if ${}^\bullet U \subseteq M$. Firings of transitions are atomic operations, and transitions may fire concurrently in steps. We assume that all transitions participating in a step should differ, hence, only the sets (not multisets) of transitions may fire. Thus, we do not allow self-concurrency, i.e., firing of transitions concurrently to themselves. This restriction is introduced because we would like to avoid technical difficulties while calculating probabilities for multisets of transitions as we shall see after the following formal definitions.

Let M be a marking of an LDTSPN N. A transition $t \in Ena(M)$ fires with probability $\Omega_N(t)$ when no other transitions conflicting with it are enabled. Let ${}^\bullet U \subseteq M$. The probability that the transitions from U *try to fire* in M is

$$PF(U,M) = \prod_{t\in U} \Omega_N(t) \cdot \prod_{u\in Ena(M)\setminus U} (1-\Omega_N(u)).$$

In the case $U = \emptyset$ we define

$$PF(\emptyset,M) = \begin{cases} \prod_{u\in Ena(M)}(1-\Omega_N(u)), & Ena(M) \neq \emptyset; \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $PF(U,M)$ could be interpreted as a *joint* probability of independent events. Each such an event is interpreted as trying or not trying to fire of a particular transition from U. The multiplication in the definition is used because it reflects the probability of event intersection. When no transitions are enabled in M, we have $PF(\emptyset,M) = 1$, since we stay in M in this case.

Concurrent firing of the transitions from U changes the marking M to $\widetilde{M} = M - {}^\bullet U + U^\bullet$, denoted by $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT(U,M)$ and

$$PT(U,M) = \frac{PF(U,M)}{\sum_{\{V \mid {}^\bullet V\subseteq M\}} PF(V,M)}.$$

In the case $U = \emptyset$ we have $M = \widetilde{M}$ and

$$PT(\emptyset,M) = \frac{PF(\emptyset,M)}{\sum_{\{V \mid {}^\bullet V\subseteq M\}} PF(V,M)}.$$

Thus, $PT(U,M)$ is the probability that the set U tries to fire *normalized* by the probability to fire for *any* set enabled in M. The denominator of the fraction above is a summation since it reflects the probability of the event union.

We write $M \xrightarrow{U} \widetilde{M}$ if $\exists \mathcal{P}\ M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$ and $M \to \widetilde{M}$ if $\exists U\ M \xrightarrow{U} \widetilde{M}$. For one-element transition set $U = \{t\}$ we write $M \xrightarrow{t}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{t} \widetilde{M}$.

Note that for all markings of an LDTSPN N the sum of outgoing probabilities is equal to one. More formally, $\forall M \in I\!N_f^{P_N}\ PT(\emptyset,M) + \sum_{\{U \mid {}^\bullet U\subseteq M\}} PT(U,M) = 1$. This obviously follows from the definition of $PT(U,M)$ and guarantees that it defines a probability distribution.

Definition 4.2 *Let N be an LDTSPN.*

- *The* reachability set *of N, denoted by $RS(N)$, is the minimal set of markings such that*

- $M_N \in RS(N)$;
- *if* $M \in RS(N)$ *and* $M \to \widetilde{M}$ *then* $\widetilde{M} \in RS(N)$.

- *The* reachability graph *of* N, *denoted by* $RG(N)$, *is a directed labeled graph with the set of nodes* $RS(N)$ *and an arc labeled with* $(U, \mathcal{P})$ *between nodes* M *and* $\widetilde{M}$ *if* $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$.

- *The* underlying discrete time Markov chain (DTMC) *of* N, *denoted by* $DTMC(N)$, *has the state space* $RS(N)$ *and the transitions* $M \to_{\mathcal{P}} \widetilde{M}$, *if* $M \to \widetilde{M}$, *where* $\mathcal{P} = PM(M, \widetilde{M})$ *and*

$$PM(M, \widetilde{M}) = \sum_{\{U \mid M \xrightarrow{U} \widetilde{M}\}} PT(U, M).$$

Since $PM(M, \widetilde{M})$ is the probability for *any* transition set to change marking M to $\widetilde{M}$, we use summation in the definition. Note that $\forall M \in RS(N)$ $\sum_{\{\widetilde{M} \mid M \to \widetilde{M}\}} PM(M, \widetilde{M}) = \sum_{\{\widetilde{M} \mid M \to \widetilde{M}\}} \sum_{\{U \mid M \xrightarrow{U} \widetilde{M}\}} PT(U, M) = \sum_{\{U \mid {}^\bullet U \subseteq M\}} PT(U, M) = 1$.

Example 4.1 *In Figure 2 an LDTSPN* N *with two visible transitions* t_1 *(labeled by* a*),* t_2 *(labeled by* b*) and one invisible transition* t_3 *(labeled by* τ*) is depicted. Transition probabilities of* N *are denoted by* $\rho = \Omega_N(t_1)$, $\chi = \Omega_N(t_2)$, $\theta = \Omega_N(t_3)$. *In the figure one can see the reachability graph* $RG(N)$ *and the underlying DTMC* $DTMC(N)$ *as well.* $RS(N)$ *consists of markings* $M_1 = (1,1,0)$, $M_2 = (0,1,1)$, $M_3 = (1,0,1)$, $M_4 = (0,0,2)$.

The sojourn time vector is

$$SJ = \left(\frac{1}{\rho + \chi - \rho\chi}, \frac{1}{\chi}, \frac{1}{\rho}, \frac{1}{\theta}\right).$$

The elements $\mathcal{P}_{ij}$ $(1 \le i, j \le 4)$ *of (one-step) transition probability matrix (TPM) for* $DTMC(N)$ *are defined as*

$$\mathcal{P}_{ij} = \begin{cases} PM(M_i, M_j), & M_i \to M_j; \\ 0, & \textit{otherwise}. \end{cases}$$

Thus, the TPM is

$$\mathbf{P} = \begin{bmatrix} (1-\rho)(1-\chi) & \rho(1-\chi) & \chi(1-\rho) & \rho\chi \\ 0 & 1-\chi & 0 & \chi \\ 0 & 0 & 1-\rho & \rho \\ \theta & 0 & 0 & 1-\theta \end{bmatrix}.$$

The steady state probability mass function (PMF) $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ *for* $DTMC(N)$ *is the solution of the equation system*

$$\begin{cases} \psi(\mathbf{P} - \mathbf{E}) = \mathbf{0} \\ \psi \mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{E}$ *is the unitary matrix of dimension four and* $\mathbf{0} = (0,0,0,0)$, $\mathbf{1} = (1,1,1,1)$.

For the case $\rho = \chi = \theta$ *we have*

$$\psi = \frac{1}{5 - 3\rho}(1, 1-\rho, 1-\rho, 2-\rho).$$

The inverse of the steady state PMF for $DTMC(N)$ *is its mean recurrence time vector*

$$RC = \left(5 - 3\rho, \frac{5-3\rho}{1-\rho}, \frac{5-3\rho}{1-\rho}, \frac{5-3\rho}{2-\rho}\right).$$

Each element of RC *is the mean number of steps to return to the corresponding marking. For instance, one can see that the average time to come back to the initial marking* $M_N = M_1$ *in the long-term behaviour belongs in the interval* $(2; 5)$, *since* $\rho \in (0; 1)$.

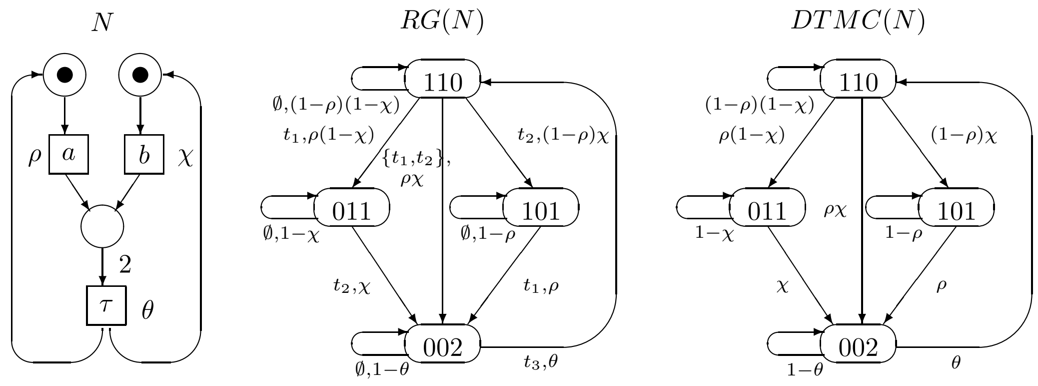

Figure 2: LDTSPN, its reachability graph and the underlying DTMC

4.2 Algebra of dts-boxes

Now we propose discrete time stochastic Petri boxes and associated algebraic operations to define a net representation of $dtsPBC$ expressions.

Definition 4.3 *A* plain discrete time stochastic Petri box (plain dts-box) *is a tuple* $N = (P_N, T_N, W_N, \Lambda_N)$, *where*

- P_N *and* T_N *are finite sets of* places *and* transitions, *respectively, such that* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$*;*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \rightarrow I\!N$ *is a function describing the* weights of arcs *between places and transitions and vice versa;*
- Λ_N *is the* place and transition labeling *function such that* $\Lambda_N : P_N \rightarrow \{\mathsf{e}, \mathsf{i}, \mathsf{x}\}$ *(it specifies* entry, internal *and* exit *places, respectively) and* $\Lambda_N : T_N \rightarrow \mathcal{SL}$ *(it associates activities with transitions).*

Moreover, $\forall t \in T_N\ {}^\bullet t \neq \emptyset \neq t^\bullet,\ {}^\bullet t \cap t^\bullet = \emptyset$*. In addition, for the set of* entry *places of* N *defined as* ${}^\circ N = \{p \in P_N \mid \Lambda_N(p) = \mathsf{e}\}$ *and the set of* exit *places of* N *defined as* $N^\circ = \{p \in P_N \mid \Lambda_N(p) = \mathsf{x}\}$ *the following holds:* ${}^\circ N \neq \emptyset \neq N^\circ,\ {}^\bullet({}^\circ N) = \emptyset = (N^\circ)^\bullet$*.*

A *marked plain dts-box* is a pair (N, M_N), where N is a plain dts-box and $M_N \in I\!N_f^{P_N}$ is the *initial marking*. We shall use the following notation: $\overline{N} = (N, {}^\circ N)$ and $\underline{N} = (N, N^\circ)$. Note that a marked plain dts-box $(P_N, T_N, W_N, \Lambda_N, M_N)$ could be interpreted as the LDTSPN $(P_N, T_N, W_N, \Omega_N, L_N, M_N)$, where functions Ω_N and L_N are defined as follows: $\forall t \in T_N\ \Omega_N(t) = \Omega(\Lambda_N(t)),\ L_N(t) = \mathcal{L}(\Lambda_N(t))$. In this case, the label τ of silent transitions from the LDTSPN corresponds to the multiaction part $\emptyset$ of activities that label unobservable transitions of the corresponding dts-box. The behaviour of marked dts-boxes follows from the firing rule of LDTSPNs. A plain dts-box N is *safe*, if $\overline{N}$ is so, i.e., $\forall M \in RS(\overline{N})\ M \subseteq P_N$. A plain dts-box N is *clean* if $N^\circ \subseteq M \ \Rightarrow\ M = N^\circ$, i.e., if there are tokens in exit places, then all exit places and only they have tokens.

To define semantic function that associates a plain dts-box with every static expression of $dtsPBC$, we need to propose the *enumeration* function $Enu : T_N \rightarrow I\!N^*$. It associates the numbers with transitions of plain dts-box N in accordance with the enumeration of activities from left to right in the syntax of the underlying static expression. In the case of synchronization, the function associates concatenation of the numbers of the transitions it comes from with the resulting new transition.

The structure of the plain dts-box corresponding to a static expression is constructed as in PBC, see [12, 13, 6]. I.e., we use simultaneous refinement and relabeling meta-operator (net refinement) in addition to the *operator dts-boxes* corresponding to the algebraic operations of $dtsPBC$ and featuring transformational transition relabelings. Thus, the resulting plain dts-boxes are safe and clean. In the definition of denotational semantics, we shall use standard constructions used for PBC. For convenience, we only use slightly different notation: ϱ, Θ and u stand for ρ (relabeling), Ω (operator box) and v (transition name) from PBC setting, respectively.

The relabeling relations $\varrho \subseteq I\!N_f^{\mathcal{SL}} \times \mathcal{SL}$ are defined as follows:

- $\varrho_{id} = \{(\{(\alpha, \rho)\}, (\alpha, \rho) \mid (\alpha, \rho) \in \mathcal{SL}\}$ is the *identity* relabeling keeping the interface as it is;
- $\varrho_{[f]} = \{(\{(\alpha, \rho)\}, (f(\alpha), \rho) \mid (\alpha, \rho) \in \mathcal{SL}\}$;

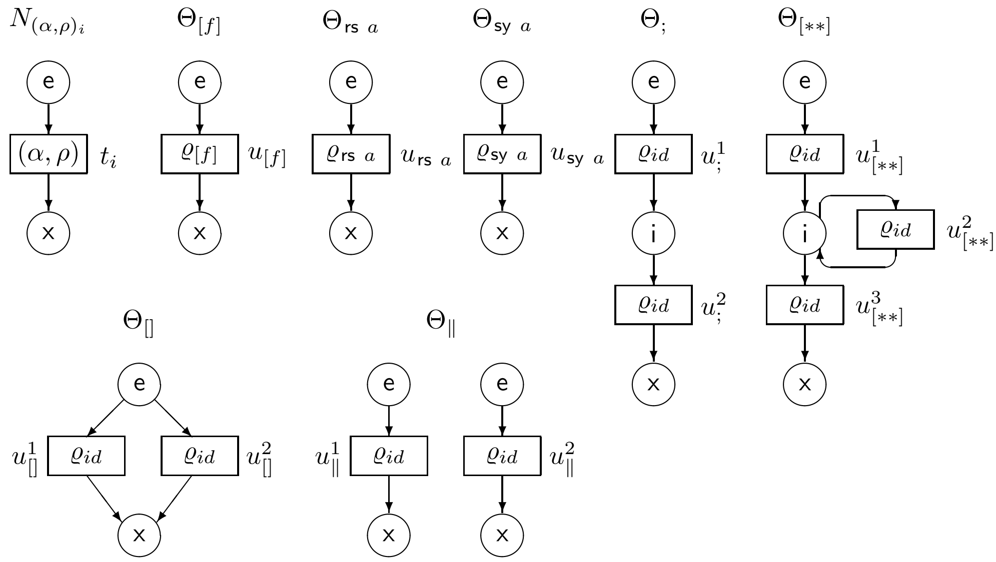

Figure 3: The plain and operator dts-boxes

- $\varrho_{\mathsf{rs}\ a} = \{(\{(\alpha,\rho)\},(\alpha,\rho) \mid (\alpha,\rho) \in \mathcal{SL},\ a,\hat{a} \notin \mathcal{A}(\alpha)\}$;
- $\varrho_{\mathsf{sy}\ a}$ is the least relabeling relation contained in ϱ_{id} such that if $(\Gamma,\{(\alpha+\{a\},\rho)\} \in \varrho_{\mathsf{sy}\ a}$ and $(\Delta,\{(\beta+\{\hat{a}\},\chi)\} \in \varrho_{\mathsf{sy}\ a}$ then $(\Gamma+\Delta,\{(\alpha+\beta,\rho\cdot\chi)\} \in \varrho_{\mathsf{sy}\ a}$.

The plain and operator dts-boxes are presented in Figure 3. Note that the the symbol i is usually omitted.

Now we define the enumeration function Enu for every operator of $dtsPBC$. Let $Box_{dts}(E) = (P_E,T_E,W_E,\Omega_E,L_E)$ be the plain dts-box corresponding to a static expression E, and Enu_E be the enumeration function for T_E.

- $Box_{dts}(E \circ F) = \Theta_\circ(Box_{dts}(E), Box_{dts}(F)),\ \circ \in \{;\,,[],\|\}$. Since we do not introduce new transitions, we preserve the initial enumeration:

$$Enu(t) = \begin{cases} Enu_E(t), & t \in T_E; \\ Enu_F(t), & t \in T_F. \end{cases}$$

- $Box_{dts}(E[f]) = \Theta_{[f]}(Box_{dts}(E))$. Since we only change the labels of some multiactions by a bijection, we preserve the initial enumeration:

$$Enu(t) = Enu_E(t),\ t \in T_E.$$

- $Box_{dts}(E \ \mathsf{rs}\ a) = \Theta_{\mathsf{rs}\ a}(Box_{dts}(E))$. Since we remove all transitions labeled with multiactions containing a or $\hat{a}$, this does not change the enumeration of the remaining transitions:

$$Enu(t) = Enu_E(t),\ t \in T_E,\ a,\hat{a} \notin L_E(t).$$

- $Box_{dts}(E \ \mathsf{sy}\ a) = \Theta_{\mathsf{sy}\ a}(Box_{dts}(E))$. Note that $\forall v,w \in T_E$ such that $L_E(v) = \alpha+\{a\},\ L_E(w) = \beta+\{\hat{a}\}$, the new transition t resulting from synchronization of v and w has the label $L(t) = \alpha+\beta$, probability $\Omega(t) = \Omega_E(v)\cdot\Omega_E(w)$ and enumeration $Enu(t) = Enu_E(v)\cdot Enu_E(w)$. Thus, the enumeration is defined as

$$Enu(t) = \begin{cases} Enu_E(t), & t \in T_E; \\ Enu_E(v)\cdot Enu_E(w), & t \text{ results from synchronization of } v \text{ and } w. \end{cases}$$

To avoid introducing redundant transitions generated by synchronizing the same transition set in a different order, we only consider a single one of them in the plain dts-box.

- $Box_{dts}([E{*}F{*}K]) = \Theta_{[**]}(Box_{dts}(E), Box_{dts}(F), Box_{dts}(K))$. Since we do not introduce new transitions, we preserve the initial enumeration:

$$Enu(t) = \begin{cases} Enu_E(t), & t \in T_E; \\ Enu_F(t), & t \in T_F; \\ Enu_K(t), & t \in T_K. \end{cases}$$

Now we can formally define the denotational semantics as a homomorphism.

Definition 4.4 *Let $(\alpha, \rho) \in \mathcal{SL}$, $a \in Act$ and $E, F, K \in RegStatExpr$. The* denotational semantics *of dtsPBC is a mapping Box_{dts} from RegStatExpr into the area of plain dts-boxes defined as follows:*

1. $Box_{dts}((\alpha, \rho)_i) = N_{(\alpha,\rho)_i}$;
2. $Box_{dts}(E \circ F) = \Theta_\circ(Box_{dts}(E), Box_{dts}(F)),\ \circ \in \{;, [], \|\}$;
3. $Box_{dts}(E[f]) = \Theta_{[f]}(Box_{dts}(E))$;
4. $Box_{dts}(E \circ a) = \Theta_{\circ a}(Box_{dts}(E)),\ \circ \in \{\mathsf{rs}, \mathsf{sy}\}$;
5. $Box_{dts}([E * F * K]) = \Theta_{[**]}(Box_{dts}(E), Box_{dts}(F), Box_{dts}(K))$.

The dts-boxes of dynamic expressions can be defined as well. For $E \in RegStatExpr$, let $Box_{dts}(\overline{E}) = \overline{Box_{dts}(E)}$ and $Box_{dts}(\underline{E}) = \underline{Box_{dts}(E)}$. Note that any dynamic expression can be decomposed into overlined or underlined static expressions or those without overlines and underlines, and the definition of dts-boxes is compositional.

Isomorphism is a coincidence of systems up to renaming of their components or states. Let $\simeq$ denote isomorphism between transition systems or DTMCs and reachability graphs. Due to the space restrictions, we omit the corresponding definitions as they resemble that of the isomorphism between transition systems. Note that the names of transitions of the dts-box corresponding to a static expression could be identified with the enumerated activities of the latter.

Theorem 4.1 *For any static expression E*

$$TS(\overline{E}) \simeq RG(Box_{dts}(\overline{E})).$$

Proof. As for the qualitative (functional) behaviour, we have the same isomorphism as in PBC.

The quantitative behaviour is the same by the following reasons. First, the activities of a static expression have probability parts coinciding with the probabilities of the transitions belonging to the corresponding plain dts-box. Second, in both semantics, conflicts are resolved via the same probability functions. ☐

Proposition 4.1 *For any static expression E*

$$DTMC(\overline{E}) \simeq DTMC(Box_{dts}(\overline{E})).$$

Proof. By Theorem 4.1 and definitions of underlying DTMC for dynamic expressions and LDTSPNs, since transition probabilities of the associated DTMCs are the sums of those belonging to transition systems or reachability graphs. ☐

Example 4.2 *Let E be from Example 3.1. In Figure 4 the marked dts-box $N = Box_{dts}(\overline{E})$, its reachability graph $RG(N)$ and the underlying DTMC $DTMC(N)$ are presented. It is easy to see that $TS(\overline{E})$ and $RG(N)$ are isomorphic, as well as $DTMC(\overline{E})$ and $DTMC(N)$.*

Consider the next example that demonstrates synchronization.

Example 4.3 *Let $E_1 = (\{a\}, \rho)$, $E_2 = (\{\hat{a}\}, \chi)$ and $E = (E_1 \| E_2)\ \mathsf{sy}\ a = ((\{a\}, \rho)\|(\{\hat{a}\}, \chi))\ \mathsf{sy}\ a$. In Figure 5 the transition system $TS(\overline{E})$ and the underlying DTMC $DTMC(\overline{E})$ are presented. In Figure 6 the marked dts-box $N = Box_{dts}(\overline{E})$, its reachability graph $RG(N)$ and the underlying DTMC $DTMC(N)$ are presented. It is easy to see that $TS(\overline{E})$ and $RG(N)$ are isomorphic, as well as $DTMC(\overline{E})$ and $DTMC(N)$.*

The probabilities $\mathcal{P}_{ij}$ $(1 \le i, j \le 4)$ are calculated as follows. Note that the symbol sy inscribes probability of the transition generated by synchronization, and the symbol $\|$ inscribes that of the transition corresponding

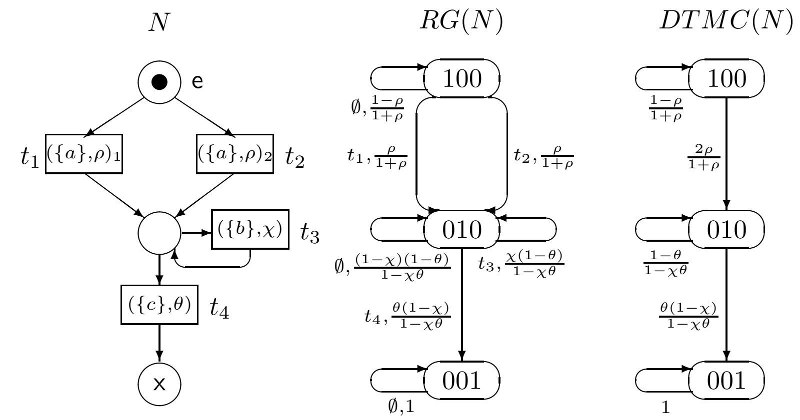

Figure 4: The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$, its reachability graph and the underlying DTMC

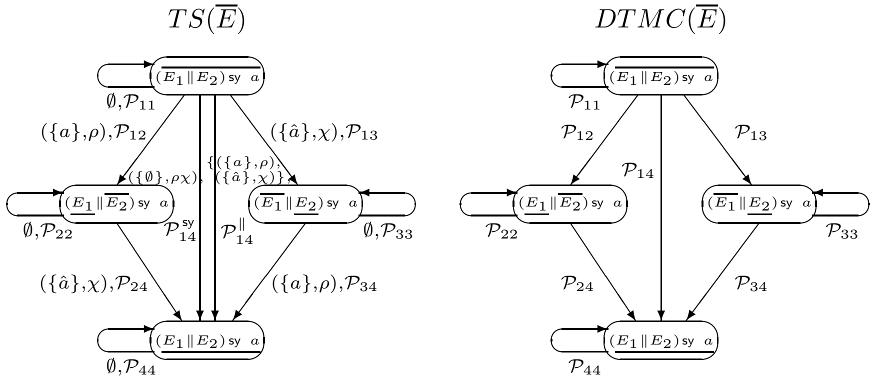

Figure 5: The transition system and the underlying DTMC of $\overline{E}$ for $E = ((\{a\},\rho)\|(\{\hat{a}\},\chi))$ sy a

to the concurrent execution of two activities. To avoid complex notation, we use the normalization factor $\mathcal{N} = \frac{1}{1-\rho^2\chi-\rho\chi^2+\rho^2\chi^2}$.

