Algebra $dtsPBC$: a discrete time stochastic extension of Petri box calculus

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Abstract: In [MVF01], a continuous time stochastic extension $sPBC$ of finite Petri box calculus PBC [BDH92] was proposed. In [MVCC03], iteration operator was added to $sPBC$.

Algebra $sPBC$ has an interleaving semantics, but PBC has a step one.

We constructed a discrete time stochastic extension $dtsPBC$ of finite PBC [Tar05] and enriched it with iteration [Tar06].

The step operational semantics is defined in terms of labeled probabilistic transition systems.

The denotational semantics is defined in terms of a subclass of labeled DTSPNs (LDTSPNs) called discrete time stochastic Petri boxes (dts-boxes).

We propose a variety of stochastic equivalences and investigate their interrelations.

It is explained how to use the equivalences for transition systems reduction.

A logical characterization of the equivalences is presented via probabilistic modal logics.

We demonstrate how to apply the equivalences to compare stationary behaviour.

A congruence relation is defined. The case studies of performance evaluation are presented.

Keywords: stochastic Petri nets, stochastic process algebras, Petri box calculus, iteration, discrete time, transition systems, operational semantics, dts-boxes, denotational semantics, empty loops, stochastic equivalences, reduction, modal logics, stationary behaviour, congruence, performance evaluation.

Contents

Introduction

Previous work

- Continuous time (subsets of $\mathbb{R}_+$): interleaving semantics
  - *Continuous time stochastic Petri nets (CTSPNs)* [Mol82,FN85]:

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    Continuous time Markov chain (CTMC).
  - *Generalized stochastic Petri nets (GSPNs)* [MCB84,CMBC93]:

    exponential and zero transition firing delays,

    Semi-Markov chain (SMC).
- Discrete time (subsets of $\mathbb{N}$): step semantics
  - *Discrete time stochastic Petri nets (DTSPNs)* [Mol85,ZG94]:

    geometric transition firing delays,

    Discrete time Markov chain (DTMC).

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- GSPNs [Buc98]
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- $PEPA$ [Hil96]
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Algebra PBC and its extensions

- *Petri box calculus* PBC [BDH92]
- *Time Petri box calculus* $tPBC$ [Kou00]
- *Timed Petri box calculus* $TPBC$ [MF00]
- *Stochastic Petri box calculus* $sPBC$ [MVF01,MVCC03]
- *Ambient Petri box calculus* $APBC$ [FM03]
- *Arc time Petri box calculus* $atPBC$ [Nia05]
- *Generalized stochastic Petri box calculus* $gsPBC$ [MVCR08]
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Stochastic equivalences

- Probabilistic transition systems (PTSs) [BM89,Chr90,LS91,BHe97,KN98]
- CTMCs [HR94,Hil94]
- Markov process algebras (MPAs) [Buc94,BKe01]
- CTSPNs [Buc95]
- GSPNs [Buc98]
- Stochastic automata (SAs) [Buc99]
- Stochastic event structures (SESs) [MCW03]

Syntax

The *set of all finite multisets* over X is $\mathbb{N}_f^X$.

The *set of all subsets* of X is 2^X.

$Act = \{a, b, \ldots\}$ is the set of *elementary actions*.

$\widehat{Act} = \{\hat{a}, \hat{b}, \ldots\}$ is the set of *conjugated actions (conjugates)* s.t. $a \neq \hat{a}$ and $\hat{\hat{a}} = a$.

$\mathcal{A} = Act \cup \widehat{Act}$ is the set of *all actions*.

$\mathcal{L} = \mathbb{N}_f^{\mathcal{A}}$ is the set of *all multiactions*.

The *alphabet* of $\alpha \in \mathcal{L}$ is $\mathcal{A}(\alpha) = \{x \in \mathcal{A} \mid \alpha(x) > 0\}$.

An *activity (stochastic multiaction)* is a pair (α, ρ), where
$\alpha \in \mathcal{L}$ and $\rho \in (0; 1)$ is the probability of multiaction α.

$\mathcal{SL}$ is the set of *all activities*.

The *alphabet* of $(\alpha, \rho) \in \mathcal{SL}$ is $\mathcal{A}(\alpha, \rho) = \mathcal{A}(\alpha)$.