$\mathcal{P}_{11} = \mathcal{N}(1-\rho)(1-\chi)(1-\rho\chi)$	$\mathcal{P}_{12} = \mathcal{N}\rho(1-\chi)(1-\rho\chi)$	$\mathcal{P}_{13} = \mathcal{N}\chi(1-\rho)(1-\rho\chi)$
$\mathcal{P}_{14}^{\mathsf{sy}} = \mathcal{N}\rho\chi(1-\rho)(1-\chi)$	$\mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(1-\rho\chi)$	$\mathcal{P}_{22} = 1-\chi$
$\mathcal{P}_{24} = \chi$	$\mathcal{P}_{33} = 1-\rho$	$\mathcal{P}_{34} = \rho$
$\mathcal{P}_{44} = 1$	$\mathcal{P}_{14} = \mathcal{P}_{14}^{\mathsf{sy}} + \mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(2-\rho-\chi)$	

Consider the case $\rho = \chi = \frac{1}{2}$. *Then the transition probabilities will be the following:*

$$\mathcal{P}_{11} = \mathcal{P}_{12} = \mathcal{P}_{13} = \mathcal{P}_{14}^{\|} = \frac{3}{13},\ \mathcal{P}_{14}^{\mathsf{sy}} = \frac{1}{13},\ \mathcal{P}_{22} = \mathcal{P}_{24} = \mathcal{P}_{33} = \mathcal{P}_{34} = \frac{1}{2},\ \mathcal{P}_{44} = 1,\ \mathcal{P}_{14} = \frac{4}{13}.$$

5 Stochastic equivalences

In this section, we propose a number of stochastic equivalences of expressions. Semantic equivalence $=_{ts}$ is too strict in many cases, hence, we need weaker equivalence notions to compare behaviour of processes specified by algebraic formulas.

To identify processes with intuitively similar behavior, and to be able to apply standard constructions and techniques, we should abstract from infinite internal behaviour. Since *dtsPBC* is a stochastic extension of finite part of *PBC* with iteration, the only source of infinite silent behaviour are empty loops, i.e., the transitions which do not change states and have empty multiaction parts of their labels. During such the abstraction, we should collect the probabilities of the empty loops. Note that the resulting probabilities are those defined for

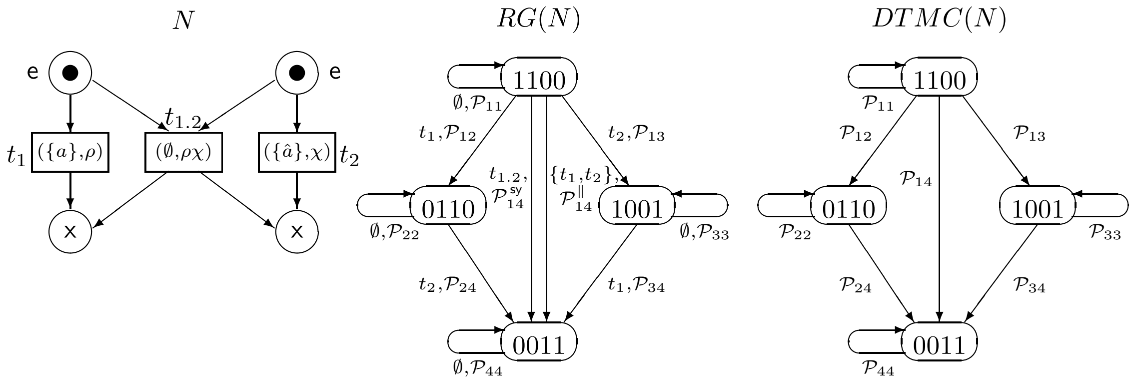

Figure 6: The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = ((\{a\},\rho)\|(\{\hat{a}\},\chi))$ sy a, its reachability graph and the underlying DTMC

infinite number of empty steps. In the following, we explain how to abstract from empty loops both in the algebraic setting of $dtsPBC$ and in the net one of LDTSPNs.

5.1 Empty loops in transition systems

Let G be a dynamic expression. Transition system $TS(G)$ can have loops going from a state to itself which are labeled by the empty set and have non-zero probability. Such the *empty loop* $s \stackrel{\emptyset}{\rightarrow}_{\mathcal{P}} s$ appears when no activities occur at a time step, and this happens with some positive probability. Obviously, in this case the current state remains unchanged.

Let G be a dynamic expression and $s \in DR(G)$.

The *probability to stay in* s *due to* k $(k \geq 1)$ *empty loops* is

$$(PT(\emptyset, s))^k.$$

The *probability to execute in* s *a non-empty multiset of activities* $\Gamma \in Exec(s) \setminus \{\emptyset\}$ *after possible empty loops* is

$$PT^*(\Gamma, s) = PT(\Gamma, s) \cdot \sum_{k=0}^{\infty} (PT(\emptyset, s))^k = \frac{PT(\Gamma, s)}{1 - PT(\emptyset, s)} = SJ(s) \cdot PT(\Gamma, s).$$

The value $k = 0$ in the summation above corresponds to the case when no empty loops occur.

Note that after abstraction from transition probabilities with empty multisets of activities, the remaining transition probabilities are normalized. In order to calculate transition probabilities $PT(\Gamma, s)$, we had to normalize $PF(\Gamma, s)$. Then, to obtain transition probabilities of non-empty steps $PT^*(\Gamma, s)$, we have to normalize $PT(\Gamma, s)$. Thus, we have two-stage normalization as a result.

Note that $PT^*(\Gamma, s) \leq 1$, hence, it is really a probability, since $PT(\emptyset, s) + PT(\Gamma, s) \leq PT(\emptyset, s) + \sum_{\Delta \in Exec(s) \setminus \{\emptyset\}} PT(\Delta, s) = \sum_{\Delta \in Exec(s)} PT(\Delta, s) = 1$. Moreover, $PT^*(\Gamma, s)$ defines a probability distribution, i.e., $\forall s \in DR(G)$ $\sum_{\Gamma \in Exec(s) \setminus \{\emptyset\}} PT^*(\Gamma, s) = 1$.

Definition 5.1 *The* (labeled probabilistic) transition system without empty loops $TS^*(G)$ *has the state space* $DR(G)$ *and the transitions* $s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$*, if* $s \stackrel{\Gamma}{\rightarrow} \tilde{s}$*,* $\Gamma \neq \emptyset$ *and* $\mathcal{P} = PT^*(\Gamma, s)$*.*

Note that $TS^*(G)$ describes the viewpoint of a person who observes steps only if they include non-empty multisets of activities.

We write $s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$ if $\exists \mathcal{P}$ $s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \twoheadrightarrow \tilde{s}$ if $\exists \Gamma$ $s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$. For one-element transition set $\Gamma = \{(\alpha, \rho)\}$ we write $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}$.

We decided to consider empty loops followed by a non-empty step only just for convenience. Alternatively, we could take a non-empty step succeeded by empty loops or a non-empty step preceded and succeeded by empty loops. In all these three cases our sequence begins or/and ends with loops which do not change states. At the same time, the overall probabilities of the evolutions can differ since empty loops have positive probabilities. To avoid inconsistency of definitions and too complex description, we consider sequences ending with a non-empty step. It resembles in some sense a construction of branching bisimulation [28].

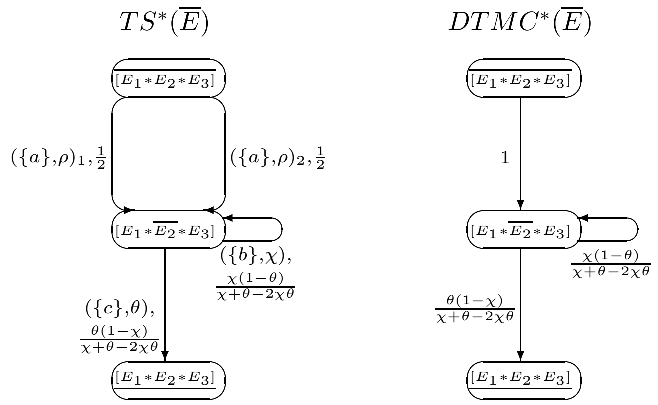

Figure 7: The transition system and the underlying DTMC without empty loops of $\overline{E}$ from Example 3.1

Transition systems without empty loops of static expressions can be defined as well. For $E \in \mathit{RegStatExpr}$ let $TS^*(E) = TS^*(\overline{E})$.

Definition 5.2 *Two dynamic expressions G and G' are* isomorphic with respect to transition systems without empty loops*, denoted by $G =_{ts*} G'$, if $TS^*(G) \simeq TS^*(G')$.*

Definition 5.3 *The* underlying DTMC without empty loops *$DTMC^*(G)$ has the state space $DR(G)$ and the transitions $s \twoheadrightarrow_{\mathcal{P}} \tilde{s}$, if $s \twoheadrightarrow \tilde{s}$, where $\mathcal{P} = PM^*(s,\tilde{s})$ and*

$$PM^*(s,\tilde{s}) = \sum_{\{\Gamma | s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}} PT^*(\Gamma, s).$$

Underlying DTMCs without empty loops of static expressions can be defined as well. For $E \in \mathit{RegStatExpr}$ let $DTMC^*(E) = DTMC^*(\overline{E})$.

When concurrency aspects are not relevant, interleaving behaviour is to be considered. Interleaving semantics abstracts from steps with more than one element. After such an abstracting, one has to normalize probabilities of the remaining one-element steps. We need to do it since the sum of outgoing probabilities should always be equal to one for each marking to form a probability distribution. For this, a special *interleaving transition relation* is proposed. Let G be a dynamic expression, $s, \tilde{s} \in DR(G)$ and $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}$. We write $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = PT_i^*((\alpha,\rho), s)$ and

$$PT_i^*((\alpha,\rho), s) = \frac{PT^*(\{(\alpha,\rho)\}, s)}{\sum_{\{(\beta,\chi)\} \in Exec(s)} PT^*(\{(\beta,\chi)\}, s)}.$$

Example 5.1 *Let E be from Example 3.1. In Figure 7 the transition system $TS^*(\overline{E})$ and the underlying DTMC $DTMC^*(\overline{E})$ without empty loops are presented.*

Let us demonstrate how the transition probabilities of non-empty steps are calculated. For instance, we have $PT(\emptyset, s_1) = \frac{1-\rho}{1+\rho}$ and $\frac{1}{1-PT(\emptyset,s_1)} = \frac{1+\rho}{2\rho}$. Hence, since $PT^*(\{(\{a\},\rho)_1\}, s_1) = \frac{\rho}{1+\rho}$, we have $PT^*(\{(\{a\},\rho)_1\}, s_1) = \frac{PT(\{(\{a\},\rho)_1\},s_1)}{1-PT(\emptyset,s_1)} = \frac{1+\rho}{2\rho} \cdot \frac{\rho}{1+\rho} = \frac{1}{2}$. According to the same pattern, we obtain $PT^*(\{(\{a\},\rho)_2\}, s_1) = \frac{1}{2}$. The other probabilities are calculated in a similar way.

5.2 Empty loops in reachability graphs

Let N be an LDTSPN. Reachability graph $RG(N)$ can have loops going from a marking to itself which are labeled by an emptyset and have non-zero probability. Such the *empty loop* $M \stackrel{\emptyset}{\rightarrow}_{\mathcal{P}} M$ appears when no transitions fire at a time step, and this happens with some positive probability. Obviously, in this case the current marking remains unchanged.

Let N be an LDTSPN and $M \in RS(N)$.

The *probability to stay in M due to k $(k \geq 1)$ empty loops* is

$$(PT(\emptyset, M))^k.$$

The *probability to execute in M a non-empty transition set $U \neq \emptyset$, ${}^\bullet U \subseteq M$ after possible empty loops* is

$$PT^*(U,M) = PT(U,M) \cdot \sum_{k=0}^{\infty} (PT(\emptyset,M))^k = \frac{PT(U,M)}{1 - PT(\emptyset,M)} = SJ(M) \cdot PT(U,M).$$

The value $k = 0$ in the summation above corresponds to the case when no empty loops occur.

Note that $PT^*(U,M) \leq 1$, hence, it is really a probability, since $PT(\emptyset,M) + PT(U,M) \leq PT(\emptyset,M) + \sum_{\{V|^\bullet V \subseteq M\}} PT(V,M) = 1$. Moreover, $PT^*(U,M)$ defines a probability distribution, i.e., $\forall M \in RS(N)$ $\sum_{U \in Ena(M)} PT^*(U,M) = 1$.

Definition 5.4 *The* reachability graph without empty loops $RG^*(N)$ *has the set of nodes $RS(N)$ and the arcs corresponding to the transitions $M \xrightarrow{U}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$, if $M \xrightarrow{U} \widetilde{M}$, $U \neq \emptyset$ and $\mathcal{P} = PT^*(U,M)$.*

Note that $RG^*(N)$ describes the viewpoint of a person who observes steps only if they include non-empty transition sets.

We write $M \xrightarrow{U}\!\!\!\!\rightarrow \widetilde{M}$ if $\exists \mathcal{P}$ $M \xrightarrow{U}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$ and $M \twoheadrightarrow \widetilde{M}$ if $\exists U$ $M \xrightarrow{U}\!\!\!\!\rightarrow \widetilde{M}$. For one-element transition set $U = \{t\}$ we write $M \xrightarrow{t}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{t}\!\!\!\!\rightarrow \widetilde{M}$.

Definition 5.5 *The* underlying DTMC without empty loops $DTMC^*(N)$ *has the state space $RS(N)$ and the transitions $M \twoheadrightarrow_{\mathcal{P}} \widetilde{M}$, if $M \twoheadrightarrow \widetilde{M}$, where $\mathcal{P} = PM^*(M,\widetilde{M})$ and*

$$PM^*(M,\widetilde{M}) = \sum_{\{U \in Ena(M) | M \xrightarrow{U}\!\!\!\!\rightarrow \widetilde{M}\}} PT^*(U,M).$$

The *interleaving transition relation* is proposed as follows. Let N be an LDTSPN, $M, \widetilde{M} \in RS(N)$ and $M \xrightarrow{t}\!\!\!\!\rightarrow \widetilde{M}$. We write $M \xrightarrow{t}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT_i^*(t,M)$ and

$$PT_i^*(t,M) = \frac{PT^*(\{t\},M)}{\sum_{u \in Ena(M)} PT^*(\{u\},M)}.$$

Theorem 5.1 *For any static expression E*

$$TS^*(\overline{E}) \simeq RG^*(Box_{dts}(\overline{E})).$$

Proof. As Theorem 4.1. □

Proposition 5.1 *For any static expression E*

$$DTMC^*(\overline{E}) \simeq DTMC^*(Box_{dts}(\overline{E})).$$

Proof. As Proposition 4.1. □

Note that Theorem 5.1 guarantees that the net versions of algebraic equivalences could be easily defined. For every equivalence on the empty loops free transition system of a dynamic expression, a similarly defined analogue exists on the empty loops free reachability graph of the corresponding dts-box.

Example 5.2 *Let E be from Example 3.1 and N be from Example 4.2. In Figure 8 the reachability graph $RG^*(N)$ and the underlying DTMC $DTMC^*(N)$ without from empty loops are presented. It is easy to see that $TS^*(\overline{E})$ and $RG^*(N)$ are isomorphic as well as $DTMC^*(\overline{E})$ and $DTMC^*(N)$.*

Consider the next example that demonstrates synchronization.

Example 5.3 *Let E and N be those from Example 4.3. In Figure 9 the transition system $TS^*(\overline{E})$ and the underlying DTMC $DTMC^*(\overline{E})$ without empty loops are presented. In Figure 10 the reachability graph $RG^*(N)$ and the underlying DTMC $DTMC^*(N)$ without from empty loops are presented. It is easy to see that $TS^*(\overline{E})$ and $RG^*(N)$ are isomorphic as well as $DTMC^*(\overline{E})$ and $DTMC^*(N)$.*

The probabilities $\mathcal{P}_{ij}^*$ $(1 \leq i,j \leq 4)$ are calculated as follows. Note that the symbol sy *inscribes probability of the transition generated by synchronization, and the symbol $\|$ inscribes that of the transition corresponding to the concurrent execution of two activities. To avoid complex notation, we use the normalization factor $\mathcal{N}^* = \frac{1}{\rho+\chi-2\rho^2\chi-2\rho\chi^2+2\rho^2\chi^2}$. The probabilities $\mathcal{P}_{ij}$ $(1 \leq i,j \leq 4)$ are taken from Example 4.3.*

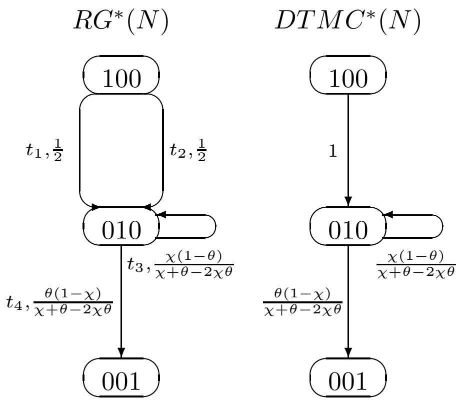

Figure 8: The reachability graph and the underlying DTMC without empty loops of N from Example 4.2

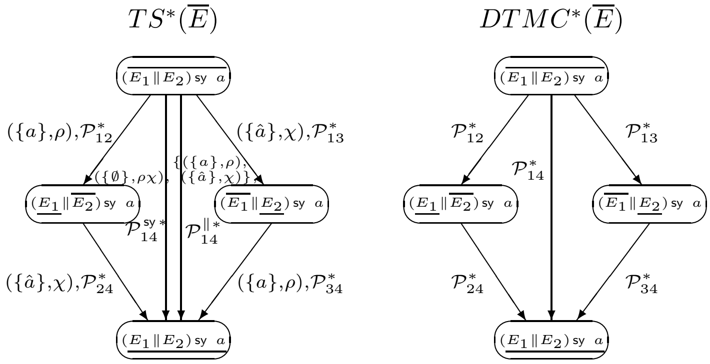

Figure 9: The transition system and the underlying DTMC without empty loops of $\overline{E}$ from Example 4.3

$$
\begin{array}{ll}
\mathcal{P}^*_{12} = \frac{\mathcal{P}_{12}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho(1-\chi)(1-\rho\chi) & \mathcal{P}^*_{13} = \frac{\mathcal{P}_{13}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\chi(1-\rho)(1-\rho\chi) \\
\mathcal{P}^{\mathsf{sy}\,*}_{14} = \frac{\mathcal{P}^{\mathsf{sy}}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho)(1-\chi) & \mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho\chi) \\
\mathcal{P}^*_{24} = \frac{\mathcal{P}_{24}}{1-\mathcal{P}_{22}} = 1 & \mathcal{P}^*_{34} = \frac{\mathcal{P}_{34}}{1-\mathcal{P}_{33}} = 1 \\
\mathcal{P}^*_{14} = \mathcal{P}^{\mathsf{sy}\,*}_{14} + \mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\mathsf{sy}}_{14}+\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(2-\rho-\chi) &
\end{array}
$$

Consider the case $\rho = \chi = \frac{1}{2}$. Then the transition probabilities will be the following:

$$\mathcal{P}^*_{12} = \mathcal{P}^*_{13} = \mathcal{P}^{\|*}_{14} = \frac{3}{10},\ \mathcal{P}^{\mathsf{sy}\,*}_{14} = \frac{1}{10},\ \mathcal{P}^*_{24} = \mathcal{P}^*_{34} = 1,\ \mathcal{P}^*_{14} = \frac{2}{5}.$$

5.3 Stochastic trace equivalences

Trace equivalences are the least discriminating ones. In the trace semantics, the behavior of a system is associated with the set of all possible sequences of activities, i.e., protocols of work or computations. Thus, the points of choice of an external observer between several extensions of a particular computation are not taken into account.

Formal definitions of stochastic trace relations resemble those of trace equivalences for standard Petri nets [66] or process algebras, but additionally we have to take into account the probabilities of sequences of (multisets of) multiactions. First, we have to multiply occurrence probabilities for all (multisets of) activities along every path starting from the initial state of the transition system corresponding to a dynamic expression. The product is the probability of the sequence of multiaction parts of the (multisets of) activities along the path. Second, we should calculate a sum of probabilities for all paths corresponding to the same sequence of multiaction parts.

For $\Gamma \in I\!N_f^{\mathcal{SL}}$, we define its *multiaction part* by $\mathcal{L}(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma} \alpha$. Note that $\mathcal{L}(\Gamma) \in I\!N_f^{\mathcal{L}}$, i.e, $\mathcal{L}(\Gamma)$ is a multiset of multiactions.

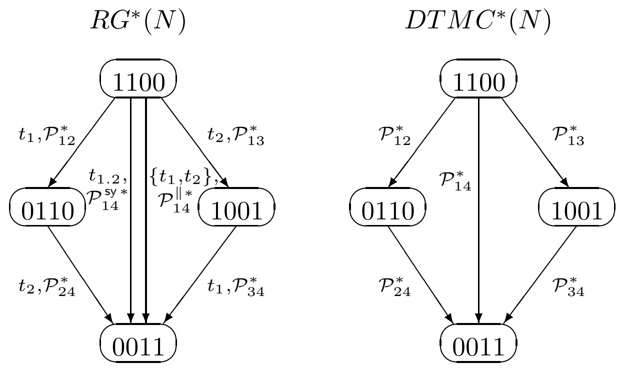

Figure 10: The reachability graph and the underlying DTMC without empty loops of N from Example 4.3

Definition 5.6 *An* interleaving stochastic trace *of a dynamic expression G is a pair $(\sigma, PT^*(\sigma))$, where $\sigma = \alpha_1 \cdots \alpha_n \in \mathcal{L}^*$ and*

$$PT^*(\sigma) = \sum_{\{(\alpha_1,\rho_1),\ldots,(\alpha_n,\rho_n) \mid [G]_\simeq = s_0 \stackrel{(\alpha_1,\rho_1)}{\to} s_1 \stackrel{(\alpha_2,\rho_2)}{\to} \cdots \stackrel{(\alpha_n,\rho_n)}{\to} s_n\}} \prod_{i=1}^{n} PT_i^*((\alpha_i,\rho_i), s_{i-1}).$$

We denote a set of all interleaving stochastic traces *of a dynamic expression G by $IntStochTraces(G)$. Two dynamic expressions G and G' are* interleaving stochastic trace equivalent*, denoted by $G \equiv_{is} G'$, if*

$$IntStochTraces(G) = IntStochTraces(G').$$

Definition 5.7 *A* step stochastic trace *of a dynamic expression G is a pair $(\Sigma, PT^*(\Sigma))$, where $\Sigma = A_1 \cdots A_n \in (I\!N_f^{\mathcal{L}})^*$ and*

$$PT^*(\Sigma) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq = s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \le i \le n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}).$$

We denote a set of all step stochastic traces *of a dynamic expression G by $StepStochTraces(G)$. Two dynamic expressions G and G' are* step stochastic trace equivalent*, denoted by $G \equiv_{ss} G'$, if*

$$StepStochTraces(G) = StepStochTraces(G').$$

5.4 Stochastic bisimulation equivalences

Bisimulation equivalences respect completely the particular points of choice in the behavior of a modeled system. We intend to present a parameterized definition of stochastic bisimulation equivalences.

To define stochastic bisimulation equivalences, we have to consider a bisimulation as an *equivalence* relation which partitions the states of the *union* of the transition systems $TS^*(G)$ and $TS^*(G')$ of two dynamic expressions G and G' to be compared. For G and G' to be bisimulation equivalent, the initial states of their transition systems, $[G]_\simeq$ and $[G']_\simeq$, are to be related by a bisimulation having the following transfer property: two states are related if in each of them the same (multisets of) multiactions can occur, and the resulting states *belong to the same equivalence class*. In addition, sums of probabilities for all such occurrences should be the same for both states. Thus, in our definitions, we follow the approach of [40, 41]. The difference between bisimulation and trace equivalences is that we do not consider *all possible* occurrences of (multisets of) multiactions from the initial states, but only such that lead (stepwise) to the states *belonging to the same equivalence class*.

First, we introduce several helpful notations. Let G be a dynamic expression and $\mathcal{H} \subseteq DR(G)$. Then for some $s \in DR(G)$ and $A \in I\!N_f^{\mathcal{L}}$ we write $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM_A^*(s, \mathcal{H})$ and

$$PM_A^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s},\ \mathcal{L}(\Gamma) = A\}} PT^*(\Gamma, s).$$

Thus, $PM_A^*(s, \mathcal{H})$ is the overall probability to come into the set of states $\mathcal{H}$ starting in s via steps with multiaction part A. The summation above reflects the probability of the event union.