The operations: *sequential execution* $;$, *choice* $[]$, *parallelism* $\|$, *relabeling* $[f]$, *restriction* rs, *synchronization* sy and *iteration* $[**]$.

Sequential execution and choice have the standard interpretation.

Parallelism does not include synchronization unlike that in standard process algebras.

Relabeling functions $f : \mathcal{A} \to \mathcal{A}$ are bijections preserving conjugates: $\forall x \in \mathcal{A}\ f(\hat{x}) = \widehat{f(x)}$. For $\alpha \in \mathcal{L}$, let $f(\alpha) = \sum_{x\in\alpha} f(x)$.

Restriction over an action a: any process behaviour containing a or its conjugate $\hat{a}$ is not allowed.

Let $\alpha, \beta \in \mathcal{L}$ be two multiactions s.t. for $a \in Act$ we have $a \in \alpha$ and $\hat{a} \in \beta$ or $\hat{a} \in \alpha$ and $a \in \beta$. Synchronization of α and β by a is $\alpha \oplus_a \beta = \gamma$:

$$\gamma(x) = \begin{cases} \alpha(x) + \beta(x) - 1, & x = a \text{ or } x = \hat{a}; \\ \alpha(x) + \beta(x), & \text{otherwise}. \end{cases}$$

In the iteration, the initialization subprocess is executed first,

then the body one is performed zero or more times, finally, the termination one is executed.

Static expressions specify the structure of processes.

Definition 1 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$. *A* static expression *of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}E{*}E].$$

$StatExpr$ is the set of *all static expressions* of $dtsPBC$.

Definition 2 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$. *A* regular static expression *of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}D{*}E],$$

where $D ::= \ (\alpha, \rho) \mid D;E \mid D[]D \mid D[f] \mid D \text{ rs } a \mid D \text{ sy } a \mid [D{*}D{*}E].$

$RegStatExpr$ is the set of *all regular static expressions* of $dtsPBC$.

Dynamic expressions specify the states of processes.

Dynamic expressions are combined from static ones annotated with upper or lower bars.

The *underlying static expression* of a dynamic one: removing all upper and lower bars.

Definition 3 *Let $a \in Act$ and $E \in StatExpr$. A dynamic expression of $dtsPBC$ is*

$$G ::= \overline{E} \mid \underline{E} \mid G;E \mid E;G \mid G[]E \mid E[]G \mid G\|G \mid G[f] \mid G \text{ rs } a \mid G \text{ sy } a \mid$$
$$[G{*}E{*}E] \mid [E{*}G{*}E] \mid [E{*}E{*}G].$$

$DynExpr$ is the set of *all dynamic expressions* of $dtsPBC$.

A *regular dynamic expression*: its underlying static expression is regular.

$RegDynExpr$ is the set of *all regular dynamic expressions* of $dtsPBC$.

We shall consider regular expressions only and omit the word “regular”.

Operational semantics

Inaction rules

Inaction rules: execution of the empty multiset of activities.

Let $E, F, K \in RegStatExpr$ and $a \in Act$.

Inaction rules for overlined and underlined static expressions

$$\overline{E;F} \xrightarrow{\emptyset} \overline{E};F \qquad \underline{E};F \xrightarrow{\emptyset} E;\overline{F} \qquad E;\underline{F} \xrightarrow{\emptyset} \underline{E;F}$$

$$\overline{E[]F} \xrightarrow{\emptyset} \overline{E}[]F \qquad \overline{E[]F} \xrightarrow{\emptyset} E[]\overline{F} \qquad \underline{E}[]F \xrightarrow{\emptyset} \underline{E[]F}$$

$$E[]\underline{F} \xrightarrow{\emptyset} \underline{E[]F} \qquad \overline{E\|F} \xrightarrow{\emptyset} \overline{E}\|\overline{F} \qquad \underline{E}\|\underline{F} \xrightarrow{\emptyset} \underline{E\|F}$$

$$\overline{E[f]} \xrightarrow{\emptyset} \overline{E}[f] \qquad \underline{E}[f] \xrightarrow{\emptyset} \underline{E[f]} \qquad \overline{E \text{ rs } a} \xrightarrow{\emptyset} \overline{E} \text{ rs } a$$

$$\underline{E} \text{ rs } a \xrightarrow{\emptyset} \underline{E \text{ rs } a} \qquad \overline{E \text{ sy } a} \xrightarrow{\emptyset} \overline{E} \text{ sy } a \qquad \underline{E} \text{ sy } a \xrightarrow{\emptyset} \underline{E \text{ sy } a}$$

$$\overline{[E*F*K]} \xrightarrow{\emptyset} [\overline{E}*F*K] \qquad [\underline{E}*F*K] \xrightarrow{\emptyset} [E*\overline{F}*K] \qquad [E*\underline{F}*K] \xrightarrow{\emptyset} [E*\overline{F}*K]$$

$$[E*\underline{F}*K] \xrightarrow{\emptyset} [E*F*\overline{K}] \qquad [E*F*\underline{K}] \xrightarrow{\emptyset} \underline{[E*F*K]}$$

Let $E, F \in RegStatExpr,\ G, H, \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

Inaction rules for arbitrary dynamic expressions

$$G \stackrel{\emptyset}{\rightarrow} G \qquad \frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{G\circ E\stackrel{\emptyset}{\rightarrow}\widetilde{G}\circ E} \qquad \frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{E\circ G\stackrel{\emptyset}{\rightarrow}E\circ\widetilde{G}} \qquad \frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G}}{G\|H\stackrel{\emptyset}{\rightarrow}\widetilde{G}\|H}$$

$$\frac{H\stackrel{\emptyset}{\rightarrow}\widetilde{H}}{G\|H\stackrel{\emptyset}{\rightarrow}G\|\widetilde{H}} \qquad \frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G}}{G[f]\stackrel{\emptyset}{\rightarrow}\widetilde{G}[f]} \qquad \frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G},\ \circ\in\{\text{rs},\text{sy}\}}{G\circ a\stackrel{\emptyset}{\rightarrow}\widetilde{G}\circ a} \qquad \frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G}}{[G*E*F]\stackrel{\emptyset}{\rightarrow}[\widetilde{G}*E*F]}$$

$$\frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G}}{[E*G*F]\stackrel{\emptyset}{\rightarrow}[E*\widetilde{G}*F]} \qquad \frac{G\stackrel{\emptyset}{\rightarrow}\widetilde{G}}{[E*F*G]\stackrel{\emptyset}{\rightarrow}[E*F*\widetilde{G}]}$$

An *operative regular dynamic expression* G: no inaction rule can be applied to it, with the exception of $G \stackrel{\emptyset}{\rightarrow} G$.

$OpRegDynExpr$ is the set of *all operative regular dynamic expressions* of $dtsPBC$.

Definition 4 *$\approx\ = (\stackrel{\emptyset}{\rightarrow} \cup \stackrel{\emptyset}{\leftarrow})^*$ is the structural equivalence of dynamic expressions in $dtsPBC$. G and G' are* structurally equivalent, *$G\approx G'$, if they can be reached each from other by applying inaction rules in forward or backward direction.*

Action rules

Action rules: execution of non-empty multisets of activities.

For $\Gamma \in I\!N_f^{\mathcal{SL}}$, let $f(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma}(f(\alpha), \rho)$.

The *alphabet* of $\Gamma \in I\!N_f^{\mathcal{SL}}$ is $\mathcal{A}(\Gamma) = \cup_{(\alpha,\rho)\in\Gamma}\mathcal{A}(\alpha)$.

Let $(\alpha, \rho), (\beta, \chi) \in \mathcal{SL},\ E, F \in RegStatExpr,\ G, H \in OpRegDynExpr,$ $\widetilde{G}, \widetilde{H} \in RegDynExpr,\ a \in Act$ and $\Gamma, \Delta \in I\!N_f^{\mathcal{SL}} \setminus \{\emptyset\}$.