We write $s \xrightarrow{A}\!\!\!\!\rightarrow \mathcal{H}$ if $\exists \mathcal{P}\ s \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \mathcal{H}$.
We write $s \twoheadrightarrow_{\mathcal{P}} \mathcal{H}$ if $\exists A\ s \xrightarrow{A}\!\!\!\!\rightarrow \mathcal{H}$, where $\mathcal{P} = PM^*(s, \mathcal{H})$ and

$$PM^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{\Gamma}\!\!\!\!\rightarrow \tilde{s}\}} PT^*(\Gamma, s).$$

We propose the *interleaving transition relation* $s \overset{\alpha}{\rightharpoonup}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM^*_{i\alpha}(s, \mathcal{H})$ and

$$PM^*_{i\alpha}(s, \mathcal{H}) = \sum_{\{(\alpha,\rho) \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{(\alpha,\rho)}\!\!\!\!\rightarrow \tilde{s}\}} PT^*_i((\alpha, \rho), s).$$

Definition 5.8 *Let G be a dynamic expression. An* equivalence *relation $\mathcal{R} \subseteq DR(G)^2$ is a* $\star$-stochastic bisimulation *between states s_1 and s_2 from $DR(G)$, $\star \in$*{interleaving, step}*, denoted by $\mathcal{R} : s_1 \underline{\leftrightarrow}_{\star s} s_2,\ \star \in \{i, s\}$, if $\forall \mathcal{H} \in DR(G)/_{\mathcal{R}}$*

- *$\forall x \in \mathcal{L}$ and $\hookrightarrow = \rightharpoonup$, if $\star = i$;*
- *$\forall x \in I\!N_f^{\mathcal{L}}$ and $\hookrightarrow = \twoheadrightarrow$, if $\star = s$;*

$$s_1 \overset{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{H}\ \Leftrightarrow\ s_2 \overset{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{H}.$$

Two states s_1 and s_2 are $\star$-stochastic bisimulation equivalent*, $\star \in$*{interleaving, step}*, denoted by $s_1 \underline{\leftrightarrow}_{\star s} s_2$, if $\exists \mathcal{R} : s_1 \underline{\leftrightarrow}_{\star s} s_2,\ \star \in \{i, s\}$.*

To introduce bisimulation between dynamic expressions G and G', we should consider a "composite" set of states $DR(G) \cup DR(G')$.

Definition 5.9 *Let G, G' be dynamic expressions. A relation $\mathcal{R} \subseteq (DR(G) \cup DR(G'))^2$ is a* $\star$-stochastic bisimulation *between G and G', $\star \in$*{interleaving, step}*, denoted by $\mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'$, if $\mathcal{R} : [G]_\simeq \underline{\leftrightarrow}_{\star s} [G']_\simeq,\ \star \in \{i, s\}$.*

Two dynamic expressions G and G' are $\star$-stochastic bisimulation equivalent*, $\star \in$*{interleaving, step}*, denoted by $G \underline{\leftrightarrow}_{\star s} G'$, if $\exists \mathcal{R} : G \underline{\leftrightarrow}_{\star s} G',\ \star \in \{i, s\}$.*

5.5 Stochastic isomorphism

Stochastic isomorphism is a relation that is weaker than the equivalence with respect to the isomorphism of the associated transition systems without empty loops. The main idea of the following definition is to collect probabilities of all transitions between the same pair of states such that the transition labels have the same multiaction parts. We use summation, since it is the probability of event union.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$ such that $s \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \{\tilde{s}\}$. In this case, we write $s \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \tilde{s}$.

Definition 5.10 *Let G, G' be dynamic expressions. A mapping $\beta : DR(G) \rightarrow DR(G')$ is a* stochastic isomorphism *between G and G', denoted by $\beta : G =_{sto} G'$, if*

1. *β is a bijection such that $\beta([G]_\simeq) = [G']_\simeq$;*
2. *$\forall s, \tilde{s} \in DR(G)\ \forall A \in I\!N_f^{\mathcal{L}}\ s \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \tilde{s}\ \Leftrightarrow\ \beta(s) \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \beta(\tilde{s})$.*

Two dynamic expressions G and G' are stochastically isomorphic*, denoted by $G =_{sto} G'$, if $\exists \beta : G =_{sto} G'$.*

5.6 Interrelations of the stochastic equivalences

Now we intend to compare the introduced stochastic equivalences and obtain the lattice of their interrelations.

Proposition 5.2 *Let $\star \in \{i, s\}$. For dynamic expressions G and G' the following holds:*

$$G \underline{\leftrightarrow}_{\star s} G'\ \Rightarrow\ G \equiv_{\star s} G'.$$

Proof. See Appendix A. □

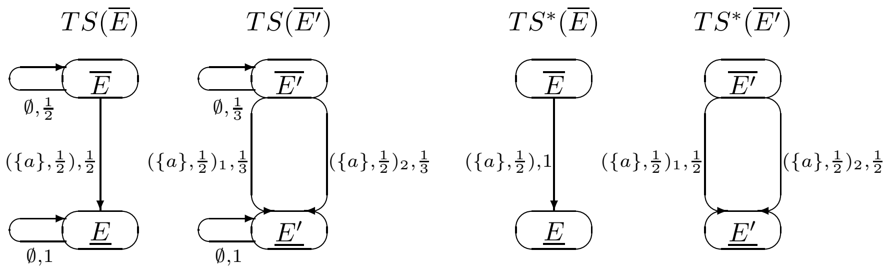

Figure 11: A problem with stochastic isomorphism based on transition systems with empty loops

Proposition 5.3 *For dynamic expressions G and G' the following holds:*

$$G =_{ts*} G' \;\Leftrightarrow\; G =_{ts} G'.$$

Proof. ($\Leftarrow$) It is enough to note that the abstraction from empty loops is based on transition probabilities which are the same for isomorphic transition systems.

($\Rightarrow$) Note that $TS(G)$ and $TS^*(G)$ (as well as $TS(G')$ and $TS^*(G')$) differ by presence of empty loops and by values of transition probabilities only. The sets of states, the labeling area, the non-empty multisets of activities which label the transitions and the initial states coincide. We have isomorphism of $TS^*(G)$ and $TS^*(G')$. For a state s of $TS^*(G)$, let s' be the state of $TS^*(G')$ such that these two states are related by the isomorphism of $TS^*(G)$ and $TS^*(G')$. Then $Exec(s) = \{\Gamma \mid \exists \tilde{s}\ s \xrightarrow{\Gamma}\!\!\!\!\rightarrow \tilde{s}\} \cup \{\emptyset\} = \{\Gamma \mid \exists \tilde{s}'\ s' \xrightarrow{\Gamma}\!\!\!\!\rightarrow \tilde{s}'\} \cup \{\emptyset\} = Exec(s')$. Note that in the previous equality we can always find the pairs of states s and s' related by the isomorphism of $TS^*(G)$ and $TS^*(G')$. Further, the definition of $PT(\Gamma, s)$ depends on $Exec(s)$ only rather than on concrete s. Thus, for each state s of $TS(G)$ the probabilities of outgoing transitions will be the same as for the corresponding state s' of $TS(G')$. Hence, $TS(G)$ and $TS(G')$ are isomorphic. □

Note that though isomorphism of transition systems with and without empty loops appears to be the same relation, the equivalences defined on these two types of transition systems could be different. This is the case when the relations abstract from concrete activities which can happen (more exactly, from their probability parts) and take into account the overall probabilities to execute multiactions only. It is clear that the equivalences defined through transition systems with empty loops imply the relations based on those without empty loops, but the reverse implication is not valid.

For instance, we have defined stochastic isomorphism with the use of transition systems without empty loops. We can define the corresponding relation based on transition systems with empty loops as well. Then the latter equivalence will be strictly stronger than the former. As mentioned above, we decided to abstract from empty loops because of the difficulties with infinite internal behavior. Now we can explain another reason for this decision: the equivalences based on transition systems with empty loops are rather cumbersome. The following example shows why.

Example 5.4 *Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2})_1[](\{a\}, \frac{1}{2})_2$. Then $\overline{E} =_{sto} \overline{E'}$, but $\overline{E}$ is not equivalent to $\overline{E'}$ according to the stronger version of stochastic isomorphism, since the probability of the only non-empty transition in $TS(\overline{E})$ is $\frac{1}{2}$ whereas the probability of both non-empty transitions in $TS(\overline{E'})$ is $\frac{1}{3}$, and $\frac{1}{2} \neq \frac{1}{3} + \frac{1}{3}$. On the other hand, the probability of the only non-empty transition in $TS^*(\overline{E})$ is 1, the probability of both non-empty transitions in $TS^*(\overline{E'})$ is $\frac{1}{2}$, and $1 = \frac{1}{2} + \frac{1}{2}$. The transition systems with and without empty loops of $\overline{E}$ and $\overline{E}'$ are presented in Figure 11.*

In the continuous time setting of $sPBC$ there are no problems with equivalences like in the example above, but only interleaving relations can be introduced. On the other hand, the concurrency information from expressions has to be preserved in their transition systems to define correctly the congruence relation [49, 50, 53].

In the following, the symbol '_' will denote "nothing", and the equivalences subscribed by it are considered as those without any subscription.

Theorem 5.2 *Let $\leftrightarrow, \leftrightarrow\!\!\!\!\rightarrow \in \{\equiv, \underline{\leftrightarrow}, =, \simeq\}$ and $\star, \star\star \in \{\_, is, ss, sto, ts\}$. For dynamic expressions G and G'*

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \leftrightarrow\!\!\!\!\rightarrow_{\star\star} G'$$

iff in the graph in Figure 12 there exists a directed path from $\leftrightarrow_{\star}$ to $\leftrightarrow\!\!\!\!\rightarrow_{\star\star}$.

Proof. ($\Leftarrow$) Let us check the validity of implications in the graph in Figure 12.

$$
\begin{tikzcd}
 & \simeq \arrow[d] \\
 & =_{ts} \arrow[d] \\
 & =_{sto} \arrow[d] \\
\underline{\leftrightarrow}_{is} \arrow[d] & \underline{\leftrightarrow}_{ss} \arrow[l] \arrow[d] \\
\equiv_{is} & \equiv_{ss} \arrow[l]
\end{tikzcd}
$$

Figure 12: Interrelations of the stochastic equivalences

- The implications $\leftrightarrow_{ss} \to \leftrightarrow_{is}$, $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$ are valid, since single activities are one-element multisets.
- The implications $\underline{\leftrightarrow}_{\star s} \to \equiv_{\star s}$, $\star \in \{i, s\}$, are valid by Proposition 5.2.
- The implication $=_{sto} \to \underline{\leftrightarrow}_{ss}$ is proved as follows. Let $\beta : G =_{sto} G'$. Then it is easy to see that $\mathcal{S} : G \underline{\leftrightarrow}_{ss} G'$, where $\mathcal{S} = \{(s, \beta(s)) \mid s \in DR(G)\}$.
- The implication $=_{ts} \to =_{sto}$ is valid, since stochastic isomorphism is that of empty loops free transition systems up to merging of transitions with labels having identical multiaction parts.
- The implication $\simeq \to =_{ts}$ is valid, since the transition system of a dynamic formula is defined based on its isomorphism class.

($\Rightarrow$) An absence of additional nontrivial arrows (not resulting from the combination of the existing ones) in the graph in Figure 12 is proved by the following examples.

- Let $E = (\{a\}, \frac{1}{2}) \| (\{b\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{b\}, \frac{1}{2}); (\{a\}, \frac{1}{2}))$. Then $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$, but $\overline{E} \not\equiv_{ss} \overline{E'}$, since only in $TS^*(\overline{E'})$ multiactions $\{a\}$ and $\{b\}$ cannot be executed concurrently.
- Let $E = (\{a\}, \frac{1}{2}); ((\{b\}, \frac{1}{2})[](\{c\}, \frac{1}{2}))$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{a\}, \frac{1}{2}); (\{c\}, \frac{1}{2}))$. Then $\overline{E} \equiv_{ss} \overline{E'}$, but $\overline{E} \underline{\not\leftrightarrow}_{is} \overline{E'}$, since only in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.
- Let $E = (\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})[](\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})$. Then $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$, but $\overline{E} \neq_{sto} \overline{E'}$, since only in $TS^*(\overline{E})$ there is a transition with multiaction part of label $\{a\}$ and probability 1 that is single one between its start and final states such that the transition has no corresponding transition set in $TS^*(\overline{E'})$. Note that in $TS^*(\overline{E'})$, the only transition with the same multiaction part of label has probability $\frac{1}{2}$.
- Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2})_1[](\{a\}, \frac{1}{2})_2$. Then $\overline{E} =_{sto} \overline{E'}$, but $\overline{E} \neq_{ts} \overline{E'}$, since only $TS(\overline{E'})$ has two transitions.
- Let $E = (\{a\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E} =_{ts} \overline{E'}$, but $\overline{E} \not\simeq \overline{E'}$, since $\overline{E}$ and $\overline{E'}$ cannot be reached from each other by applying inaction rules. $\square$

Example 5.5 *In Figure 13 the marked dts-boxes corresponding to the dynamic expressions from equivalence examples of Theorem 5.2 are presented, i.e.,* $N = Box_{dts}(\overline{E})$ *and* $N' = Box_{dts}(\overline{E'})$ *for each picture (a)–(e). Since all the equivalences of dynamic expressions can be transferred to the corresponding marked dts-boxes, we depict also the net analogues (denoted by the same symbols) of the algebraic equivalences which relate the nets.*

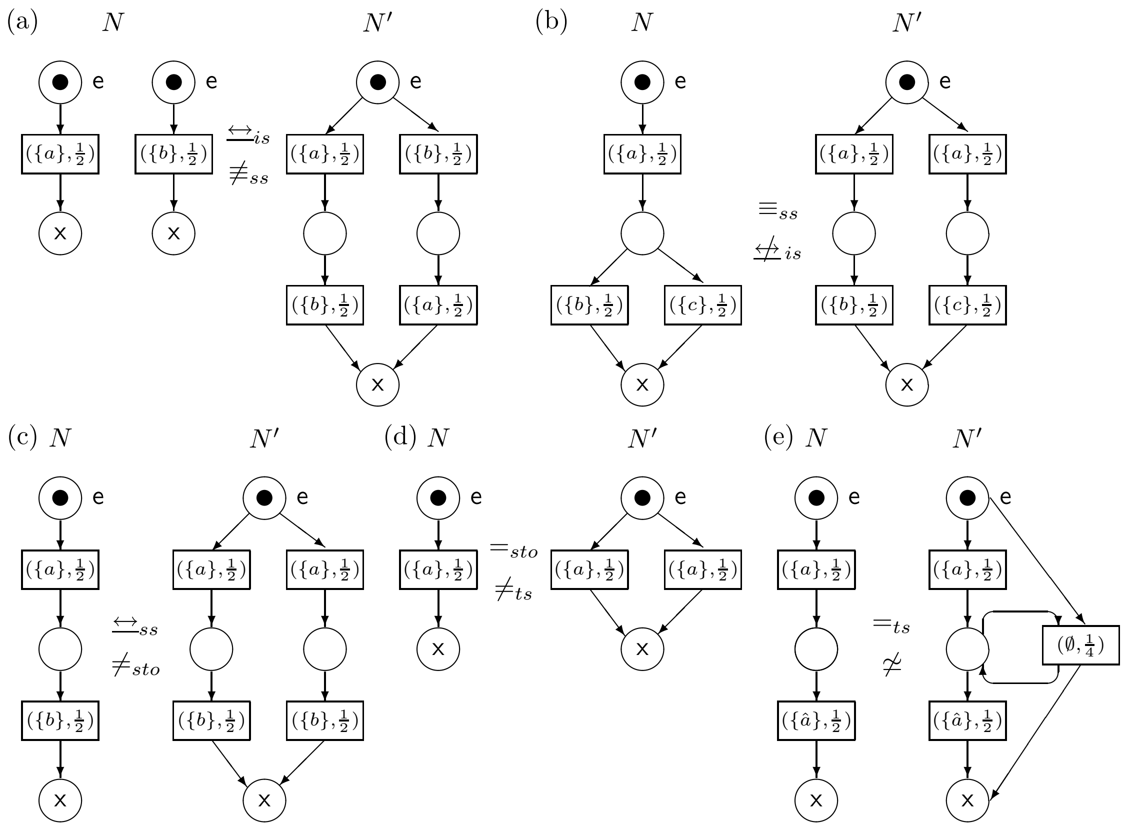

Figure 13: Dts-boxes of the dynamic expressions from equivalence examples of Theorem 5.2

The following example shows how the stochastic equivalences can be used to simplify process specifications. Accordingly, the net analogues of the relations can be used for net reduction.

Example 5.6 *Let* $E = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))\|((\{c\}, \frac{1}{2}); (\{d\}, \frac{1}{2}))$ *and* $E' = (((\{a, x\}, \frac{1}{2}); ((\{b, y_1\}, \frac{1}{2})[](\{b, y_2\}, \frac{1}{2})))$ $\|((\{a, \hat{x}\}, \frac{1}{2}); ((\{b, \widehat{y_2}, y_2'\}, \frac{1}{2})[](\{d, v_1\}, \frac{1}{2})))\|((\{c, z\}, \frac{1}{2}); ((\{b, \widehat{y_2'}\}, \frac{1}{2})[](\{d, \widehat{v_1}, v_1'\}, \frac{1}{2})))\|((\{c, z\}, \frac{1}{2}); ((\{d, v_1\}, \frac{1}{2})[]$ $(\{d, \widehat{v_1'}\}, \frac{1}{2})))\|((\{b, \widehat{y_1}\}, \frac{1}{2})[](\{d, \widehat{v_2}\}, \frac{1}{2})))$ sy x sy y_1 sy y_2 sy y_2' sy z sy v_1 sy v_1' sy v_2 rs x rs y_1 rs y_2 rs y_2' rs z rs v_1 rs v_1' rs v_2. *Then* $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$, *but* $\overline{E} \neq_{sto} \overline{E'}$, *since* $TS^*(\overline{E'})$ *has more states than* $TS^*(\overline{E})$. *It is clear that the syntax of* E *is is much simpler than that of* E', *but both static expressions have the same semantics induced by* $\underline{\leftrightarrow}_{ss}$. *Hence,* E *is a simplification of* E' *with respect to* $\underline{\leftrightarrow}_{ss}$.

In Figure 14 $N = Box_{dts}(\overline{E})$ *and* $N' = Box_{dts}(\overline{E'})$. *In addition, we depict the net analogues of the algebraic equivalences. Thus,* N *is a reduction of* N' *up to the net version of* $\underline{\leftrightarrow}_{ss}$.

6 Logical characterization

In this section, a logical characterization of stochastic bisimulation equivalences is accomplished via formulas of probabilistic modal logics. The results obtained could be interpreted as an operational characterization of the corresponding logical equivalences. Dynamic expressions are considered as logically equivalent if they satisfy the same formulas.

6.1 Logic $iPML$

The probabilistic modal logic PML has been introduced in [40] on probabilistic transition systems without invisible actions for logical interpretation of the interleaving probabilistic bisimulation equivalence. On the basis of PML, we propose a new interleaving modal logic $iPML$ used for characterization of the interleaving stochastic bisimulation equivalence.

Definition 6.1 *Let* $\top$ *denote the truth and* $\alpha \in \mathcal{L}$, $\mathcal{P} \in (0; 1]$. *A* formula *of* $iPML$ *is defined as follows:*

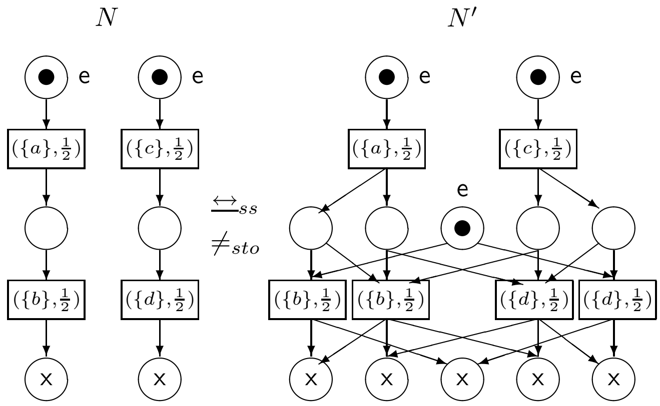

Figure 14: Reduction of a dts-box up to $\underline{\leftrightarrow}_{ss}$

$$\Phi ::= \ \top \mid \neg\Phi \mid \Phi \wedge \Phi \mid \langle\alpha\rangle_{\mathcal{P}}\Phi.$$

We define $\langle\alpha\rangle\Phi = \exists\mathcal{P}\ \langle\alpha\rangle_{\mathcal{P}}\Phi$.

iPML denotes the set of *all formulas of the logic iPML.*

Definition 6.2 *Let G be a dynamic expression and $s \in DR(G)$. The* satisfaction relation $\models_G \subseteq DR(G) \times$ **iPML** *is defined as follows:*

1. $s \models_G \top$ *— always;*
2. $s \models_G \neg\Phi$, *if* $s \not\models_G \Phi$;
3. $s \models_G \Phi \wedge \Psi$, *if* $s \models_G \Phi$ *and* $s \models_G \Psi$;
4. $s \models_G \langle\alpha\rangle_{\mathcal{P}}\Phi$, *if* $\exists\mathcal{H} \subseteq DR(G)\ s \xrightarrow{\alpha}_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\tilde{s} \in \mathcal{H}\ \tilde{s} \models_G \Phi$.

Note that $\langle\alpha\rangle_{\mathcal{Q}}\Phi$ implies $\langle\alpha\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

Definition 6.3 *We write $G \models_G \Phi$, if $[G]_{\simeq} \models_G \Phi$. Two dynamic expressions G and G' are* logically equivalent *in iPML, denoted by $G =_{iPML} G'$, if $\forall\Phi \in$* **iPML** *$G \models_G \Phi \ \Leftrightarrow\ G' \models_{G'} \Phi$.*

Let G be a dynamic expression and $s \in DR(G)$, $\alpha \in \mathcal{L}$. The set of states reached from s by execution of multiaction α, the *image set*, is defined as $Image(s, \alpha) = \{\tilde{s} \mid \exists\{(\alpha, \rho)\} \in Exec(s)\ s \overset{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}\}$. A dynamic expression G is an *image-finite* one, if $\forall s \in DR(G)\ \forall\alpha \in \mathcal{L}\ |Image(s, \alpha)| < \infty$.

Theorem 6.1 *For image-finite dynamic expressions G and G'*

$$G \underline{\leftrightarrow}_{is} G' \ \Leftrightarrow\ G =_{iPML} G'.$$

Proof. As the subsequent Theorem 6.2, but with state changes due to execution of single multiactions and the interleaving transition relation. □

Hence, in the interleaving semantics, we obtained a logical characterization of the stochastic bisimulation relation or, symmetrically, an operational characterization of the probabilistic modal logic equivalence.

Example 6.1 *Let $E = (\{a\}, \frac{1}{2}); ((\{b\}, \frac{1}{2})[](\{c\}, \frac{1}{2}))$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{a\}, \frac{1}{2}); (\{c\}, \frac{1}{2}))$. Then $\overline{E} \neq_{iPML} \overline{E'}$, because for $\Phi = \langle\{a\}\rangle_1\langle\{b\}\rangle_{\frac{1}{2}}\top$ we have $\overline{E} \models_{\overline{E}} \Phi$, but $\overline{E'} \not\models_{\overline{E'}} \Phi$, since only in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.*

6.2 Logic $sPML$

On the basis of PML, we propose a new step modal logic $sPML$ used for characterization of the step stochastic bisimulation equivalence.

Definition 6.4 *Let $\top$ denote the truth and $A \in \mathbb{N}_f^{\mathcal{L}}$, $\mathcal{P} \in (0;1]$. A* formula *of sPML is defined as follows:*

$$\Phi ::= \ \top \mid \neg\Phi \mid \Phi \wedge \Phi \mid \langle A\rangle_{\mathcal{P}}\Phi.$$

We define $\langle A\rangle\Phi = \exists\mathcal{P}\ \langle A\rangle_{\mathcal{P}}\Phi$.
sPML denotes the set of *all formulas of the logic sPML.*

Definition 6.5 *Let G be a dynamic expression and $s \in DR(G)$. The* satisfaction relation $\models_G \subseteq DR(G)\times\mathbf{sPML}$ *is defined as follows:*

1. $s \models_G \top$ *— always;*
2. $s \models_G \neg\Phi$, *if* $s \not\models_G \Phi$;
3. $s \models_G \Phi \wedge \Psi$, *if* $s \models_G \Phi$ *and* $s \models_G \Psi$;
4. $s \models_G \langle A\rangle_{\mathcal{P}}\Phi$, *if* $\exists\mathcal{H} \subseteq DR(G)\ s \xrightarrow{A}_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\tilde{s} \in \mathcal{H}\ \tilde{s} \models_G \Phi$.