Action rules

$$\textbf{B}\ \ \overline{(\alpha,\rho)} \stackrel{\{(\alpha,\rho)\}}{\longrightarrow} \underline{(\alpha,\rho)} \qquad \textbf{SC1}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G},\ \circ \in \{;, []\}}{G \circ E \stackrel{\Gamma}{\to} \widetilde{G} \circ E} \qquad \textbf{SC2}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G},\ \circ \in \{;, []\}}{E \circ G \stackrel{\Gamma}{\to} E \circ \widetilde{G}}$$

$$\textbf{P1}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G}}{G \| H \stackrel{\Gamma}{\to} \widetilde{G} \| H} \qquad \textbf{P2}\ \ \frac{H \stackrel{\Gamma}{\to} \widetilde{H}}{G \| H \stackrel{\Gamma}{\to} G \| \widetilde{H}} \qquad \textbf{P3}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G},\ H \stackrel{\Delta}{\to} \widetilde{H}}{G \| H \stackrel{\Gamma+\Delta}{\longrightarrow} \widetilde{G} \| \widetilde{H}}$$

$$\textbf{L}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G}}{G[f] \stackrel{f(\Gamma)}{\longrightarrow} \widetilde{G}[f]} \qquad \textbf{Rs}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G},\ a, \hat{a} \notin \mathcal{A}(\Gamma)}{G \text{ rs } a \stackrel{\Gamma}{\to} \widetilde{G} \text{ rs } a} \qquad \textbf{I1}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G}}{[G * E * F] \stackrel{\Gamma}{\to} [\widetilde{G} * E * F]}$$

$$\textbf{I2}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G}}{[E * G * F] \stackrel{\Gamma}{\to} [E * \widetilde{G} * F]} \qquad \textbf{I3}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G}}{[E * F * G] \stackrel{\Gamma}{\to} [E * F * \widetilde{G}]} \qquad \textbf{Sy1}\ \ \frac{G \stackrel{\Gamma}{\to} \widetilde{G}}{G \text{ sy } a \stackrel{\Gamma}{\to} \widetilde{G} \text{ sy } a}$$

$$\textbf{Sy2}\ \ \frac{G \text{ sy } a \stackrel{\Gamma+\{(\alpha,\rho)\}+\{(\beta,\chi)\}}{\longrightarrow} \widetilde{G} \text{ sy } a,\ a \in \alpha,\ \hat{a} \in \beta}{G \text{ sy } a \stackrel{\Gamma+\{(\alpha \oplus_a \beta, \rho \cdot \chi)\}}{\longrightarrow} \widetilde{G} \text{ sy } a}$$

Transition systems

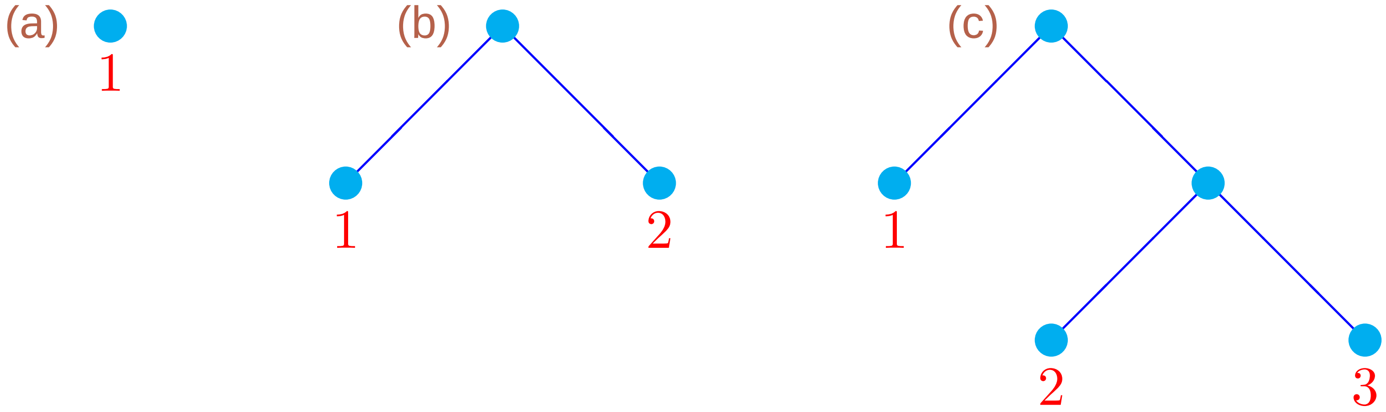
Definition 5 *Let $\iota \in I\!N$. The numbering of expressions is*

$$\iota ::= \iota \mid (\iota)(\iota).$$

Num is the set of *all numberings* of expressions.

The *content* of a numbering $\iota \in Num$ is

$$Cont(\iota) = \begin{cases} \{\iota\}, & \iota \in I\!N; \\ Cont(\iota_1) \cup Cont(\iota_2), & \iota = (\iota_1)(\iota_2). \end{cases}$$



The binary trees encoded with the numberings $1,\ (1)(2)$ and $(1)((2)(3))$

$[G]_\approx = \{H \mid G \approx H\}$ is the equivalence class of G w.r.t. structural equivalence.

Definition 6 *The derivation set $DR(G)$ of a dynamic expression G is the minimal set:*

- $[G]_\approx \in DR(G)$;
- *if* $[H]_\approx \in DR(G)$ *and* $\exists\Gamma\ H \stackrel{\Gamma}{\rightarrow} \widetilde{H}$ *then* $[\widetilde{H}]_\approx \in DR(G)$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

The set of *all multisets of activities executable from s* is $Exec(s) = \{\Gamma \mid \exists H \in s\ \exists\widetilde{H}\ H \stackrel{\Gamma}{\rightarrow} \widetilde{H}\}$.

Let $\Gamma \in Exec(s) \setminus \{\emptyset\}$. The *probability that the multiset of activities Γ is ready for execution in s*:

$$PF(\Gamma, s) = \prod_{(\alpha,\rho)\in\Gamma} \rho \cdot \prod_{\{\{(\beta,\chi)\}\in Exec(s) \mid (\beta,\chi)\not\in\Gamma\}} (1-\chi).$$

In the case $\Gamma = \emptyset$ we define

$$PF(\emptyset, s) = \begin{cases} \prod_{\{(\beta,\chi)\}\in Exec(s)} (1-\chi), & Exec(s) \neq \{\emptyset\}; \\ 1, & \text{otherwise}. \end{cases}$$

Let $\Gamma \in Exec(s)$. The *probability to execute the multiset of activities* Γ *in* s:

$$PT(\Gamma, s) = \frac{PF(\Gamma, s)}{\sum_{\Delta \in Exec(s)} PF(\Delta, s)}.$$

The *probability to move from* s *to* $\tilde{s}$ *by executing any multiset of activities*:

$$PM(s, \tilde{s}) = \sum_{\{\Gamma \mid \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}} PT(\Gamma, s).$$

Definition 7 *The (labeled probabilistic) transition system of a dynamic expression G is* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, *where*

- *the set of states is* $S_G = DR(G)$;
- *the set of labels is* $L_G \subseteq \mathbb{N}_f^{\mathcal{SL}} \times (0; 1]$;
- *the set of transitions is* $\mathcal{T}_G = \{(s, (\Gamma, PT(\Gamma, s)), \tilde{s}) \mid s \in DR(G),\ \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}$;
- *the initial state is* $s_G = [G]_{\approx}$.

A transition $(s, (\Gamma, \mathcal{P}), \tilde{s}) \in \mathcal{T}_G$ is written as $s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s}$.

We write $s \xrightarrow{\Gamma} \tilde{s}$ if $\exists \mathcal{P}\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s}$ and $s \rightarrow \tilde{s}$ if $\exists \Gamma\ s \xrightarrow{\Gamma} \tilde{s}$.

Definition 8 *Let G, G' be dynamic expressions and* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, $TS(G') = (S_{G'}, L_{G'}, \mathcal{T}_{G'}, s_{G'})$ *be their transition systems. A mapping $\beta : S_G \rightarrow S_{G'}$ is an isomorphism between $TS(G)$ and $TS(G')$,* $\beta : TS(G) \simeq TS(G')$, *if*

1. β *is a bijection s.t.* $\beta(s_G) = s_{G'}$;
2. $\forall s, \tilde{s} \in S_G\ \forall \Gamma\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \xrightarrow{\Gamma}_{\mathcal{P}} \beta(\tilde{s})$.

$TS(G)$ *and* $TS(G')$ *are isomorphic,* $TS(G) \simeq TS(G')$, *if* $\exists \beta : TS(G) \simeq TS(G')$.