Note that $\langle A\rangle_{\mathcal{Q}}\Phi$ implies $\langle A\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

Definition 6.6 *We write $G \models_G \Phi$, if $[G]_\simeq \models_G \Phi$. Two dynamic expressions G and G' are* logically equivalent *in sPML, denoted by $G =_{sPML} G'$, if $\forall\Phi \in \mathbf{sPML}\ G \models_G \Phi\ \Leftrightarrow\ G' \models_{G'} \Phi$.*

Let G be a dynamic expression and $s \in DR(G),\ A \in \mathbb{N}_f^{\mathcal{L}}$. The set of states reached from s by execution of a multiset of multiactions A, the *image set*, is defined as $Image(s,A) = \{\tilde{s} \mid \exists\Gamma \in Exec(s)\ \mathcal{L}(\Gamma) = A,\ s \xrightarrow{\Gamma} \tilde{s}\}$. A dynamic expression G is an *image-finite* one, if $\forall s \in DR(G)\ \forall A \in \mathbb{N}_f^{Act}\ |Image(s,A)| < \infty$.

Theorem 6.2 *For image-finite dynamic expressions G and G'*

$$G\underline{\leftrightarrow}_{ss}G'\ \Leftrightarrow\ G =_{sPML} G'.$$

Proof. ($\Leftarrow$) To simplify the presentation, we propose the *indicator function* Ξ that recovers a dynamic expression by a state belonging to its derivation set. For a dynamic expression G and $s \in DR(G)$ we define $\Xi(s) = G$.

Let us define the equivalence relation $\mathcal{R} = \{(s_1,s_2) \in (DR(G)\cup DR(G'))^2 \mid \forall\Phi \in \mathbf{sPML}\ s_1 \models_{\Xi(s_1)} \Phi\ \Leftrightarrow\ s_2 \models_{\Xi(s_2)} \Phi\}$. We have $([G]_\simeq,[G']_\simeq) \in \mathcal{R}$. Let us prove that $\mathcal{R}$ is a step stochastic bisimulation.

Assume that $[G]_\simeq \xrightarrow{A}_{\mathcal{P}} \mathcal{H} \in (DR(G)\cup DR(G'))/_{\mathcal{R}}$. Let $[G']_\simeq \xrightarrow{A}_{\mathcal{P}'_1} s'_1,\ldots,[G']_\simeq \xrightarrow{A}_{\mathcal{P}'_i} s'_i,[G']_\simeq \xrightarrow{A}_{\mathcal{P}'_{i+1}} s'_{i+1},\ldots,[G']_\simeq \xrightarrow{A}_{\mathcal{P}'_n} s'_n$ be changes of the state $[G']_\simeq$ as a result of execution of the multiset of multiactions A. Since dynamic expression G' is an image-finite one, the number of such state changes is finite. The state changes are ordered so that $s'_1,\ldots s'_i \in \mathcal{H}$ and $s'_{i+1},\ldots s'_n \notin \mathcal{H}$.

Then $\exists\Phi_{i+1},\ldots,\Phi_n \in \mathbf{sPML}$ such that $\forall j\ (i+1 \leq j \leq n)\ \forall s \in \mathcal{H}\ s \models_{\Xi(s)} \Phi_j$, but $s'_j \not\models_{G'} \Phi_j$. We have $[G]_\simeq \models_G \langle A\rangle_{\mathcal{P}}(\wedge_{j=i+1}^n\Phi_j)$ and $[G']_\simeq \models_{G'} \langle A\rangle_{(1-\sum_{j=1}^i\mathcal{P}'_j)}\neg(\wedge_{j=i+1}^n\Phi_j)$.

Assume that $\mathcal{P} > \sum_{j=1}^i\mathcal{P}'_j$. Then $[G']_\simeq \models_{G'} \langle A\rangle_{(1-\mathcal{P})}\neg(\wedge_{j=i+1}^n\Phi_j)$ and $[G']_\simeq \not\models_{G'} \langle A\rangle_{\mathcal{P}}(\wedge_{j=i+1}^n\Phi_j)$ what contradicts to $([G]_\simeq,[G']_\simeq) \in \mathcal{R}$. Hence, $\mathcal{P} \leq \sum_{j=1}^i\mathcal{P}'_j$. Consequently, $[G']_\simeq \xrightarrow{A}_{\mathcal{P}'} \mathcal{H},\ \mathcal{P} \leq \sum_{j=1}^i\mathcal{P}'_j \leq \mathcal{P}'$. By symmetry of $\underline{\leftrightarrow}_{ss}$, we have $\mathcal{P} \geq \mathcal{P}'$. Thus, $\mathcal{P} = \mathcal{P}'$, and $\mathcal{R}$ is a step stochastic bisimulation.

($\Rightarrow$) It is sufficient to consider only the case $\langle A\rangle_{\mathcal{P}}\Phi$, since all other cases are trivial. Let for dynamic expressions G and G' $G\underline{\leftrightarrow}_{ss}G'$. Then $[G]_\simeq\underline{\leftrightarrow}_{ss}[G']_\simeq$. Assume that $[G]_\simeq \models_G \langle A\rangle_{\mathcal{P}}\Phi$. Then $\exists\mathcal{H} \subseteq DR(G)\cup DR(G')$ such that $[G]_\simeq \xrightarrow{A}_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ and $\forall s \in \mathcal{H}\ s \models_{\Xi(s)} \Phi$.

Let us define $\widetilde{\mathcal{H}} = \bigcup\{\overline{\mathcal{H}} \in (DR(G)\cup DR(G'))/_{\underline{\leftrightarrow}_{ss}} \mid \overline{\mathcal{H}}\cap\mathcal{H} \neq \emptyset\}$. Then $\forall\tilde{s} \in \widetilde{\mathcal{H}}\ \exists s \in \mathcal{H}\ s\underline{\leftrightarrow}_{ss}\tilde{s}$. Since $\forall s \in \mathcal{H}\ s \models_{\Xi(s)} \Phi$, we have $\forall\tilde{s} \in \widetilde{\mathcal{H}}\ \tilde{s} \models_{\Xi(\tilde{s})} \Phi$ by the induction hypothesis.

Since $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$, we have $[G]_\simeq \xrightarrow{A}_{\widetilde{\mathcal{Q}}} \widetilde{\mathcal{H}},\ \widetilde{\mathcal{Q}} \geq \mathcal{Q}$. Since $\widetilde{\mathcal{H}}$ is the union of the equivalence classes with respect to $\underline{\leftrightarrow}_{ss}$, we have $[G]_\simeq\underline{\leftrightarrow}_{ss}[G']_\simeq$ implies $[G']_\simeq \xrightarrow{A}_{\widetilde{\mathcal{Q}}} \widetilde{\mathcal{H}}$. Since $\widetilde{\mathcal{Q}} \geq \mathcal{Q} \geq \mathcal{P}$, we have $[G']_\simeq \models_{G'} \langle A\rangle_{\mathcal{P}}\Phi$. Therefore, G' satisfies all the formulas which G does. By symmetry of $\underline{\leftrightarrow}_{ss}$, G satisfies all the formulas which G' does. Thus, the sets of formulas satisfiable for G and G' coincide. $\square$

Hence, in the step semantics, we obtained a logical characterization of the stochastic bisimulation relation or, symmetrically, an operational characterization of the probabilistic modal logic equivalence.

Example 6.2 *Let* $E = (\{a\}, \frac{1}{2}) \| (\{b\}, \frac{1}{2})$ *and* $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{b\}, \frac{1}{2}); (\{a\}, \frac{1}{2}))$. *Then* $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$ *but* $\overline{E} \not=_{sPML} \overline{E'}$, *because for* $\Phi = \langle\{a, b\}\rangle_{\frac{1}{3}} \top$ *we have* $\overline{E} \models_{\overline{E}} \Phi$, *but* $\overline{E'} \not\models_{\overline{E'}} \Phi$, *since only in* $TS^*(\overline{E'})$ *multiactions* $\{a\}$ *and* $\{b\}$ *cannot be executed concurrently.*

7 Stationary behaviour

Let us examine how the proposed equivalences can be used to compare behaviour of stochastic processes in their steady states. In this section, we consider only formulas specifying stochastic processes with an infinite behavior, thus, the expressions with iteration operator. We suppose that the underlined DTMC of each such an expression is irreducible or contains at least only one irreducible subset of states to guarantee an existence of the steady state.

7.1 Theoretical background

Let G be a dynamic expression. The elements $\mathcal{P}^*_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of (one-step) transition probability matrix (TPM) $\mathbf{P}^*$ for $DTMC^*(G)$ are defined as

$$\mathcal{P}^*_{ij} = \begin{cases} PM^*(s_i, s_j), & s_i \to s_j; \\ 0, & \text{otherwise.} \end{cases}$$

The transient (k-step, $k \in I\!N$) probability mass function (PMF) $\psi^*[k] = (\psi^*_1[k], \ldots, \psi^*_n[k])$ for $DTMC^*(G)$ is the solution of the equation system

$$\psi^*[k] = \psi^*[0](\mathbf{P}^*)^k,$$

where $\psi^*[0] = (\psi^*_1[0], \ldots, \psi^*_n[0])$ is the initial PMF defined as

$$\psi^*_i[0] = \begin{cases} 1, & s_i = [G]_{\simeq}; \\ 0, & \text{otherwise.} \end{cases}$$

Note also that $\psi^*[k+1] = \psi^*[k]\mathbf{P}^*$ $(k \in I\!N)$.

The steady state PMF $\psi^* = (\psi^*_1, \ldots, \psi^*_n)$ for $DTMC^*(G)$ is the solution of the equation system

$$\begin{cases} \psi^*(\mathbf{P}^* - \mathbf{E}) = \mathbf{0} \\ \psi^*\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{E}$ is the unitary matrix of dimension n and $\mathbf{0}$ is a vector with n values 0, $\mathbf{1}$ is that with n values 1.

When $DTMC^*(G)$ has the steady state, we have $\psi^* = \lim_{k\to\infty} \psi^*[k]$.

For $s \in DR(G)$ with $s = s_i$ $(1 \leq i \leq n)$ we define $\psi^*[k](s) = \psi^*_i[k]$ $(k \in I\!N)$ and $\psi^*(s) = \psi^*_i$.

7.2 Steady state and equivalences

The following proposition demonstrates that for two dynamic expressions related by $\underline{\leftrightarrow}_{ss}$ the steady state probabilities to come in an equivalence class coincide. One can also interpret the result stating that the mean recurrence time for an equivalence class is the same for both expressions.

Proposition 7.1 *Let* G, G' *be dynamic expressions with* $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$. *Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} {\psi'}^*(s').$$

Proof. See Appendix B. □

We define the expression $\mathsf{Stop} = (\{c\}, \frac{1}{2})$ rs c specifying the special process analogous to the one used in the examples of [49, 50, 53]. The latter is a continuous time stochastic analogue of the stop process proposed in [6]. Stop is a discrete time stochastic analogue of the stop, it is only able to perform empty loops with probability 1 and never terminates. Note that in the specification of Stop one could use an arbitrary action from $\mathcal{A}$ and any conditional probability belonging to the interval $(0; 1)$.

The following example demonstrates that the result of Proposition 7.1 does not hold for $\underline{\leftrightarrow}_{is}$.

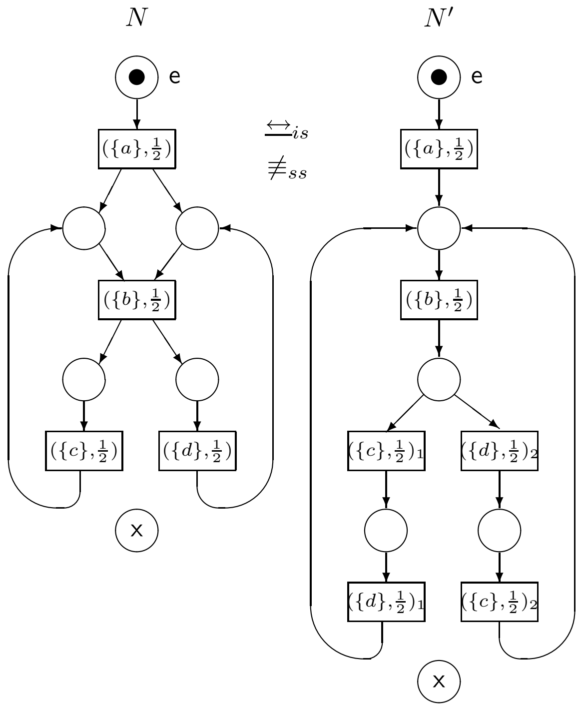

Figure 15: $\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady state probabilities to come in an equivalence class

Example 7.1 *Let* $E = [(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); ((\{c\},\frac{1}{2})\|(\{d\},\frac{1}{2}))) * \mathsf{Stop}]$ *and*
$E' = [(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1; (\{d\},\frac{1}{2})_1)[\,](((\{d\},\frac{1}{2})_2; (\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]$. *We have* $\overline{E}\underline{\leftrightarrow}_{is}\overline{E'}$.

$DR(\overline{E})$ consists of isomorphism classes

$s_1 = [[\overline{(\{a\},\frac{1}{2})} * ((\{b\},\frac{1}{2}); ((\{c\},\frac{1}{2})\|(\{d\},\frac{1}{2}))) * \mathsf{Stop}]]_\simeq$,
$s_2 = [[(\{a\},\frac{1}{2}) * (\overline{(\{b\},\frac{1}{2})}; ((\{c\},\frac{1}{2})\|(\{d\},\frac{1}{2}))) * \mathsf{Stop}]]_\simeq$,
$s_3 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (\overline{(\{c\},\frac{1}{2})}\|\overline{(\{d\},\frac{1}{2})})) * \mathsf{Stop}]]_\simeq$,
$s_4 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (\underline{(\{c\},\frac{1}{2})}\|\overline{(\{d\},\frac{1}{2})})) * \mathsf{Stop}]]_\simeq$,
$s_5 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (\overline{(\{c\},\frac{1}{2})}\|\underline{(\{d\},\frac{1}{2})})) * \mathsf{Stop}]]_\simeq$.

$DR(\overline{E'})$ consists of isomorphism classes

$s'_1 = [[\overline{(\{a\},\frac{1}{2})} * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1; (\{d\},\frac{1}{2})_1)[\,](((\{d\},\frac{1}{2})_2; (\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]]_\simeq$,
$s'_2 = [[(\{a\},\frac{1}{2}) * (\overline{(\{b\},\frac{1}{2})}; (((\{c\},\frac{1}{2})_1; (\{d\},\frac{1}{2})_1)[\,](((\{d\},\frac{1}{2})_2; (\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]]_\simeq$,
$s'_3 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); \overline{(((\{c\},\frac{1}{2})_1; (\{d\},\frac{1}{2})_1)[\,](((\{d\},\frac{1}{2})_2; (\{c\},\frac{1}{2})_2))}) * \mathsf{Stop}]]_\simeq$,
$s'_4 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1; \overline{(\{d\},\frac{1}{2})_1})[\,](((\{d\},\frac{1}{2})_2; (\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]]_\simeq$,
$s'_5 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1; (\{d\},\frac{1}{2})_1)[\,](((\{d\},\frac{1}{2})_2; \overline{(\{c\},\frac{1}{2})_2}))) * \mathsf{Stop}]]_\simeq$.

The steady state PMFs ψ^* for $DTMC^*(\overline{E})$ and ${\psi'}^*$ for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ {\psi'}^* = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right).$$

Consider the equivalence class $\mathcal{H} = \{s_3, s'_3\}$. We have $\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$ whereas $\sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} {\psi'}^*(s') = {\psi'}^*(s'_3) = \frac{1}{3}$. Thus, $\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady state probabilities to come in an equivalence class.

In Figure 15 the marked dts-boxes corresponding to the dynamic expressions above are presented, i.e., $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$. In addition, we depict the net analogues of the algebraic equivalences.

The following example demonstrates that the result of Proposition 7.1 does not even hold for $\underline{\leftrightarrow}_{is}$ plus $\equiv_{ss}$.

Example 7.2 *Let* $E = [(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); ((\{c\},\frac{1}{2})\|(\{d\},\frac{1}{2}))) * \mathsf{Stop}]$ *and*
$E' = [(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[\,]((((\{c\},\frac{1}{2})_2; (\{d\},\frac{1}{2})_2)[\,](((\{d\},\frac{1}{2})_3; (\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]$. *We have* $\overline{E}\underline{\leftrightarrow}_{is}\overline{E'}$ *and* $\overline{E} \equiv_{ss} \overline{E'}$.

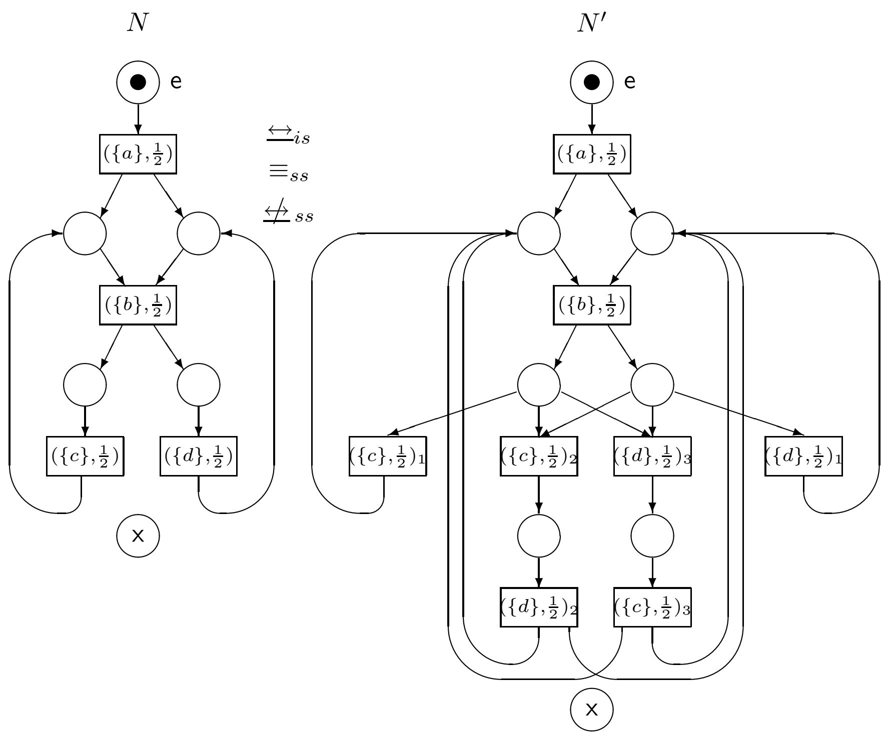

Figure 16: $\underline{\leftrightarrow}_{is}$ plus $\equiv_{ss}$ do not guarantee a coincidence of steady state probabilities to come in an equivalence class

$DR(\overline{E})$ is specified in the Example 7.1. $DR(\overline{E'})$ consists of isomorphism classes
$s_1' = [[\overline{(\{a\},\frac{1}{2})} * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[](((\{c\},\frac{1}{2})_2; (\{d\},\frac{1}{2})_2)[]((\{d\},\frac{1}{2})_3; (\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]]_\simeq$,
$s_2' = [[(\{a\},\frac{1}{2}) * (\overline{(\{b\},\frac{1}{2})}; (((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[](((\{c\},\frac{1}{2})_2; (\{d\},\frac{1}{2})_2)[]((\{d\},\frac{1}{2})_3; (\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]]_\simeq$,
$s_3' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); \overline{(((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[](((\{c\},\frac{1}{2})_2; (\{d\},\frac{1}{2})_2)[]((\{d\},\frac{1}{2})_3; (\{c\},\frac{1}{2})_3))}))) * \mathsf{Stop}]]_\simeq$,
$s_4' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\underline{(\{c\},\frac{1}{2})_1}\|\overline{(\{d\},\frac{1}{2})_1}))[](((\{c\},\frac{1}{2})_2; (\{d\},\frac{1}{2})_2)[]((\{d\},\frac{1}{2})_3; (\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]]_\simeq$,
$s_5' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\underline{(\{c\},\frac{1}{2})_1}\|\underline{(\{d\},\frac{1}{2})_1}))[](((\{c\},\frac{1}{2})_2; (\{d\},\frac{1}{2})_2)[]((\{d\},\frac{1}{2})_3; (\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]]_\simeq$,
$s_6' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[](((\{c\},\frac{1}{2})_2; \overline{(\{d\},\frac{1}{2})_2})[]((\{d\},\frac{1}{2})_3; (\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]]_\simeq$,
$s_7' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[](((\{c\},\frac{1}{2})_2; (\{d\},\frac{1}{2})_2)[]((\{d\},\frac{1}{2})_3; \overline{(\{c\},\frac{1}{2})_3}))))) * \mathsf{Stop}]]_\simeq$.

The steady state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ \psi'^* = \left(0, \frac{13}{38}, \frac{13}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}\right).$$

Consider the equivalence class $\mathcal{H} = \{s_3, s_3'\}$. We have $\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$ whereas $\sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'^*(s') = \psi'^*(s_3') = \frac{13}{38}$. Thus, $\underline{\leftrightarrow}_{is}$ plus $\equiv_{ss}$ do not guarantee a coincidence of steady state probabilities to come in an equivalence class.

In Figure 16 the marked dts-boxes corresponding to the dynamic expressions above are presented, i.e., $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$. In addition, we depict the net analogues of the algebraic equivalences.

By Proposition 7.1, $\underline{\leftrightarrow}_{ss}$ preserves the quantitative properties of stationary behaviour (the level of DTMCs). Now we intend to demonstrate that the qualitative properties of stationary behaviour (the level of transition systems) are preserved as well.

Definition 7.1 *A* step trace *of a dynamic expression G is a chain $\Sigma = A_1 \cdots A_n \in (I\!N_f^{\mathcal{L}})^*$ where $\exists s \in DR(G)\ s \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \le i \le n)$. Then* the probability of the step trace Σ to start in the state s *is*

$$PT^*(\Sigma, s) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid s=s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\le i\le n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}).$$

The following theorem demonstrates that for two dynamic expressions related by $\underline{\leftrightarrow}_{ss}$ the steady state probabilities to come in an equivalence class and start a step trace from it coincide.

Theorem 7.1 *Let G, G' be dynamic expressions with $\mathcal{R} : G\underline{\leftrightarrow}_{ss}G'$ and Σ be a step trace. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$\sum_{s\in\mathcal{H}\cap DR(G)} \psi^*(s)PT^*(\Sigma, s) = \sum_{s'\in\mathcal{H}\cap DR(G')} {\psi'}^*(s')PT^*(\Sigma, s').$$

Proof. See Appendix C. $\square$

Note that in the proof of Theorem 7.1 a limit construction us used to go from transient to stationary case. Thus, the result of this theorem is valid as well if we replace steady state probabilities with transient ones in its statement.

Example 7.3 *Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$ and $E' = [(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$. We have $\overline{E} =_{sto} \overline{E'}$, hence, $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$.*

$DR(\overline{E})$ consists of isomorphism classes

$s_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_{\simeq}$,

$s_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; ((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_{\simeq}$,

$s_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); \overline{((\{c\}, \frac{1}{2})_1[](\{c\}, \frac{1}{2})_2)}) * \mathsf{Stop}]]_{\simeq}$.

$DR(\overline{E'})$ consists of isomorphism classes

$s'_1 = [[\overline{(\{a\}, \frac{1}{2})} * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_{\simeq}$,

$s'_2 = [[(\{a\}, \frac{1}{2}) * \overline{(((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))} * \mathsf{Stop}]]_{\simeq}$,

$s'_3 = [[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; \overline{(\{c\}, \frac{1}{2})_1})[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_{\simeq}$,

$s'_4 = [[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; \overline{(\{c\}, \frac{1}{2})_2})) * \mathsf{Stop}]]_{\simeq}$.

The steady state PMFs ψ^* for $DTMC^*(\overline{E})$ and ${\psi'}^*$ for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{1}{2}, \frac{1}{2}\right),\ {\psi'}^* = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Consider the equivalence class $\mathcal{H} = \{s_3, s'_3, s'_4\}$. One can see that the steady state probabilities for $\mathcal{H}$ coincide: $\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = {\psi'}^*(s'_3) + {\psi'}^*(s'_4) = \sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} {\psi'}^*(s')$. Let $\Sigma = \{\{c\}\}$. The steady state probabilities to come in the equivalence class $\mathcal{H}$ and start the step trace Σ from it coincide as well: $\psi^*(s_3)(PT^*(\{(\{c\}, \frac{1}{2})_1\}, s_3) + PT^*(\{(\{c\}, \frac{1}{2})_2\}, s_3)) = \frac{1}{2}\cdot(\frac{1}{2}+\frac{1}{2}) = \frac{1}{2} = \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 1 = {\psi'}^*(s'_3)PT^*(\{(\{c\}, \frac{1}{2})_1\}, s'_3) + {\psi'}^*(s'_4)PT^*(\{(\{c\}, \frac{1}{2})_2\}, s'_4)$.

In Figure 17 the marked dts-boxes corresponding to the dynamic expressions above are presented, i.e., $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$. In addition, we depict the net analogues of the algebraic equivalences.

8 Preservation by algebraic operations

An important question concerning equivalence relations is whether two compound expressions always remain equivalent if they are constructed from pairwise equivalent subexpressions. The equivalence having the mentioned property of preservation by algebraic operations is called a congruence. To be a congruence is a desirable property but not an obligatory one, since many important behavioural equivalences are not congruences. As a rule, a congruence relation is too strict, i.e., it differentiates too many formulas. This is the reason why a weaker but more interesting equivalence notion that is not a congruence is preferred in many cases when process behaviour is to be compared.

Definition 8.1 *Let $\leftrightarrow$ be an equivalence of dynamic expressions. Two static expressions E and E' are* equivalent with respect to $\leftrightarrow$*, denoted by $E \leftrightarrow E'$, if $\overline{E} \leftrightarrow \overline{E'}$.*

Let us investigate which algebraic equivalences we proposed are congruences on static expressions. By definition, $\simeq$ is a congruence. The following example demonstrates that all the equivalences between $\equiv_{is}$ and $=_{sto}$ are not congruences.

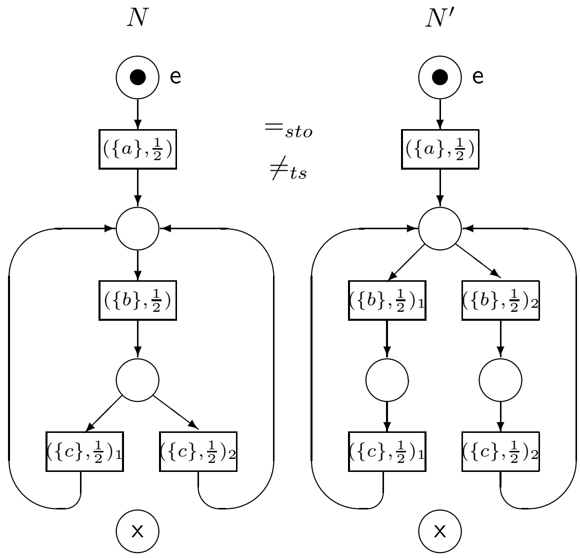

Figure 17: $\underline{\leftrightarrow}_{ss}$ implies a coincidence of the steady state probabilities to come in an equivalence class and start a trace from it

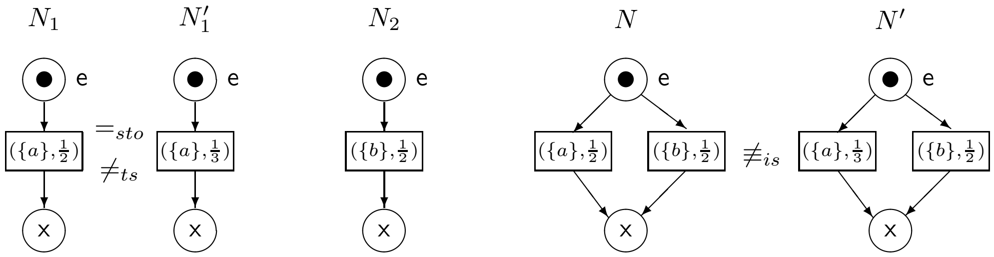

Figure 18: The equivalences between $\equiv_{is}$ and $=_{sto}$ are not congruences

Example 8.1 *Let* $E = (\{a\}, \frac{1}{2})$, $E' = (\{a\}, \frac{1}{3})$ *and* $F = (\{b\}, \frac{1}{2})$. *We have* $\overline{E} =_{sto} \overline{E'}$, *since both* $TS(\overline{E})$ *and* $TS(\overline{E'})$ *have the transition with multiaction part of label* $\{a\}$ *and probability* 1. *On the other hand,* $\overline{E[]F} \not\equiv_{is} \overline{E'[]F}$, *since only in* $TS^*(\overline{E'[]F})$ *the probabilities of the transitions with multiaction part of label* $\{a\}$ *and* $\{b\}$ *are different* ($\frac{1}{3}$ *and* $\frac{2}{3}$, *respectively). Thus, no equivalence between* $\equiv_{is}$ *and* $=_{sto}$ *is a congruence.*

In Figure 18 the marked dts-boxes corresponding to the dynamic expressions above are presented, i.e., $N_1 = Box_{dts}(\overline{E})$, $N_1' = Box_{dts}(\overline{E'})$, $N_2 = Box_{dts}(\overline{F})$ *and* $N = Box_{dts}(\overline{E[]F})$, $N' = Box_{dts}(\overline{E'[]F})$. *In addition, we depict the net analogues of the algebraic equivalences.*

The following example demonstrates that the question of preservation by operations has the negative answer for $=_{ts}$ as well.

Example 8.2 *Let* $E = (\{a\}, \frac{1}{2})$, $E' = (\{a\}, \frac{1}{2});\mathsf{Stop}$ *and* $F = (\{b\}, \frac{1}{2})$. *We have* $\overline{E} =_{ts} \overline{E'}$, *since both* $TS^*(\overline{E})$ *and* $TS^*(\overline{E'})$ *have the transition with multiaction part of label* $\{a\}$ *and probability* $\frac{1}{2}$. *On the other hand,* $\overline{E;F} \not\equiv_{is} \overline{E';F}$, *since only in* $TS^*(\overline{E';F})$ *after the transition with multiaction part of label* $\{a\}$ *no other transition can fire. Thus, no equivalence between* $\equiv_{is}$ *and* $=_{ts}$ *is a congruence.*

In Figure 19 the marked dts-boxes corresponding to the dynamic expressions above are presented, i.e., $N_1 = Box_{dts}(\overline{E})$, $N_1' = Box_{dts}(\overline{E'})$, $N_2 = Box_{dts}(\overline{F})$ *and* $N = Box_{dts}(\overline{E;F})$, $N' = Box_{dts}(\overline{E';F})$. *In addition, we depict the net analogues of the algebraic equivalences.*

We suppose that for an analogue of $=_{ts}$ to be a congruence, we have to equip transition systems of expressions with two extra transitions skip and redo as in [49, 53]. This allows one to avoid difficulties demonstrated in Example 8.2 with unexpected termination due to the Stop process. At the same time, such an enrichment of transition systems does not overcome the problems explained in Example 8.1 with abstraction from empty loops. Hence, the equivalences between $\equiv_{is}$ and $=_{sto}$ defined on the basis of the enriched transition systems will still be non-congruences.

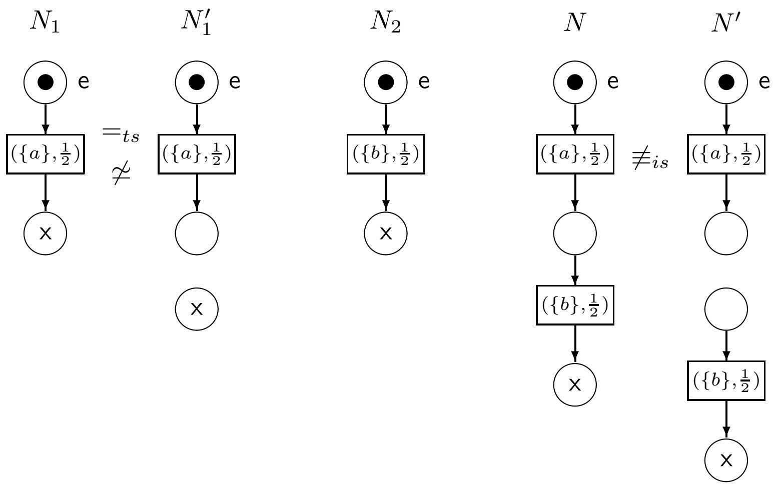

Figure 19: The equivalences between $\equiv_{is}$ and $=_{ts}$ are not congruences

To define the analogue of $=_{ts}$ mentioned above, we shall introduce a notion of sr-transition system. It has the final state and two extra transitions from the initial state to the final one and back. Note that sr-transition systems do not have the loop transitions from the final state to itself. First, we propose the rules for skip and redo. Let $E \in \mathit{RegStatExpr}$.

$$\overline{E} \stackrel{\mathsf{skip}}{\rightarrow} \underline{E} \qquad\qquad \underline{E} \stackrel{\mathsf{redo}}{\rightarrow} \overline{E}$$

Now we can define sr-transition systems of dynamic expressions in the form $\overline{E}$, where E is a static expression. This syntactic restriction is needed to take into account two additional rules above. We assume that skip has probability 0, hence, it will be never executed. On the other hand, redo has probability 1, hence, it will be immediately executed at the next time moment if it is enabled.

Definition 8.2 *Let E be a static expression and $TS(\overline{E}) = (S, L, \mathcal{T}, s)$. The* (labeled probabilistic) sr-transition system *of $\overline{E}$ is a quadruple $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$, where*

- $S_{sr} = S \cup \{[\underline{E}]_\simeq\}$;
- $L_{sr} \subseteq (I\!N_f^{\mathcal{SL}} \times (0;1]) \cup \{(\mathsf{skip}, 0), (\mathsf{redo}, 1)\}$;
- $\mathcal{T}_{sr} = \mathcal{T} \setminus \{([\underline{E}]_\simeq, (\emptyset, 1), [\underline{E}]_\simeq)\} \cup \{([\overline{E}]_\simeq, (\mathsf{skip}, 0), [\underline{E}]_\simeq), ([\underline{E}]_\simeq, (\mathsf{redo}, 1), [\overline{E}]_\simeq)\}$;
- $s_{sr} = s$.

We define a new notion of isomorphism for sr-transition systems, since we should take care of their final states.

Definition 8.3 *Let E, E' be static expressions and $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$, $TS_{sr}(\overline{E'}) = (S'_{sr}, L'_{sr}, \mathcal{T}'_{sr}, s'_{sr})$ be their sr-transition systems. A mapping $\beta : S_{sr} \to S'_{sr}$ is an* isomorphism *between $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E'})$, denoted by $\beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, if*

1. *β is a bijection such that $\beta(s_{sr}) = s'_{sr}$ and $\beta([\underline{E}]_\simeq) = [\underline{E'}]_\simeq$;*
2. *$\forall s, \tilde{s} \in S_{sr}\ \forall \Gamma\ s \stackrel{\Gamma}{\rightarrow}_{\mathcal{P}} \tilde{s} \ \Leftrightarrow\ \beta(s) \stackrel{\Gamma}{\rightarrow}_{\mathcal{P}} \beta(\tilde{s})$.*

Two sr-transition systems $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E'})$ are isomorphic, *denoted by $TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, if $\exists \beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$.*

sr-transition systems of static expressions can be defined as well. For $E \in \mathit{RegStatExpr}$ let $TS_{sr}(E) = TS_{sr}(\overline{E})$.

Example 8.3 *Let $E = (\{a\}, \frac{1}{2})$. In Figure 20 the transition systems $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E; \mathsf{Stop}})$ are presented. In the latter sr-transition system (unlike the former one) the final state can be reached by executing the transition* $(\mathsf{skip}, 0)$ *from the initial state only.*

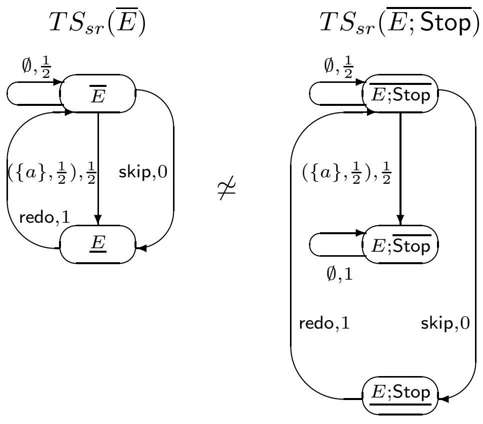

Figure 20: The sr-transition systems of $\overline{E}$ and $\overline{E;\mathsf{Stop}}$ for $E = (\{a\}, \frac{1}{2})$

Definition 8.4 *Two dynamic expressions* $\overline{E}$ *and* $\overline{E'}$ *are* isomorphic with respect to sr-transition systems*, denoted by* $\overline{E} =_{tssr} \overline{E'}$*, if* $TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$.

Note that sr-transition systems without empty loops can be defined as well as the equivalence $=_{tssr*}$ based on them. At the same time, the coincidence of $=_{tssr}$ and $=_{tssr*}$ can be proved similar to that of $=_{ts}$ and $=_{ts*}$.

It is easy to see that $=_{tssr}$ implies $=_{ts}$, since sr-transition systems have more states and transitions than usual ones. In addition, $\simeq$ implies $=_{tssr}$, since the sr-transition system of a dynamic formula is defined based on its isomorphism class. Thus, $=_{tssr}$ is "in between" $=_{ts}$ and $\simeq$ with respect to differentiating ability. To prove that the mentioned implications are strict, consider the following example.

Example 8.4

- *Let* $E = (\{a\}, \frac{1}{2})$ *and* $E' = (\{a\}, \frac{1}{2}); \mathsf{Stop}$. *We have* $\overline{E} =_{ts} \overline{E'}$ *as demonstrated in Example 8.2. On the other hand,* $\overline{E} \neq_{tssr} \overline{E'}$*, since only in* $TS_{sr}(\overline{E'})$ *after the transition with multiaction part of label* $\{a\}$ *we do not reach the final state, see Example 8.3. Thus,* $=_{tssr}$ *is strictly stronger than* $=_{ts}$.
- *Let* $E = (\{a\}, \frac{1}{2})$ *and* $E' = ((\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. *Then* $\overline{E} =_{tssr} \overline{E'}$*, since* $\overline{E} =_{ts} \overline{E'}$ *as demonstrated in the last example in the proof of Theorem 5.2, and the final states of both* $TS_{sr}(\overline{E'})$ *and* $TS_{sr}(\overline{E'})$ *are reachable from the others with "normal" transitions (i.e., not with* skip *only). On the other hand,* $\overline{E} \not\simeq \overline{E'}$. *Thus,* $\simeq$ *is strictly stronger than* $=_{tssr}$.

The following theorem demonstrates that $=_{tssr}$ is a congruence of static expressions with respect to the operations of $dtsPBC$.

Theorem 8.1 *Let* $a \in Act$ *and* $E, E', F, K \in RegStatExpr$. *If* $\overline{E} =_{tssr} \overline{E'}$ *then*

1. $\overline{E \circ F} =_{tssr} \overline{E' \circ F}$, $\overline{F \circ E} =_{tssr} \overline{F \circ E'}$, $\circ \in \{;, [], \|\}$;
2. $\overline{E[f]} =_{tssr} \overline{E'[f]}$;
3. $\overline{E \circ a} =_{tssr} \overline{E' \circ a}$, $\circ \in \{\mathsf{rs}, \mathsf{sy}\}$;
4. $\overline{[E * F * K]} =_{tssr} \overline{[E' * F * K]}$, $\overline{[F * E * K]} =_{tssr} \overline{[F * E' * K]}$, $\overline{[F * K * E]} =_{tssr} \overline{[F * K * E']}$.

Proof. First, we have no problems with termination, hence, the composite sr-transition systems build from the isomorphic ones can always execute the same multisets of activities. Second, the probabilities of the corresponding transitions of the composite systems coincide, since the probabilities are calculated from identical values. □

9 Performance evaluation

In this section with a number of case studies we demonstrate how steady state distribution can be used for performance evaluation.

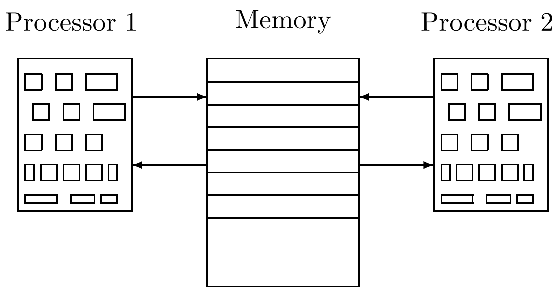

Figure 21: The diagram of the shared memory system

9.1 Shared memory system

Consider a model of two processors accessing a common shared memory described in [43] in the continuous time setting on GSPNs. We analyze this shared memory system in the discrete time setting within *dtsPBC* where concurrent execution of activities is possible. The model performs as follows. After activation of the system (turning the computer on), two processors are active, and the common memory is available. Each processor can request an access to the memory. When a processor starts an acquisition of the memory, another processor should wait until the former one ends its memory operations, and the system returns to the state with both active processors and the available common memory. The diagram of the system is depicted in Figure 21.

Let us explain the meaning of actions from syntax of the *dtsPBC* expressions which will specify the system modules. The action a corresponds to the system activation. The actions r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i. The actions b_i and e_i correspond to the beginning and the end, respectively, of the common memory access of processor i. The other actions are used for communication purpose only via synchronization, and we abstract from them later using restriction.

The static expression of the first processor is $E_1 = [(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});(\{b_1,y_1\},\frac{1}{2});(\{e_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]$. The static expression of the second processor is $E_2 = [(\{x_2\},\frac{1}{2}) * ((\{r_2\},\frac{1}{2});(\{b_2,y_2\},\frac{1}{2});(\{e_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]$. The static expression of the shared memory is $E_3 = [(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},\frac{1}{2});(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},\frac{1}{2});(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}]$. The static expression of the shared memory system with two processors is $E = (E_1\|E_2\|E_3)\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2$.

$DR(\overline{E})$ consists of isomorphism classes

$s_1 = [([\overline{(\{x_1\},\frac{1}{2})} * ((\{r_1\},\frac{1}{2});(\{b_1,y_1\},\frac{1}{2});(\{e_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]\|[\overline{(\{x_2\},\frac{1}{2})} * ((\{r_2\},\frac{1}{2});(\{b_2,y_2\},\frac{1}{2});(\{e_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]\|[\overline{(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2})} * (((\{\widehat{y_1}\},\frac{1}{2});(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},\frac{1}{2});(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,

$s_2 = [([(\{x_1\},\frac{1}{2}) * (\overline{(\{r_1\},\frac{1}{2})};(\{b_1,y_1\},\frac{1}{2});(\{e_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * (\overline{(\{r_2\},\frac{1}{2})};(\{b_2,y_2\},\frac{1}{2});(\{e_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},\frac{1}{2});(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},\frac{1}{2});(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,

$s_3 = [([(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});\overline{(\{b_1,y_1\},\frac{1}{2})};(\{e_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * (\overline{(\{r_2\},\frac{1}{2})};(\{b_2,y_2\},\frac{1}{2});(\{e_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * ((\overline{(\{\widehat{y_1}\},\frac{1}{2})};(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},\frac{1}{2});(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,

$s_4 = [([(\{x_1\},\frac{1}{2}) * (\overline{(\{r_1\},\frac{1}{2})};(\{b_1,y_1\},\frac{1}{2});(\{e_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * ((\{r_2\},\frac{1}{2});\overline{(\{b_2,y_2\},\frac{1}{2})};(\{e_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},\frac{1}{2});(\{\widehat{z_1}\},\frac{1}{2}))[](\overline{(\{\widehat{y_2}\},\frac{1}{2})};(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,

$s_5 = [([(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});(\{b_1,y_1\},\frac{1}{2});\overline{(\{e_1,z_1\},\frac{1}{2})}) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * (\overline{(\{r_2\},\frac{1}{2})};(\{b_2,y_2\},\frac{1}{2});(\{e_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},\frac{1}{2});\overline{(\{\widehat{z_1}\},\frac{1}{2})})[]((\{\widehat{y_2}\},\frac{1}{2});(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,

$s_6 = [([(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});\overline{(\{b_1,y_1\},\frac{1}{2})};(\{e_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * ((\{r_2\},\frac{1}{2});\overline{(\{b_2,y_2\},\frac{1}{2})};(\{e_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * ((\overline{(\{\widehat{y_1}\},\frac{1}{2})};(\{\widehat{z_1}\},\frac{1}{2}))[](\overline{(\{\widehat{y_2}\},\frac{1}{2})};(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,

$s_7 = [([(\{x_1\},\frac{1}{2}) * (\overline{(\{r_1\},\frac{1}{2})};(\{b_1,y_1\},\frac{1}{2});(\{e_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * ((\{r_2\},\frac{1}{2});(\{b_2,y_2\},\frac{1}{2});\overline{(\{e_2,z_2\},\frac{1}{2})}) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},\frac{1}{2});(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},\frac{1}{2});\overline{(\{\widehat{z_2}\},\frac{1}{2})})) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,

$s_8 = [([(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});(\{b_1,y_1\},\frac{1}{2});\overline{(\{e_1,z_1\},\frac{1}{2})}) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * ((\{r_2\},\frac{1}{2});(\{b_2,y_2\},\frac{1}{2});(\{e_2,z_2\},\frac{1}{2})) *$

$TS^*(\overline{E})$

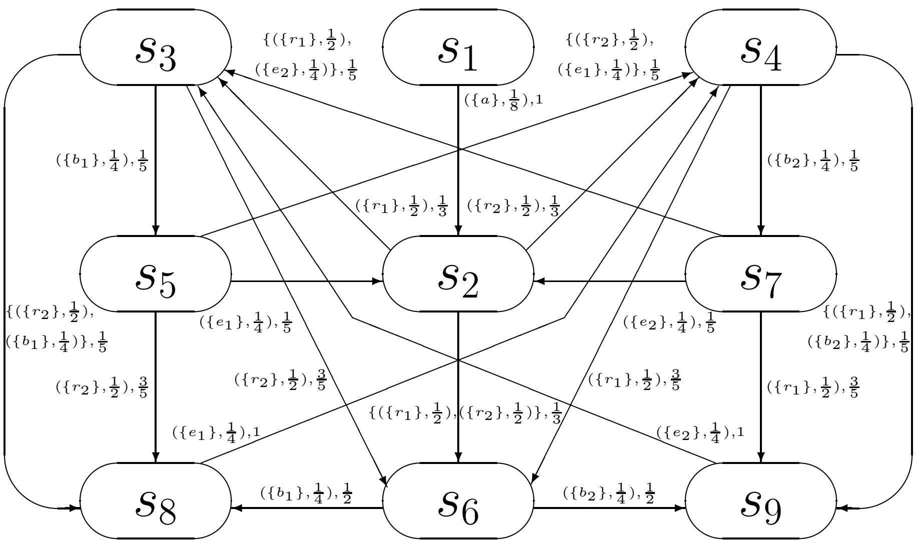

Figure 22: The transition system without empty loops of the shared memory system

$\mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2})*(((\{\widehat{y_1}\},\frac{1}{2});\overline{(\{\widehat{z_1}\},\frac{1}{2}))}[]((\{\widehat{y_2}\},\frac{1}{2});(\{\widehat{z_2}\},\frac{1}{2})))*\mathsf{Stop}])$ $\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$,
$s_9=[([(\{x_1\},\frac{1}{2})*((\{r_1\},\frac{1}{2});(\{b_1,y_1\},\frac{1}{2});(\{e_1,z_1\},\frac{1}{2}))*\mathsf{Stop}]\|[(\{x_2\},\frac{1}{2})*((\{r_2\},\frac{1}{2});(\{b_2,y_2\},\frac{1}{2});\overline{(\{e_2,z_2\},\frac{1}{2}))}*\mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2})*(((\{\widehat{y_1}\},\frac{1}{2});(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},\frac{1}{2});\overline{(\{\widehat{z_2}\},\frac{1}{2}))})*\mathsf{Stop}])$ $\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\simeq}$.

In Figure 22 the transition system without empty loops $TS^*(\overline{E})$ is presented. In Figure 23 the underlying DTMC without empty loops $DTMC^*(\overline{E})$ is presented.

The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^*=\begin{bmatrix}0&1&0&0&0&0&0&0&0\\0&0&\frac{1}{3}&\frac{1}{3}&0&\frac{1}{3}&0&0&0\\0&0&0&0&\frac{1}{5}&\frac{3}{5}&0&\frac{1}{5}&0\\0&0&0&0&0&\frac{3}{5}&\frac{1}{5}&0&\frac{1}{5}\\0&\frac{1}{5}&0&\frac{1}{5}&0&0&0&\frac{3}{5}&0\\0&0&0&0&0&0&0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{5}&\frac{1}{5}&0&0&0&0&0&\frac{3}{5}\\0&0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\end{bmatrix}.$$

In Figure 24 an alteration diagram for the transient state probabilities $\psi_i^*[k]$ $(i\in\{1,2,3,5,6,8\})$ of the shared memory system is presented for the time moments k $(0\le k\le 10)$. It is sufficient to depict the probabilities for the states s_1,s_2,s_3,s_5,s_6,s_8 only, since the corresponding values coincide for s_3,s_4 as well as for s_5,s_7 as well as for s_8,s_9.