For $E \in RegStatExpr$, let $TS(E) = TS(\overline{E})$.

Definition 9 *G and G' are equivalent w.r.t. transition systems, $G =_{ts} G'$, if $TS(G) \simeq TS(G')$.*

Definition 10 *The underlying discrete time Markov chain (DTMC) of a dynamic expression G, $DTMC(G)$, has the state space $DR(G)$ and transitions $s \rightarrow_{\mathcal{P}} \tilde{s}$, if $s \rightarrow \tilde{s}$ and $\mathcal{P} = PM(s, \tilde{s})$.*

For $E \in RegStatExpr$, let $DTMC(E) = DTMC(\overline{E})$.

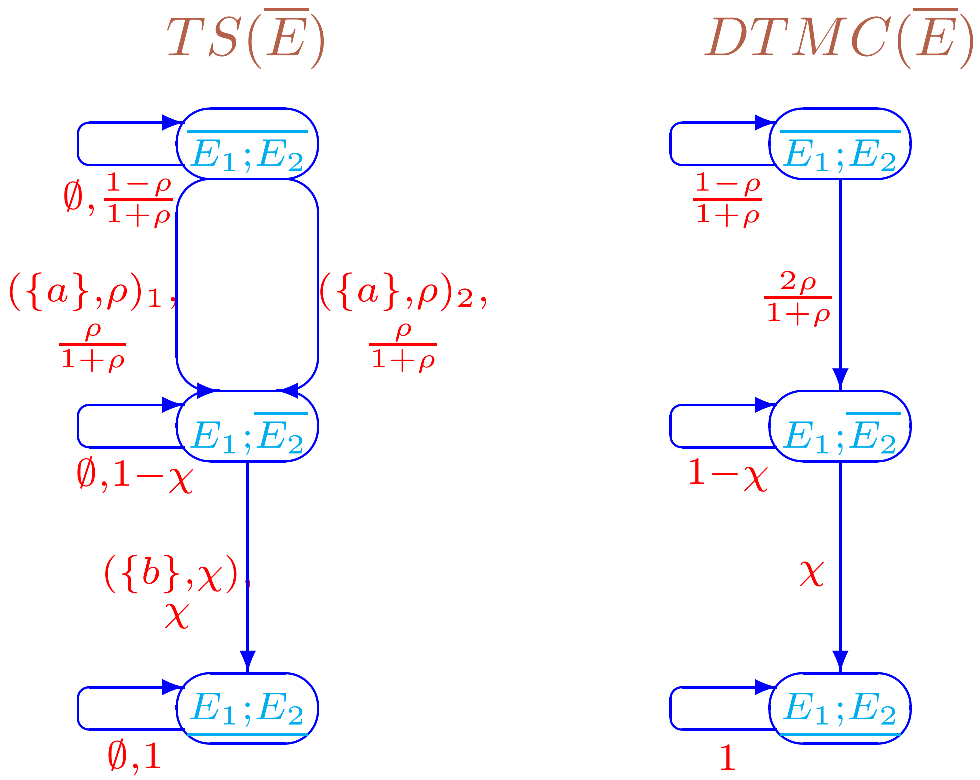
For a dynamic expression G, a discrete random variable is associated with every state of $DTMC(G)$.

The random values (residence time in the states) are geometrically distributed:
the probability to stay in the state $s \in DR(G)$ for $k-1$ moments and leave it at moment $k \geq 1$ is $PM(s,s)^{k-1}(1 - PM(s,s))$.

The mean value formula: the *average sojourn time in the state s* is

$$SJ(s) = \frac{1}{1 - PM(s,s)}.$$

The *average sojourn time vector* SJ of G is that with the elements $SJ(s),\ s \in DR(G)$.