The steady state PMF ψ^* for $DTMC^*(\overline{E})$ is

$$\psi^*=\left(0,\frac{3}{209},\frac{75}{418},\frac{75}{418},\frac{15}{418},\frac{46}{209},\frac{15}{418},\frac{35}{209},\frac{35}{209}\right).$$

We can now calculate performance indices.

- The average recurrence time in the state s_2, where no processor requests the memory, called the *average system run-through*, is $\frac{1}{\psi_2^*}=\frac{209}{3}=69\frac{2}{3}$.
- The common memory is available in the states s_2,s_3,s_4,s_6 only. The steady state probability that the memory is available is $\psi_2^*+\psi_3^*+\psi_4^*+\psi_6^*=\frac{3}{209}+\frac{75}{418}+\frac{75}{418}+\frac{46}{209}=\frac{124}{209}$. Then the steady state probability that the memory is used (i.e., not available) called the *shared memory utilization* is $1-\frac{124}{209}=\frac{85}{209}$.

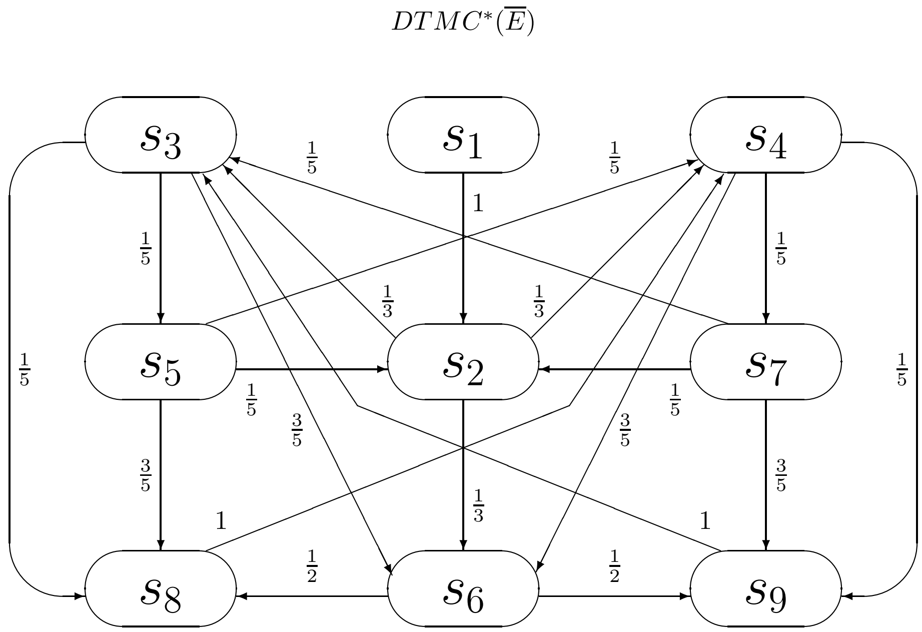

Figure 23: The underlying DTMC without empty loops of the shared memory system

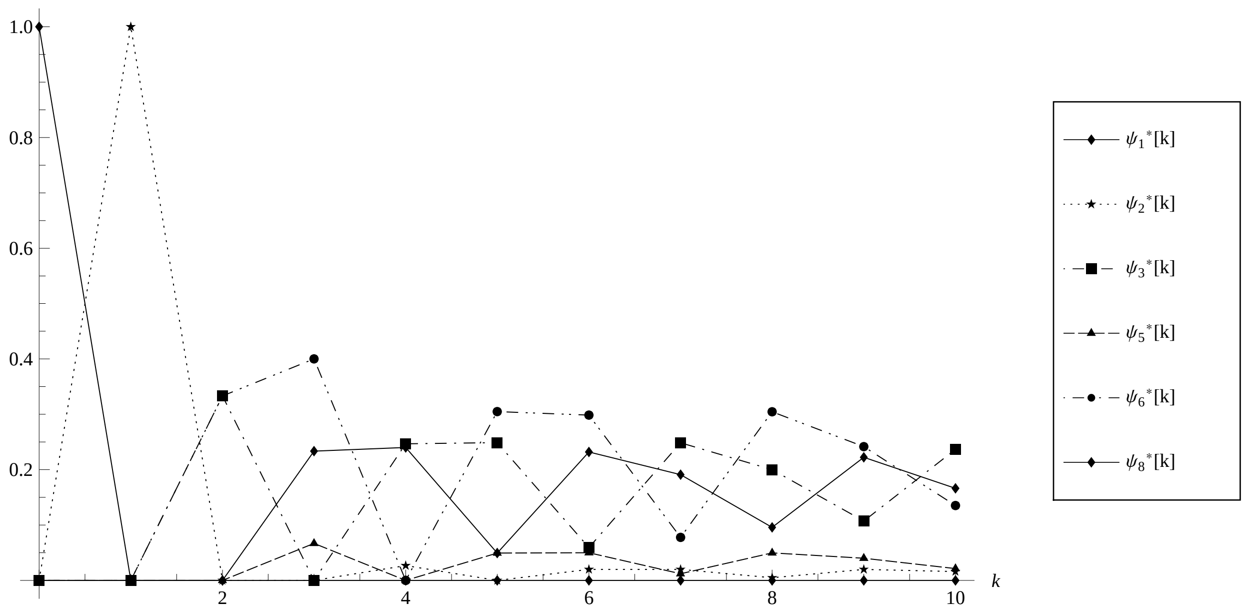

Figure 24: Transient state probabilities of the shared memory system

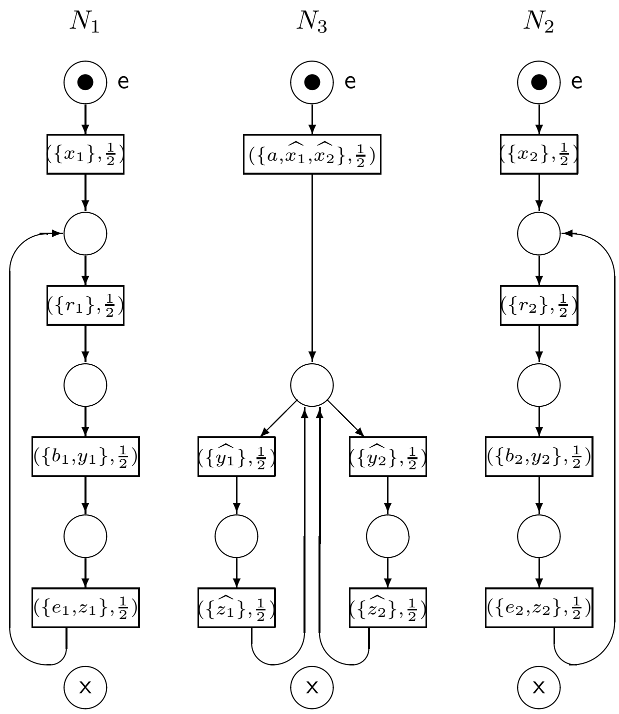

Figure 25: The marked dts-boxes of two processors and shared memory

- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is possible from the states s_2, s_4, s_7 only. The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$. Thus, the *steady state probability of the shared memory request from the first processor* is $\psi_2^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_4^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_7^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) = \frac{3}{209} \cdot \left(\frac{1}{3} + \frac{1}{3}\right) + \frac{75}{418} \cdot \left(\frac{3}{5} + \frac{1}{5}\right) + \frac{15}{418} \cdot \left(\frac{3}{5} + \frac{1}{5}\right) = \frac{38}{209}$.

In Figure 25 the marked dts-boxes corresponding to the dynamic expressions of two processors and shared memory are depicted, i.e., $N_i = Box_{dts}(\overline{E_i})$ $(1 \leq i \leq 3)$. In Figure 26 the marked dts-box corresponding to the dynamic expression of the shared memory system is presented, i.e., $N = Box_{dts}(\overline{E})$.

9.2 Dining philosophers system

Consider a model of five dining philosophers, for which the Petri net interpretation was proposed in [64], in the discrete time stochastic setting of *dtsPBC*. The philosophers occupy a round table, and there is one fork between every neighboring persons, hence, there are five forks on the table. A philosopher needs two forks to eat, namely, his left and right one. Hence, all five philosophers cannot eat together, since otherwise there will be not enough forks available, but only one of two of them who are not neighbors. The model performs as follows. After activation of the system (coming the philosophers in the dining room), five forks appear on the table. If the left and right forks available for a philosopher, he takes them simultaneously and begins eating. At the end of eating, the philosopher places both his forks simultaneously back on the table. The strategy to pick up and release two forks simultaneously prevents the situation when a philosopher takes one fork but is not able to pick up the second one since their neighbor has already done so. In particular, we avoid a deadlock when all the philosophers take their left (right) forks and wait until their right (left) forks will be available. The diagram of the system is depicted in Figure 27.

Let us explain the meaning of actions from syntax of the *dtsPBC* expressions which will specify the system modules. The action a corresponds to the system activation. The actions b_i and e_i correspond to the beginning and the end, respectively, of eating of philosopher i $(1 \leq i \leq 5)$. The other actions are used for communication purpose only via synchronization, and we abstract from them later using restriction. Note that the expression of each philosopher includes two alternative subexpressions such that the second one specifies a resource (fork) sharing with the right neighbor.

The static expression of the philosopher i $(1 \leq i \leq 4)$ is $E_i = [(\{x_i\}, \frac{1}{2}) * (((\{b_i, \widehat{y_i}\}, \frac{1}{2}); (\{e_i, \widehat{z_i}\}, \frac{1}{2}))[]$ $((\{y_{i+1}\}, \frac{1}{2}); (\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$. The static expression of the philosopher 5 is $E_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) *$

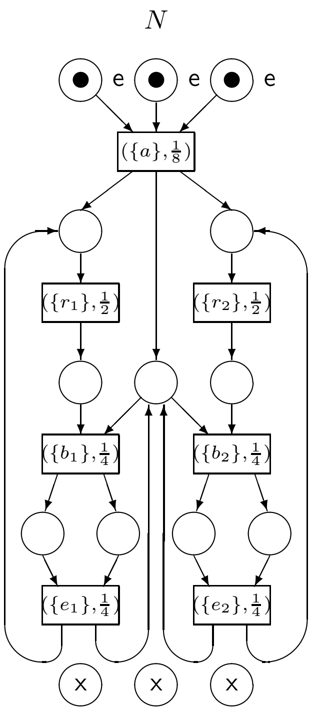

Figure 26: The marked dts-box of the shared memory system

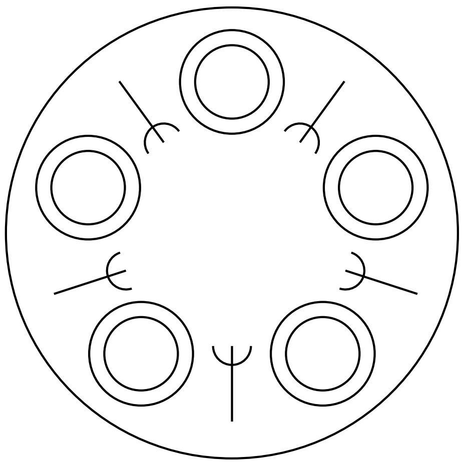

Figure 27: The diagram of the dining philosophers system

$(((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}]$. The static expression of the dining philosophers system is $E = (E_1 \| E_2 \| E_3 \| E_4 \| E_5)$ $\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5$.

$DR(\overline{E})$ consists of isomorphism classes

$s_1 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_2 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_3 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_4 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_5 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_6 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_7 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_8 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_9 = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\simeq}$,

$s_{10} = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}] \| [(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}]$

$\|[(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});$ $(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs $z_5]_{\simeq}$,

$s_{11} = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});$ $(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}]\|[(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}]$ $\|[(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});$ $(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs $z_5]_{\simeq}$,

$s_{12} = [([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))) * \mathsf{Stop}]\|[(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});$ $(\{e_2,\widehat{z_2}\},\frac{1}{2}))[]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2}))) * \mathsf{Stop}]\|[(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2}))) * \mathsf{Stop}]$ $\|[(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2}))) * \mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});$ $(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2}))) * \mathsf{Stop}])$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs $z_5]_{\simeq}$.

In Figure 28 the transition system without empty loops $TS^*(\overline{E})$ is presented. In Figure 29 the underlying DTMC without empty loops $DTMC^*(\overline{E})$ is presented.

The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\ 0 & \frac{3}{11} & 0 & \frac{3}{11} & \frac{3}{11} & \frac{1}{11} & \frac{1}{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{11} & \frac{1}{11} & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 & 0 \\ 0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 \\ 0 & \frac{1}{7} & 0 & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 \\ 0 & \frac{3}{11} & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & \frac{1}{11} & \frac{3}{11} \\ 0 & \frac{3}{11} & 0 & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} \\ 0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{7} & \frac{3}{7} & 0 \end{bmatrix}.$$

In Figure 30 an alteration diagram for the transient state probabilities $\psi_i^*[k]$ $(1 \leq i \leq 4)$ of the dining philosophers system is presented for the time moments k $(0 \leq k \leq 10)$. It is sufficient to depict the probabilities for the states $s_1,\dots,s_4$ only, since the corresponding values coincide for $s_3,s_6,s_7,s_{10},s_{11}$ as well as for s_4,s_5,s_8,s_9,s_{12}.

The steady state PMF ψ^* for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{2}{11}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}\right).$$

We can now calculate performance indices.

- The average recurrence time in the state s_2, where all the forks are available, called the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{11}{2} = 5\frac{1}{2}$.
- Nobody eats at the state s_2. Then the *fraction of time when no philosophers dine* is $\psi_2^* = \frac{2}{11}$.

  Only one philosopher eats at the states $s_3,s_6,s_7,s_{10},s_{11}$. Then the *fraction of time when only one philosopher dines* is $\psi_3^* + \psi_6^* + \psi_7^* + \psi_{10}^* + \psi_{11}^* = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}$.

  Two philosophers eat together at the states s_4,s_5,s_8,s_9,s_{12}. Then the *fraction of time when two philosophers dine* is $\psi_4^* + \psi_5^* + \psi_8^* + \psi_9^* + \psi_{12}^* = \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine with respect to when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.
- The beginning of eating of first philosopher $(\{b_1\},\frac{1}{4})$ is possible from the states s_2,s_6,s_7 only. The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b_1\},\frac{1}{4})$. Thus, the *steady state probability of the beginning of eating of first philosopher* is $\psi_2^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma,s_2) + \psi_6^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma,s_6) + \psi_7^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma,s_7) = \frac{2}{11}\cdot\left(\frac{3}{20}+\frac{1}{20}+\frac{1}{20}\right) + \frac{1}{10}\cdot\left(\frac{3}{11}+\frac{1}{11}\right) + \frac{1}{10}\cdot\left(\frac{3}{11}+\frac{1}{11}\right) = \frac{13}{110}$.

$TS^*(\overline{E})$

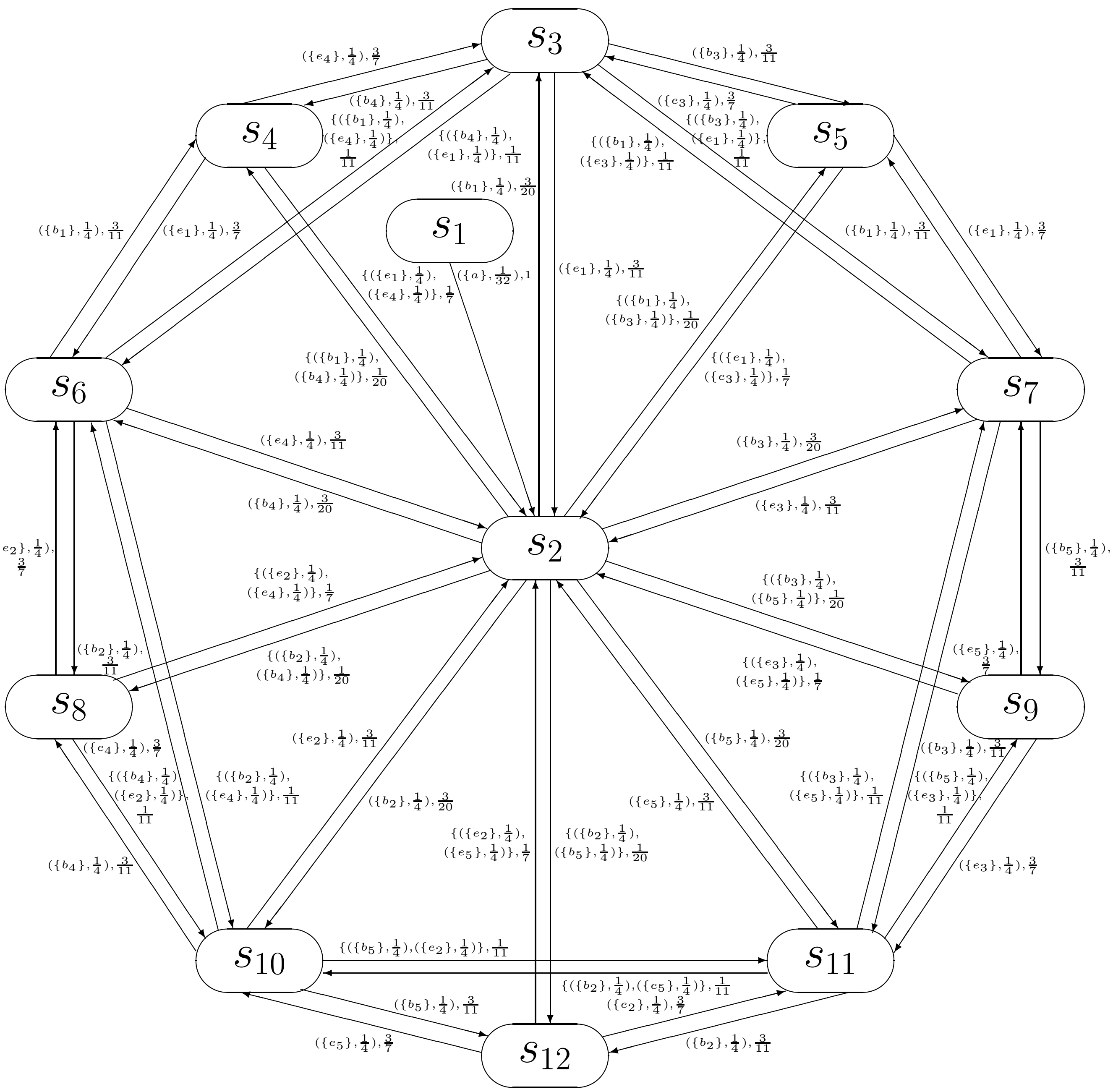

Figure 28: The transition system without empty loops of the dining philosophers system

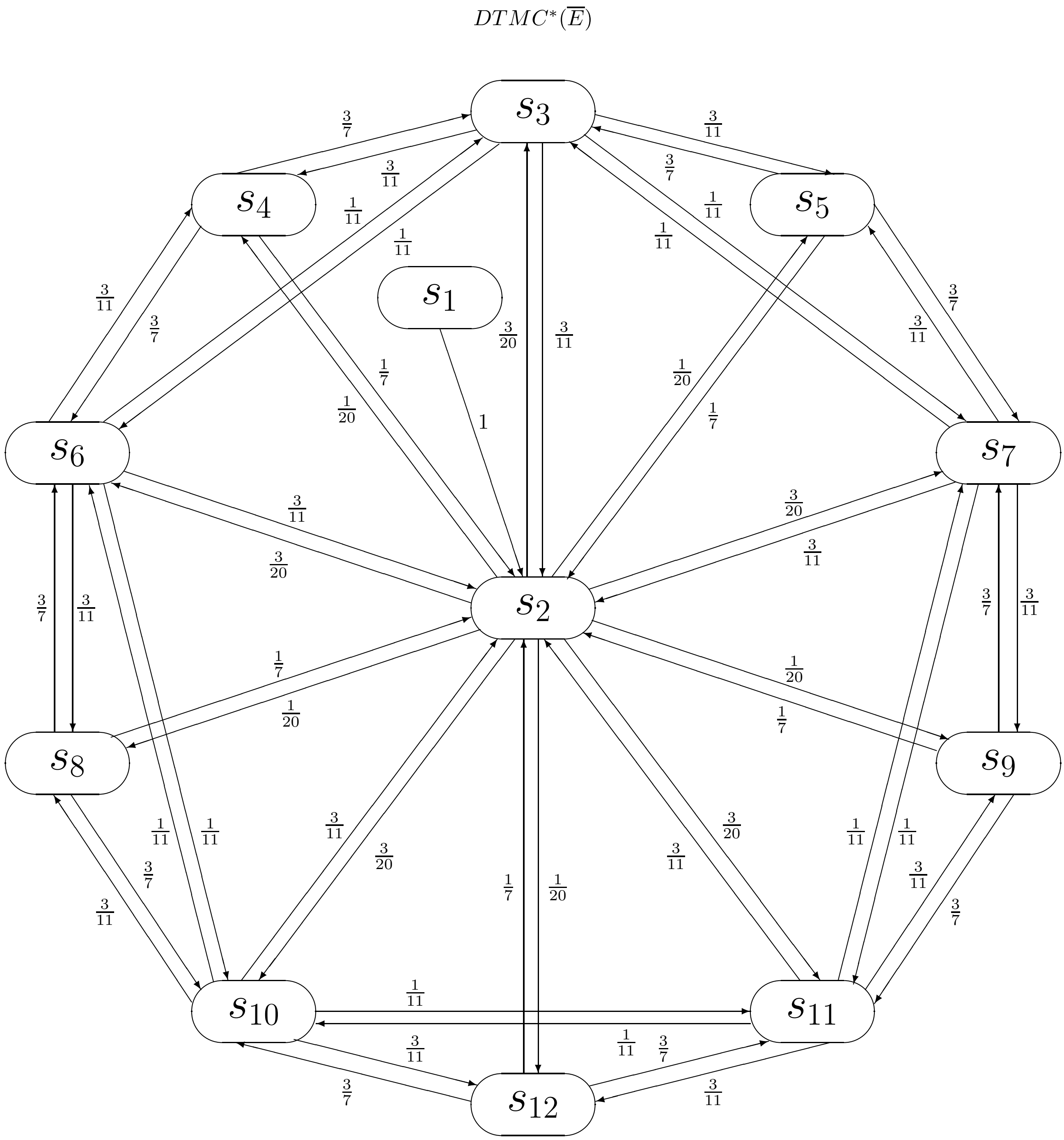

Figure 29: The underlying DTMC without empty loops of the dining philosophers system

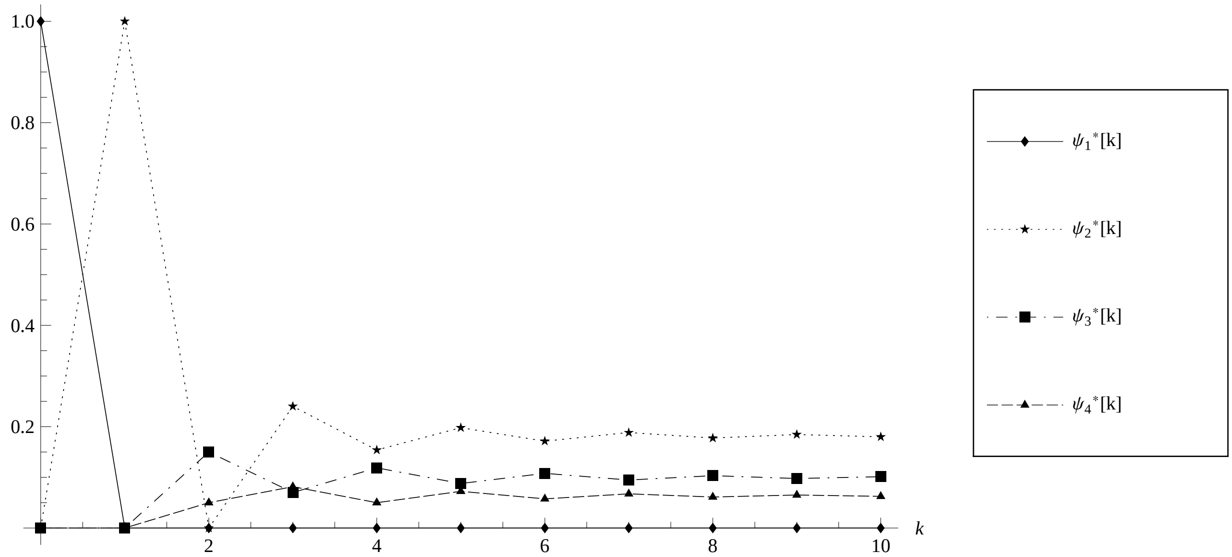

Figure 30: Transient state probabilities of the dining philosophers system

In Figure 31 the marked dts-boxes corresponding to the dynamic expressions of the dining philosophers are depicted, i.e., $N_i = Box_{dts}(\overline{E_i})$ $(1 \le i \le 5)$. In Figure 32 the marked dts-box corresponding to the dynamic expression of the dining philosophers system is presented, i.e., $N = Box_{dts}(\overline{E})$.

Let us consider a modification of the dining philosophers system with abstraction from personalities, i.e., such that all the philosophers are indistinguishable. For example, we can just see that one or two philosophers dine but cannot observe who they are. We call this system abstract dining philosophers one.

The static expression of the philosopher i $(1 \le i \le 4)$ is $F_i = [(\{x_i\}, \frac{1}{2}) * (((\{b, \widehat{y_i}\}, \frac{1}{2}); (\{e, \widehat{z_i}\}, \frac{1}{2}))[] ((\{y_{i+1}\}, \frac{1}{2}); (\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$. The static expression of the philosopher 5 is $F_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * (((\{b, \widehat{y_5}\}, \frac{1}{2}); (\{e, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2}); (\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}]$. The static expression of the abstract dining philosophers system is $F = (F_1 \| F_2 \| F_3 \| F_4 \| F_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$. We have $DTMC^*(\overline{F}) = DTMC^*(\overline{E})$. Thus, the TPM and the steady state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

The first performance index and the second group of the indices are the same for the standard and abstract systems. Let us consider the following performance index based on non-personalized viewpoint to the philosophers.

- The beginning of eating of a philosopher $(\{b\}, \frac{1}{4})$ is possible from the states $s_2, s_3, s_6, s_7, s_{10}, s_{11}$ only. The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{1}{4})$. Thus, the *steady state probability of the beginning of eating of a philosopher* is $\psi_2^* \sum_{\{\Gamma | (\{b\}, \frac{1}{4}) \in \Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma | (\{b\}, \frac{1}{4}) \in \Gamma\}} PT^*(\Gamma, s_3) + \psi_6^* \sum_{\{\Gamma | (\{b\}, \frac{1}{4}) \in \Gamma\}} PT^*(\Gamma, s_6) + \psi_7^* \sum_{\{\Gamma | (\{b\}, \frac{1}{4}) \in \Gamma\}} PT^*(\Gamma, s_7) + \psi_{10}^* \sum_{\{\Gamma | (\{b\}, \frac{1}{4}) \in \Gamma\}} PT^*(\Gamma, s_{10}) + \psi_{11}^* \sum_{\{\Gamma | (\{b\}, \frac{1}{4}) \in \Gamma\}} PT^*(\Gamma, s_{11}) = \frac{2}{11} \cdot \left(\frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) = \frac{6}{11}$.

The marked dts-boxes corresponding to the dynamic expressions of the standard and abstract dining philosophers are similar as well as the marked dts-boxes corresponding to the dynamic expression of the standard and abstract dining philosophers systems.

Let us consider a reduction of the abstract dining philosophers system.

The static expression of the philosopher 1 is $F_1' = [(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$. The static expression of the philosopher 2 is $F_2' = [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$. The static expression of the reduced abstract dining philosophers system is $F' = (F_1' \| F_2')$ sy x rs x.

$DR(\overline{F'})$ consists of isomorphism classes

$s_1' = [([\overline{(\{x\}, \frac{1}{2})} * ((\{b\}, \frac{2}{5})_1; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \| [\overline{(\{a, \hat{x}\}, \frac{1}{16})} * ((\{b\}, \frac{2}{5})_2; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])$ sy x rs $x]_{\simeq}$,

$s_2' = [([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1}; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * (\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])$ sy x rs $x]_{\simeq}$,

$s_3' = [([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * (\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])$ sy x rs $x]_{\simeq}$,

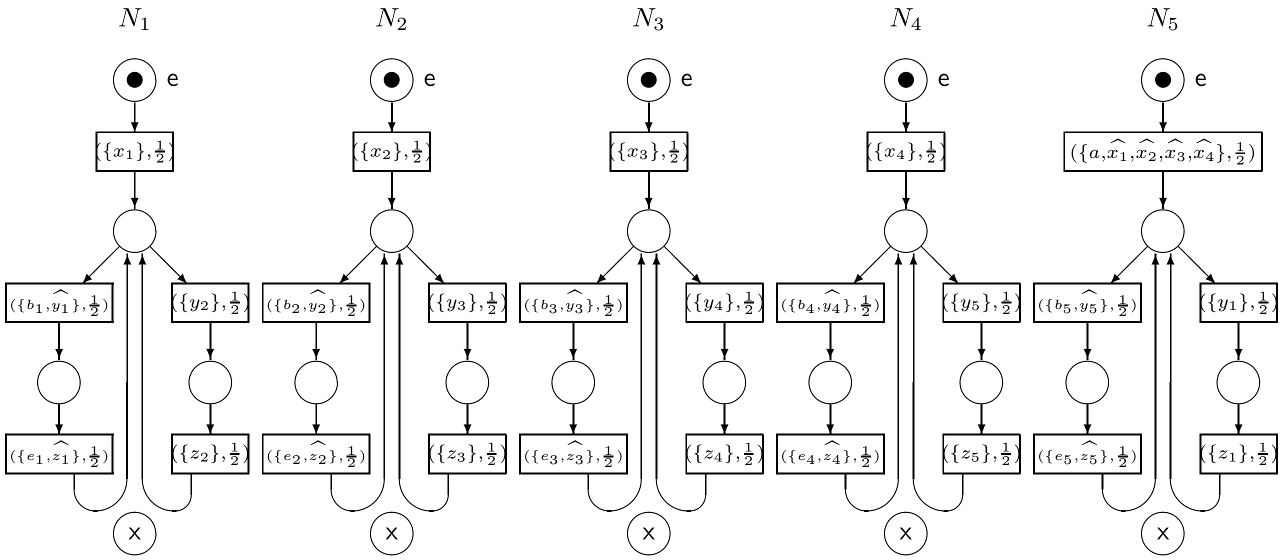

Figure 31: The marked dts-boxes of the dining philosophers

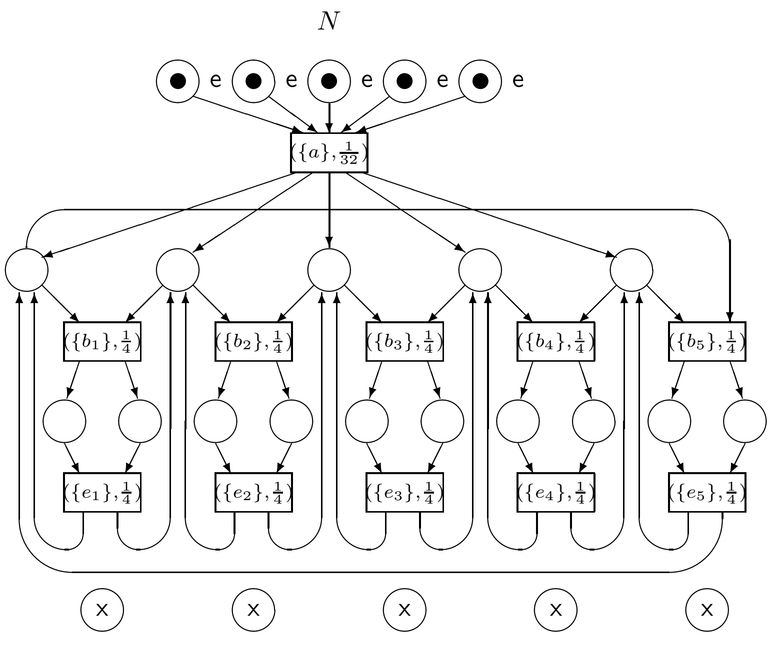

Figure 32: The marked dts-box of the dining philosophers system

$TS^*(\overline{F'})$

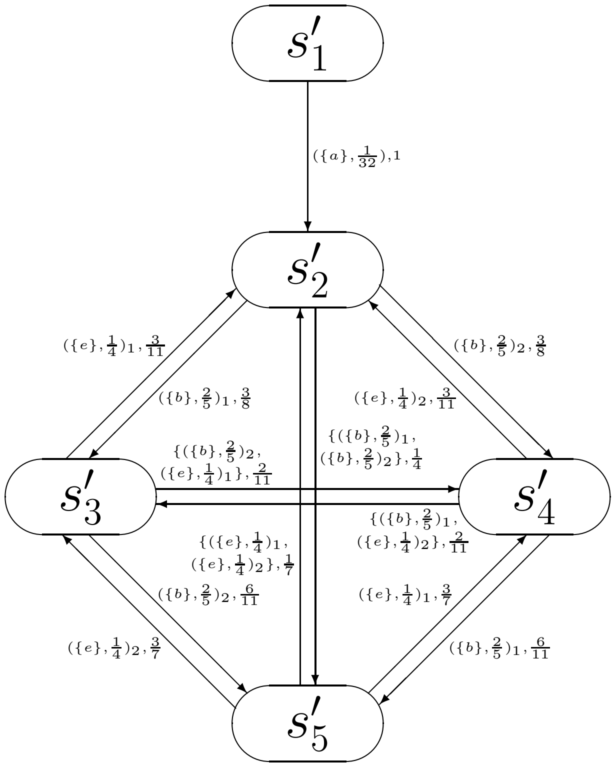

Figure 33: The transition system without empty loops of the reduced abstract dining philosophers system

$s'_4 = [([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1; (\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}]) \text{ sy } x \text{ rs } x]_{\simeq}$,
$s'_5 = [([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}]) \text{ sy } x \text{ rs } x]_{\simeq}$.

We have $\overline{F} \underline{\leftrightarrow}_{ss} \overline{F'}$ with $(DR(\overline{F}) \cup DR(\overline{F'}))/_{\underline{\leftrightarrow}_{ss}} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$, where $\mathcal{H}_1 = \{s_1, s'_1\}$ (the initial state), $\mathcal{H}_2 = \{s_2, s'_2\}$ (the system is activated and no philosophers dine), $\mathcal{H}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}, s'_3, s'_4\}$ (one philosopher dines), $\mathcal{H}_4 = \{s_4, s_5, s_8, s_9, s_{12}, s'_5\}$ (two philosophers dine). One can see that F' is a reduction of F with respect to $\underline{\leftrightarrow}_{ss}$.

In Figure 33 the transition system without empty loops $TS^*(\overline{F'})$ is presented. In Figure 34 the underlying DTMC without empty loops $DTMC^*(\overline{F'})$ is presented.

The TPM for $DTMC^*(\overline{F'})$ is

$$\mathbf{P}'^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ 0 & \frac{3}{11} & 0 & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{3}{11} & \frac{2}{11} & 0 & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{3}{7} & \frac{3}{7} & 0 \end{bmatrix}.$$

In Figure 35 an alteration diagram for the transient state probabilities $\psi'^*_i[k]$ $(i \in \{1, 2, 3, 5\})$ of the reduced abstract dining philosophers system is presented for the time moments k $(0 \leq k \leq 10)$. It is sufficient to depict the probabilities for the states s_1, s_2, s_3, s_5 only, since the corresponding values coincide for s_3, s_4.

The steady state PMF ψ'^* for $DTMC^*(\overline{F'})$ is

$$\psi'^* = \left(0, \frac{2}{11}, \frac{1}{4}, \frac{1}{4}, \frac{7}{22}\right).$$

We can now calculate performance indices.

- The average recurrence time in the state s'_2, where all the forks are available, called the *average system run-through*, is $\frac{1}{\psi'^*_2} = \frac{11}{2} = 5\frac{1}{2}$.

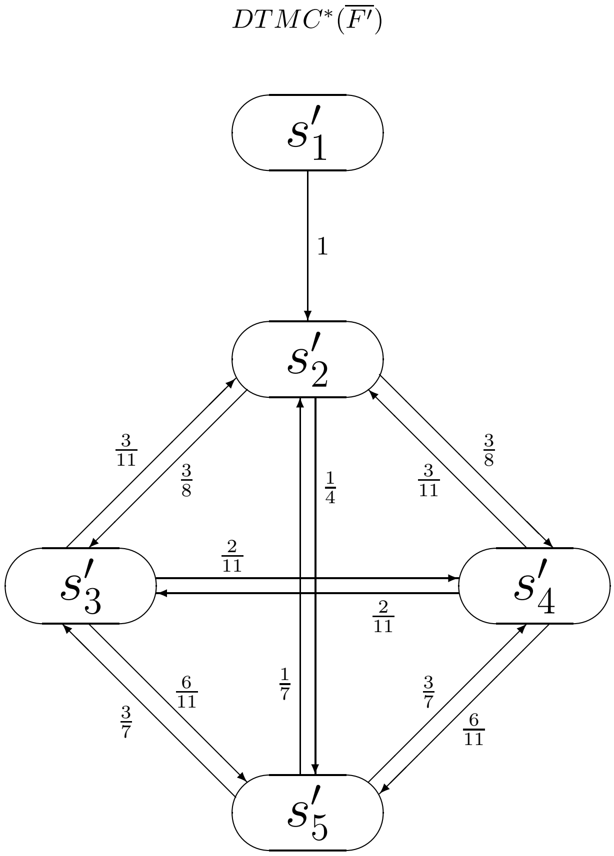

Figure 34: The underlying DTMC without empty loops of the reduced abstract dining philosophers system

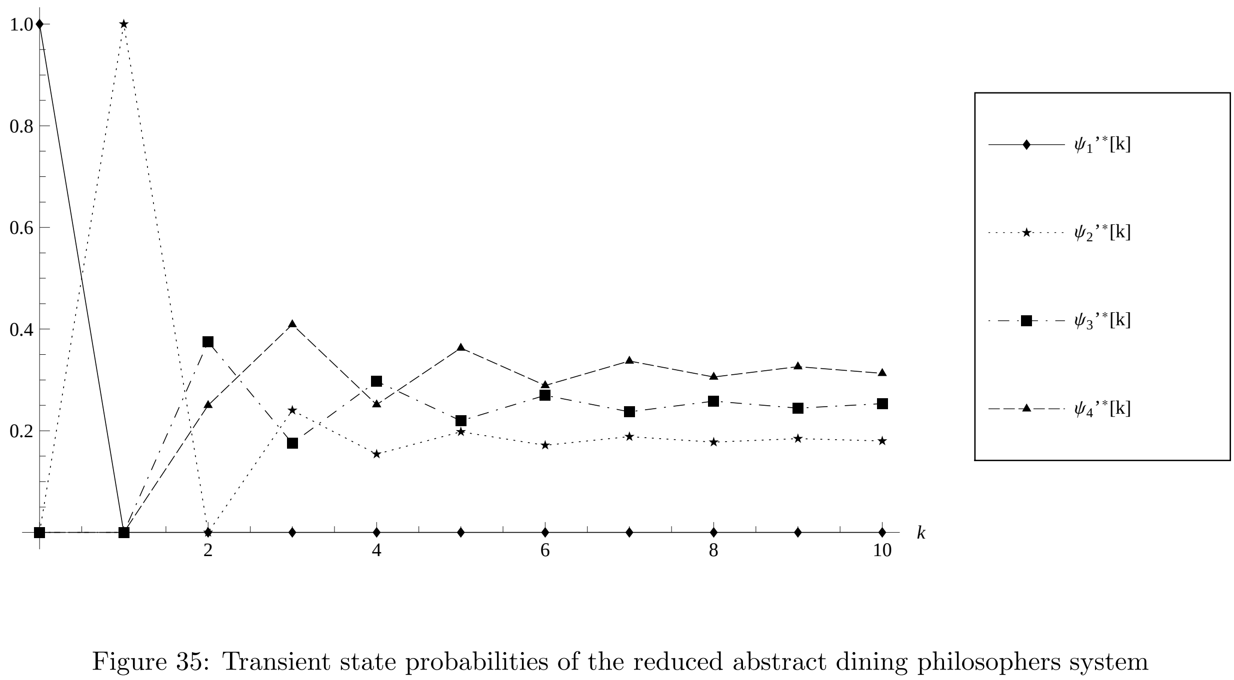

Figure 35: Transient state probabilities of the reduced abstract dining philosophers system

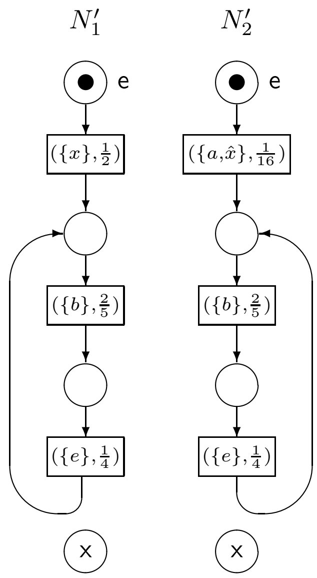

Figure 36: The marked dts-boxes of the reduced abstract dining philosophers

- Nobody eats at the state s'_2. Then the *fraction of time when no philosophers dine* is ${\psi'_2}^* = \frac{2}{11}$.

  Only one philosopher eats at the states s'_3, s'_4. Then the *fraction of time when only one philosopher dines* is ${\psi'_3}^* + {\psi'_4}^* = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

  Two philosophers eat together at the state s'_5. Then the *fraction of time when two philosophers dine* is ${\psi'_5}^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine with respect to when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of a philosopher $(\{b\}, \frac{2}{5})$ is possible from the states s'_2, s'_3, s'_4 only. The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{2}{5})$. Thus, the *steady state probability of the beginning of eating of a philosopher* is ${\psi'_2}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_2) + {\psi'_3}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_3) + {\psi'_4}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_4) = \frac{2}{11} \cdot \left(\frac{3}{8} + \frac{3}{8} + \frac{1}{4}\right) + \frac{1}{4} \cdot \left(\frac{6}{11} + \frac{2}{11}\right) + \frac{1}{4} \cdot \left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}$.

One can see that the performance indices are the same for the complete and the reduced abstract dining philosophers systems. The coincidence of the first performance index as well as the second group of indices obviously illustrates the result of Proposition 7.1. The coincidence of the third performance index is due to the Theorem 7.1: one should just apply its result to the step traces $\{\{b\}\}$, $\{\{b\},\{b\}\}$, $\{\{b\},\{e\}\}$ of the expressions $\overline{F}$ and $\overline{F'}$, and then sum the left and right parts of the three resulting equalities.

In Figure 36 the marked dts-boxes corresponding to the dynamic expressions of the reduced abstract dining philosophers are depicted, i.e., $N'_i = Box_{dts}(\overline{F'_i})$ $(1 \le i \le 2)$. In Figure 37 the marked dts-box corresponding to the dynamic expression of the reduced abstract dining philosophers system is presented, i.e., $N' = Box_{dts}(\overline{F'})$.

Note that $TS^*(\overline{F'})$ can be reduced further by merging the equivalent states s'_3 and s'_4, thus, it can be transformed into a transition system with four states only. But the resulted "minimal" reduction with respect to $\underline{\leftrightarrow}_{ss}$ of the initial transition system $TS^*(\overline{F})$ will not be anymore a transition system without empty loops corresponding to some *dtsPBC* expression. Hence, in the general case, the procedure of expressions reduction cannot be transferred smoothly from a transition systems level. The minimal equivalent expression does not always have the minimal transition system, in the case the latter can be further reduced.

Let us define the notion of minimal reduction more formally. In the following definition we consider step stochastic bisimulation equivalence between states of a dynamic expression.

Definition 9.1 *The* minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ (labeled probabilistic) transition system without empty loops *of a dynamic expression G is a quadruple $TS^*_{\underline{\leftrightarrow}_{ss}}(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$, where*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\underline{\leftrightarrow}_{ss}}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq I\!N_f^{\mathcal{L}} \times (0;1]$;

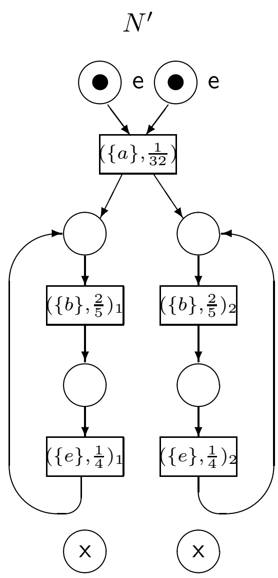

Figure 37: The marked dts-box of the reduced abstract dining philosophers system

- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{H}, (A, \mathcal{P}), \widetilde{\mathcal{H}}) \mid \exists s \in \mathcal{H}\ s \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{P}} \mathcal{H}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = \{[G]_{\simeq}\}$.

A transition $(\mathcal{H}, (A, \mathcal{P}), \widetilde{\mathcal{H}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{H} \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{P}} \widetilde{\mathcal{H}}$.

Minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ transition systems without empty loops of static expressions can be defined as well. For $E \in RegStatExpr$ let $TS^*_{\underline{\leftrightarrow}_{ss}}(E) = TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Definition 9.2 *Let G be a dynamic expression. The* minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ underlying DTMC without empty loops *of G, denoted by $DTMC^*_{\underline{\leftrightarrow}_{ss}}(G)$, has the state space $DR(G)/_{\underline{\leftrightarrow}_{ss}}$ and the transitions $\mathcal{H} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{H}}$, if $\exists s \in \mathcal{H}\ s \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{H}}$.*

Minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ underlying DTMCs without empty loops of static expressions can be defined as well. For $E \in RegStatExpr$ let $DTMC^*_{\underline{\leftrightarrow}_{ss}}(E) = DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

What concerns the abstract dining philosophers system, we have $DR(\overline{F})/_{\underline{\leftrightarrow}_{ss}} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$, where $\mathcal{K}_1 = \{s_1\}$ (the initial state), $\mathcal{K}_2 = \{s_2\}$ (the system is activated and no philosophers dine), $\mathcal{K}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}\}$ (one philosopher dines), $\mathcal{K}_4 = \{s_4, s_5, s_8, s_9, s_{12}\}$ (two philosophers dine).

In Figure 38 the minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ transition system without empty loops $TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is presented. In Figure 39 the minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ underlying DTMC without empty loops $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is presented.

Further, we can take $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ to calculate performance indices. It is easy to demonstrate that they will coincide with those calculated on the basis of $DTMC^*(\overline{F})$ or $DTMC^*(\overline{F'})$, and this is reason why we omit the details here.

Obviously, it is easier to evaluate performance with the use of a DTMC with less states, since in this case the dimension of the transition probability matrix will be smaller. Hence, to calculate steady-state probabilities, we shall solve systems of less equations. Thus, we have obtained the following method of performance analysis simplification. First, we construct the minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ underlying DTMC without empty loops. Second, we calculate steady-state probabilities and performance indices based on this minimal reduction DTMC. The indices will be the same as those calculated based on the initial unreduced DTMC.

10 Conclusion

In this paper, we considered a discrete time stochastic extension of a finite part of PBC enriched with iteration and called $dtsPBC$. The calculus has the concurrent step operational semantics based on transition systems and the denotational semantics in terms of a subclass of LDTSPNs. Within $dtsPBC$ with iteration, we have defined

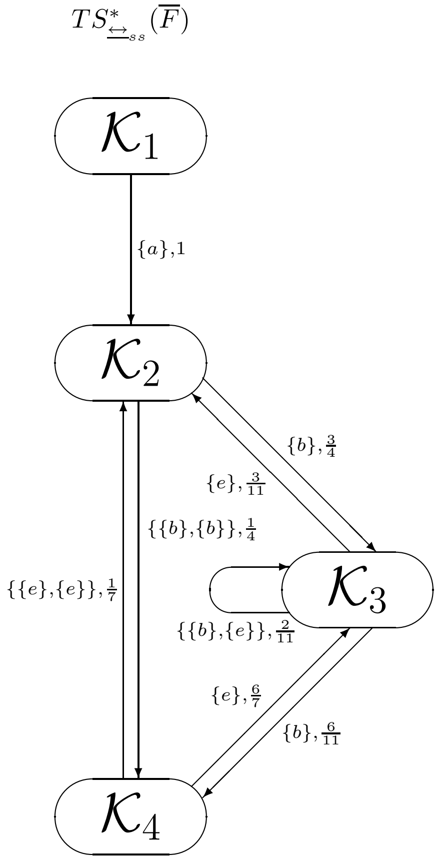

Figure 38: The minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ transition system without empty loops of the abstract dining philosophers system

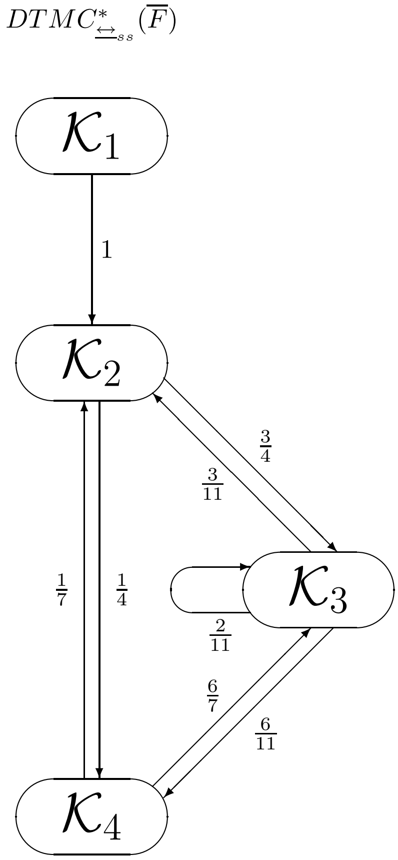

Figure 39: The minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ underlying DTMC without empty loops of the abstract dining philosophers system

a number of stochastic algebraic equivalences which have natural net analogues on LDTSPNs. The equivalences abstract from empty loops in transition systems corresponding to dynamic expressions. The diagram of interrelations for the algebraic equivalences was constructed. We presented a logical characterization of stochastic bisimulation equivalences. An applicatioof the equivalences to comparison of stationary behaviour was demonstrated. A congruence relation was proposed. Case studies of performance evaluation in the framework of the calculus were presented.

Future work consists in abstracting from the silent activities in the definitions of the equivalences, i.e., from the activities with empty multiaction part. The abstraction from empty loops and that from silent activities could be done in one step as well. The main point here is that we should collect probabilities during such the abstractions from an internal activity. As a result, we shall have the algebraic analogues of the net stochastic equivalences from [15, 16]. Moreover, we plan to extend *dtsPBC* with recursion to enhance specification power of the calculus.

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A Proof of Proposition 5.2

It is enough to prove it for $\star = s$, since $\star = i$ is a particular case of the previous one with one-element multisets of multiactions and interleaving transition relation.

Let $\mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$ and $s, \bar{s} \in \mathcal{H}$. We have $\forall \widetilde{\mathcal{H}} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}\ \forall A \in I\!N_f^{\mathcal{L}}\ s \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}} \Leftrightarrow \bar{s} \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}}$. The previous equality is valid for all $s, \bar{s} \in \mathcal{H}$, hence, we can rewrite it as $\mathcal{H} \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}}$ and denote $PM_A^*(\mathcal{H}, \widetilde{\mathcal{H}}) = PM_A^*(s, \widetilde{\mathcal{H}}) = PM_A^*(\bar{s}, \widetilde{\mathcal{H}})$. Note that transitions from the states of $DR(G)$ always lead to those from the same set, hence, $\forall s \in DR(G)\ PM_A^*(s, \widetilde{\mathcal{H}}) = PM_A^*(s, \widetilde{\mathcal{H}} \cap DR(G))$. The same is true for $DR(G')$.

Let $(A_1 \cdots A_n, \mathcal{P}) \in \mathit{StepProbTraces}(G)$. Taking into account the notes above and $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, we have $\forall \mathcal{H}_1, \ldots, \exists \mathcal{H}_n \in (DR(G) \cup DR(G'))/_{\mathcal{R}}\ [G]_\simeq \xrightarrow{A_1}\!\!\!\rightarrow_{\mathcal{Q}_1} \mathcal{H}_1 \xrightarrow{A_2}\!\!\!\rightarrow_{\mathcal{Q}_2} \cdots \xrightarrow{A_n}\!\!\!\rightarrow_{\mathcal{Q}_n} \mathcal{H}_n \Leftrightarrow [G']_\simeq \xrightarrow{A_1}\!\!\!\rightarrow_{\mathcal{Q}_1} \mathcal{H}_1 \xrightarrow{A_2}\!\!\!\rightarrow_{\mathcal{Q}_2} \cdots \xrightarrow{A_n}\!\!\!\rightarrow_{\mathcal{Q}_n} \mathcal{H}_n$.

Now we intend to prove that the sum of probabilities of all the paths starting in $[G]_\simeq$ and going through the states from $\mathcal{H}_1, \ldots, \mathcal{H}_n$ is equal to the product of $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$, which is essentially the probability of the "composite" path going through the equivalence classes $\mathcal{H}_1, \ldots, \mathcal{H}_n$ in $TS^*(G)$:

$$\sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) = \prod_{i=1}^n PM_{A_i}^*(\mathcal{H}_{i-1}, \mathcal{H}_i).$$

We prove this equality by induction on the step trace length n.

- $n = 0$

  $\sum_{\{\Gamma_1 \mid [G]_\simeq \xrightarrow{\Gamma_1} s_1,\ \mathcal{L}(\Gamma_1)=A_1,\ s_1\in\mathcal{H}_1\}} PT^*(\Gamma_1, [G]_\simeq) = PM_{A_1}^*([G]_\simeq, \mathcal{H}_1) = PM_{A_1}^*(\mathcal{H}_0, \mathcal{H}_1)$.

- $n \to n+1$

  $\sum_{\{\Gamma_1,\ldots,\Gamma_n,\Gamma_{n+1} \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n \xrightarrow{\Gamma_{n+1}} s_{n+1},\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n+1)\}} \prod_{i=1}^{n+1} PT^*(\Gamma_i, s_{i-1}) =$
  $\sum_{\{\Gamma_{n+1} \mid s_n \xrightarrow{\Gamma_{n+1}} s_{n+1},\ \mathcal{L}(\Gamma_{n+1})=A_{n+1},\ s_n\in\mathcal{H}_n,\ s_{n+1}\in\mathcal{H}_{n+1}\}} \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}}$
  $\prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) PT^*(\Gamma_{n+1}, s_n) =$
  $\sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}}$
  $\left[\prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) \sum_{\{\Gamma_{n+1} \mid s_n \xrightarrow{\Gamma_{n+1}} s_{n+1},\ \mathcal{L}(\Gamma_{n+1})=A_{n+1},\ s_n\in\mathcal{H}_n,\ s_{n+1}\in\mathcal{H}_{n+1}\}} PT^*(\Gamma_{n+1}, s_n)\right] =$
  $\sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) PM_{A_{n+1}}^*(s_n, \mathcal{H}_{n+1}) =$
  $\sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) PM_{A_{n+1}}^*(\mathcal{H}_n, \mathcal{H}_{n+1}) =$
  $PM_{A_{n+1}}^*(\mathcal{H}_n, \mathcal{H}_{n+1}) \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) =$
  $PM_{A_{n+1}}^*(\mathcal{H}_n, \mathcal{H}_{n+1}) \prod_{i=1}^n PM_{A_i}^*(\mathcal{H}_{i-1}, \mathcal{H}_i) = \prod_{i=1}^{n+1} PM_{A_i}^*(\mathcal{H}_{i-1}, \mathcal{H}_i)$.

Note that the equality we have just proved can also be applied to G'.

Now we only need to see that the summation over *all multisets of activities* is the same as the summation over *all equivalence classes*: $\mathcal{P} = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) =$
$\sum_{\mathcal{H}_1,\ldots,\mathcal{H}_n} \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\simeq \xrightarrow{\Gamma_1} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}) =$
$\sum_{\mathcal{H}_1,\ldots,\mathcal{H}_n} \prod_{i=1}^n PM_{A_i}^*(\mathcal{H}_{i-1}, \mathcal{H}_i) =$
$\sum_{\mathcal{H}_1,\ldots,\mathcal{H}_n} \sum_{\{\Gamma'_1,\ldots,\Gamma'_n \mid [G']_\simeq \xrightarrow{\Gamma'_1} \cdots \xrightarrow{\Gamma'_n} s'_n,\ \mathcal{L}(\Gamma'_i)=A_i,\ s'_i\in\mathcal{H}_i\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma'_i, s'_{i-1}) =$
$\sum_{\{\Gamma'_1,\ldots,\Gamma'_n \mid [G']_\simeq \xrightarrow{\Gamma'_1} \cdots \xrightarrow{\Gamma'_n} s'_n,\ \mathcal{L}(\Gamma'_i)=A_i,\ (1\le i\le n)\}} \prod_{i=1}^n PT^*(\Gamma'_i, s'_{i-1})$.

Hence, $(A_1 \cdots A_n, \mathcal{P}) \in \mathit{StepProbTraces}(G')$, and we have $\mathit{StepProbTraces}(G) \subseteq \mathit{StepProbTraces}(G')$. The reverse inclusion is proved by symmetry. $\square$

B Proof of Proposition 7.1

The proof is an extension of results from [19] to the process algebra framework and discrete time case.

It is enough to prove for transient PMFs only, since $\psi^* = \lim_{k\to\infty}\psi^*[k]$ and $\psi'^* = \lim_{k\to\infty}\psi'^*[k]$. We proceed by induction on k.

- $k = 0$

  The only nonzero values of the initial PMFs of $DTMC^*(G)$ and $DTMC^*(G')$ are $\psi^*[0]([G]_\simeq)$ and $\psi^*[0]([G']_\simeq)$. The only equivalence class containing $[G]_\simeq$ or $[G']_\simeq$ is $\mathcal{H} = \{[G]_\simeq, [G']_\simeq\}$. Thus, $\forall \mathcal{H} \in (DR(G)\cup DR(G'))/_{\mathcal{R}}$ $\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[0](s) = \psi^*[0]([G]_\simeq) = 1 = \psi'^*[0]([G']_\simeq) = \sum_{s'\in\mathcal{H}\cap DR(G')}\psi'^*[0](s')$.

- $k \to k+1$

  Let $\mathcal{H} \in (DR(G)\cup DR(G'))/_{\mathcal{R}}$ and $s_1, s_2 \in \mathcal{H}$. We have $\forall \widetilde{\mathcal{H}} \in (DR(G)\cup DR(G'))/_{\mathcal{R}}$ $\forall A \in I\!N_f^{\mathcal{L}}$ $s_1 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}} \Leftrightarrow s_2 \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}}$. Therefore, $PM^*(s_1,\widetilde{\mathcal{H}}) = \sum_{\{\Gamma\mid\exists\tilde{s}_1\in\widetilde{\mathcal{H}}\ s_1\xrightarrow{\Gamma}\tilde{s}_1\}} PT^*(\Gamma,s_1) = \sum_{A\in I\!N_f^{\mathcal{L}}}\sum_{\{\Gamma\mid\exists\tilde{s}_1\in\widetilde{\mathcal{H}}\ s_1\xrightarrow{\Gamma}\tilde{s}_1,\ \mathcal{L}(\Gamma)=A\}} PT^*(\Gamma,s_1) = \sum_{A\in I\!N_f^{\mathcal{L}}} PM_A^*(s_1,\widetilde{\mathcal{H}}) = \sum_{A\in I\!N_f^{\mathcal{L}}} PM_A^*(s_2,\widetilde{\mathcal{H}}) = \sum_{A\in I\!N_f^{\mathcal{L}}}\sum_{\{\Gamma\mid\exists\tilde{s}_2\in\widetilde{\mathcal{H}}\ s_2\xrightarrow{\Gamma}\tilde{s}_2,\ \mathcal{L}(\Gamma)=A\}} PT^*(\Gamma,s_2) = \sum_{\{\Gamma\mid\exists\tilde{s}_2\in\widetilde{\mathcal{H}}\ s_2\xrightarrow{\Gamma}\tilde{s}_2\}} PT^*(\Gamma,s_2) = PM^*(s_2,\widetilde{\mathcal{H}})$. Since we have the previous equality for all $s_1, s_2 \in \mathcal{H}$, we can denote $PM^*(\mathcal{H},\widetilde{\mathcal{H}}) = PM^*(s_1,\widetilde{\mathcal{H}}) = PM^*(s_2,\widetilde{\mathcal{H}})$. Note that transitions from the states of $DR(G)$ always lead to those from the same set, hence, $\forall s \in DR(G)$ $PM^*(s,\widetilde{\mathcal{H}}) = PM^*(s,\widetilde{\mathcal{H}}\cap DR(G))$. The same is true for $DR(G')$.

  By induction hypothesis, $\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[k](s) = \sum_{s'\in\mathcal{H}\cap DR(G')}\psi'^*[k](s')$. Further,
  $\sum_{\tilde{s}\in\widetilde{\mathcal{H}}\cap DR(G)}\psi^*[k+1](\tilde{s}) = \sum_{\tilde{s}\in\widetilde{\mathcal{H}}\cap DR(G)}\sum_{s\in DR(G)}\psi^*[k](s)PM^*(s,\tilde{s}) =$
  $\sum_{s\in DR(G)}\sum_{\tilde{s}\in\widetilde{\mathcal{H}}\cap DR(G)}\psi^*[k](s)PM^*(s,\tilde{s}) = \sum_{s\in DR(G)}\psi^*[k](s)\sum_{\tilde{s}\in\widetilde{\mathcal{H}}\cap DR(G)}PM^*(s,\tilde{s}) =$
  $\sum_{\mathcal{H}}\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[k](s)\sum_{\tilde{s}\in\widetilde{\mathcal{H}}\cap DR(G)}PM^*(s,\tilde{s}) =$
  $\sum_{\mathcal{H}}\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[k](s)\sum_{\tilde{s}\in\widetilde{\mathcal{H}}\cap DR(G)}\sum_{\{\Gamma\mid s\xrightarrow{\Gamma}\tilde{s}\}}PT^*(\Gamma,s) =$
  $\sum_{\mathcal{H}}\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[k](s)\sum_{\{\Gamma\mid\exists\tilde{s}\in\widetilde{\mathcal{H}}\cap DR(G)\ s\xrightarrow{\Gamma}\tilde{s}\}}PT^*(\Gamma,s) =$
  $\sum_{\mathcal{H}}\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[k](s)PM^*(s,\widetilde{\mathcal{H}}) = \sum_{\mathcal{H}}\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[k](s)PM^*(\mathcal{H},\widetilde{\mathcal{H}}) =$
  $\sum_{\mathcal{H}}PM^*(\mathcal{H},\widetilde{\mathcal{H}})\sum_{s\in\mathcal{H}\cap DR(G)}\psi^*[k](s) = \sum_{\mathcal{H}}PM^*(\mathcal{H},\widetilde{\mathcal{H}})\sum_{s'\in\mathcal{H}\cap DR(G')}\psi'^*[k](s') =$
  $\sum_{\mathcal{H}}\sum_{s'\in\mathcal{H}\cap DR(G')}\psi'^*[k](s')PM^*(\mathcal{H},\widetilde{\mathcal{H}}) = \sum_{\mathcal{H}}\sum_{s'\in\mathcal{H}'\cap DR(G')}\psi'^*[k](s')PM^*(s',\widetilde{\mathcal{H}}) =$
  $\sum_{\mathcal{H}}\sum_{s'\in\mathcal{H}\cap DR(G')}\psi'^*[k](s')\sum_{\{\Gamma\mid\exists\tilde{s}'\in\widetilde{\mathcal{H}}\cap DR(G')\ s'\xrightarrow{\Gamma}\tilde{s}'\}}PT^*(\Gamma,s') =$
  $\sum_{\mathcal{H}}\sum_{s'\in\mathcal{H}\cap DR(G')}\psi'^*[k](s')\sum_{\tilde{s}'\in\widetilde{\mathcal{H}}\cap DR(G')}\sum_{\{\Gamma\mid\exists\tilde{s}'\ s'\xrightarrow{\Gamma}\tilde{s}'\}}PT^*(\Gamma,s') =$
  $\sum_{\mathcal{H}}\sum_{s'\in\mathcal{H}\cap DR(G')}\psi'^*[k](s')\sum_{\tilde{s}'\in\widetilde{\mathcal{H}}\cap DR(G')}PM^*(s',\tilde{s}') =$
  $\sum_{s'\in DR(G')}\psi'^*[k](s')\sum_{\tilde{s}'\in\widetilde{\mathcal{H}}\cap DR(G')}PM^*(s',\tilde{s}') = \sum_{s'\in DR(G')}\sum_{\tilde{s}'\in\widetilde{\mathcal{H}}\cap DR(G')}\psi'^*[k](s')PM^*(s',\tilde{s}') =$
  $\sum_{\tilde{s}'\in\widetilde{\mathcal{H}}\cap DR(G')}\sum_{s'\in DR(G')}\psi'^*[k](s')PM^*(s',\tilde{s}') = \sum_{\tilde{s}'\in\widetilde{\mathcal{H}}\cap DR(G')}\psi'^*[k+1](\tilde{s}')$. ☐

C Proof of Theorem 7.1

The main idea of the proof is similar to that from [15, 16] but in the algebraic setting.

Let $\mathcal{H} \in (DR(G)\cup DR(G'))/_{\mathcal{R}}$ and $s, \bar{s} \in \mathcal{H}$. We have $\forall \widetilde{\mathcal{H}} \in (DR(G)\cup DR(G'))/_{\mathcal{R}}$ $\forall A \in I\!N_f^{\mathcal{L}}$ $s \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}} \Leftrightarrow \bar{s} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}}$. The previous equality is valid for all $s, \bar{s} \in \mathcal{H}$, hence, we can rewrite it as $\mathcal{H} \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \widetilde{\mathcal{H}}$ and denote $PM_A^*(\mathcal{H},\widetilde{\mathcal{H}}) = PM_A^*(s,\widetilde{\mathcal{H}}) = PM_A^*(\bar{s},\widetilde{\mathcal{H}})$. Note that transitions from the states of $DR(G)$ always lead to those from the same set, hence, $\forall s \in DR(G)$ $PM_A^*(s,\widetilde{\mathcal{H}}) = PM_A^*(s,\widetilde{\mathcal{H}}\cap DR(G))$. The same is true for $DR(G')$.

Let $\Sigma = A_1\cdots A_n$ be a step trace of G and G'. We have $\exists\mathcal{H}_0,\ldots,\exists\mathcal{H}_n \in (DR(G)\cup DR(G'))/_{\mathcal{R}}$ $\mathcal{H}_0 \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{Q}_1} \mathcal{H}_1 \xrightarrow{A_2}\!\!\!\!\rightarrow_{\mathcal{Q}_2} \cdots \xrightarrow{A_n}\!\!\!\!\rightarrow_{\mathcal{Q}_n} \mathcal{H}_n$. Now we intend to prove that the sum of probabilities of all the paths starting in every $s_0 \in \mathcal{H}_0$ and going through the states from $\mathcal{H}_1,\ldots,\mathcal{H}_n$ is equal to the product of $\mathcal{Q}_1,\ldots,\mathcal{Q}_n$:

$$\sum_{\{\Gamma_1,\ldots,\Gamma_n\mid s_0\xrightarrow{\Gamma_1}\cdots\xrightarrow{\Gamma_n}s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i\in\mathcal{H}_i\ (1\le i\le n)\}}\prod_{i=1}^n PT^*(\Gamma_i,s_{i-1}) = \prod_{i=1}^n PM_{A_i}^*(\mathcal{H}_{i-1},\mathcal{H}_i).$$

We prove this equality by induction on the step trace length n.

- $n = 0$

  $\sum_{\{\Gamma_1 | s_0 \stackrel{\Gamma_1}{\rightarrow} s_1,\ \mathcal{L}(\Gamma_1)=A_1,\ s_1 \in \mathcal{H}_1\}} PT^*(\Gamma_1, s_0) = PM^*_{A_1}(s_0, \mathcal{H}_1) = PM^*_{A_1}(\mathcal{H}_0, \mathcal{H}_1)$.

- $n \rightarrow n+1$

  $\sum_{\{\Gamma_1,\dots,\Gamma_n,\Gamma_{n+1} | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n \stackrel{\Gamma_{n+1}}{\rightarrow} s_{n+1},\ \mathcal{L}(\Gamma_i)=A_i,\ s_i \in \mathcal{H}_i\ (1 \leq i \leq n+1)\}} \prod_{i=1}^{n+1} PT^*(\Gamma_i, s_{i-1}) =$
  $\sum_{\{\Gamma_{n+1} | s_n \stackrel{\Gamma_{n+1}}{\rightarrow} s_{n+1},\ \mathcal{L}(\Gamma_{n+1})=A_{n+1},\ s_n \in \mathcal{H}_n,\ s_{n+1} \in \mathcal{H}_{n+1}\}} \sum_{\{\Gamma_1,\dots,\Gamma_n | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i \in \mathcal{H}_i\ (1 \leq i \leq n)\}}$
  $\prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}) PT^*(\Gamma_{n+1}, s_n) =$
  $\sum_{\{\Gamma_1,\dots,\Gamma_n | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i \in \mathcal{H}_i\ (1 \leq i \leq n)\}}$
  $\left[\prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}) \sum_{\{\Gamma_{n+1} | s_n \stackrel{\Gamma_{n+1}}{\rightarrow} s_{n+1},\ \mathcal{L}(\Gamma_{n+1})=A_{n+1},\ s_n \in \mathcal{H}_n,\ s_{n+1} \in \mathcal{H}_{n+1}\}} PT^*(\Gamma_{n+1}, s_n)\right] =$
  $\sum_{\{\Gamma_1,\dots,\Gamma_n | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i \in \mathcal{H}_i\ (1 \leq i \leq n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}) PM^*_{A_{n+1}}(s_n, \mathcal{H}_{n+1}) =$
  $\sum_{\{\Gamma_1,\dots,\Gamma_n | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i \in \mathcal{H}_i\ (1 \leq i \leq n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}) PM^*_{A_{n+1}}(\mathcal{H}_n, \mathcal{H}_{n+1}) =$
  $PM^*_{A_{n+1}}(\mathcal{H}_n, \mathcal{H}_{n+1}) \sum_{\{\Gamma_1,\dots,\Gamma_n | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i \in \mathcal{H}_i\ (1 \leq i \leq n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}) =$
  $PM^*_{A_{n+1}}(\mathcal{H}_n, \mathcal{H}_{n+1}) \prod_{i=1}^{n} PM^*_{A_i}(\mathcal{H}_{i-1}, \mathcal{H}_i) = \prod_{i=1}^{n+1} PM^*_{A_i}(\mathcal{H}_{i-1}, \mathcal{H}_i)$.

Let $s_0, \bar{s}_0 \in \mathcal{H}_0$. We have
$PM^*(A_1 \cdots A_n, s_0) = \sum_{\{\Gamma_1,\dots,\Gamma_n | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ (1 \leq i \leq n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}) =$
$\sum_{\mathcal{H}_1,\dots,\mathcal{H}_n} \sum_{\{\Gamma_1,\dots,\Gamma_n | s_0 \stackrel{\Gamma_1}{\rightarrow} \cdots \stackrel{\Gamma_n}{\rightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i,\ s_i \in \mathcal{H}_i\ (1 \leq i \leq n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}) =$
$\sum_{\mathcal{H}_1,\dots,\mathcal{H}_n} \prod_{i=1}^{n} PM^*_{A_i}(\mathcal{H}_{i-1}, \mathcal{H}_i) =$
$\sum_{\mathcal{H}_1,\dots,\mathcal{H}_n} \sum_{\{\overline{\Gamma}_1,\dots,\overline{\Gamma}_n | \bar{s}_0 \stackrel{\overline{\Gamma}_1}{\rightarrow} \cdots \stackrel{\overline{\Gamma}_n}{\rightarrow} \bar{s}_n,\ \mathcal{L}(\overline{\Gamma}_i)=A_i,\ \bar{s}_i \in \mathcal{H}_i\ (1 \leq i \leq n)\}} \prod_{i=1}^{n} PT^*(\overline{\Gamma}_i, \bar{s}_{i-1}) =$
$\sum_{\{\overline{\Gamma}_1,\dots,\overline{\Gamma}_n | \bar{s}_0 \stackrel{\overline{\Gamma}_1}{\rightarrow} \cdots \stackrel{\overline{\Gamma}_n}{\rightarrow} \bar{s}_n,\ \mathcal{L}(\overline{\Gamma}_i)=A_i,\ (1 \leq i \leq n)\}} \prod_{i=1}^{n} PT^*(\overline{\Gamma}_i, \bar{s}_{i-1}) = PM^*(A_1 \cdots A_n, \bar{s}_0)$.
Since we have the previous equality for all $s_0, \bar{s}_0 \in \mathcal{H}_0$, we can denote $PM^*(A_1 \cdots A_n, \mathcal{H}_0) = PM^*(A_1 \cdots A_n, s_0) = PM^*(A_1 \cdots A_n, \bar{s}_0)$.

By Proposition 7.1, $\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} {\psi'}^*(s')$. Now we can complete the proof:
$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) PT^*(\Sigma, s) = \sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) PT^*(\Sigma, \mathcal{H}) = PT^*(\Sigma, \mathcal{H}) \sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) =$
$PT^*(\Sigma, \mathcal{H}) \sum_{s' \in \mathcal{H} \cap DR(G')} {\psi'}^*(s') = \sum_{s' \in \mathcal{H} \cap DR(G')} {\psi'}^*(s') PT^*(\Sigma, \mathcal{H}) =$
$\sum_{s' \in \mathcal{H} \cap DR(G')} {\psi'}^*(s') PT^*(\Sigma, s')$. $\square$