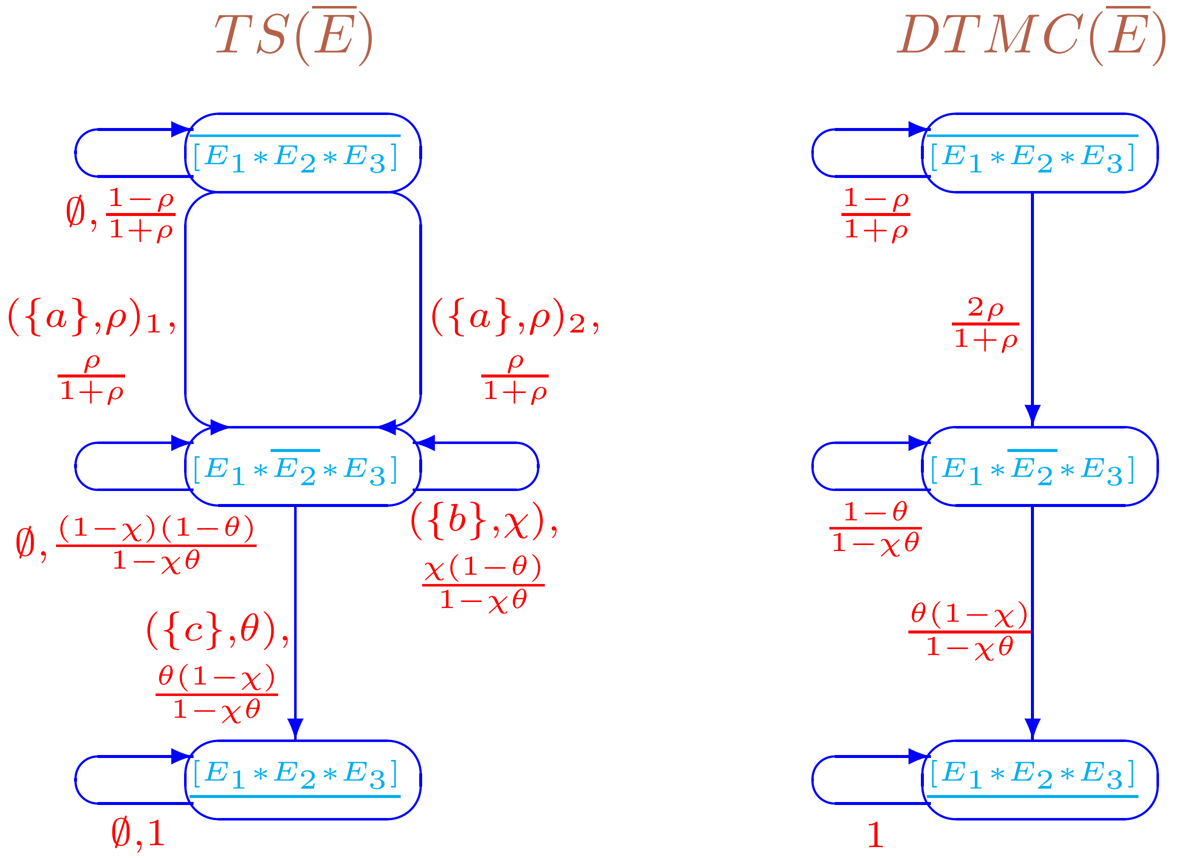


The transition system and the underlying DTMC of $\overline{E}$ for $E = ((\{a\},\rho)_1[](\{a\},\rho)_2);(\{b\},\chi)$

Let $E_1 = (\{a\},\rho)[](\{a\},\rho),\ E_2 = (\{b\},\chi)$ and $E = E_1;E_2$.

The identical activities of the composite static expression are enumerated as:

$E = ((\{a\},\rho)_1[](\{a\},\rho)_2);(\{b\},\chi)$.



EXPRIT:The transition system and the underlying DTMC of $\overline{E}$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$

Let $E_1 = (\{a\},\rho)[](\{a\},\rho),\ E_2 = (\{b\},\chi),\ E_3 = (\{c\},\theta)$ and $E = [E_1 * E_2 * E_3]$.

The identical activities of the composite static expression are enumerated as:

$E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$.

$DR(\overline{E})$ consists of $s_1 = [[\overline{E_1 * E_2 * E_3}]]_\approx,\ s_2 = [[E_1 * \overline{E_2} * E_3]]_\approx,\ s_3 = [[\underline{E_1 * E_2 * E_3}]]_\approx$.

The average sojourn time vector is $SJ = \left(\frac{1+\rho}{2\rho}, \frac{1-\chi\theta}{\theta(1-\chi)}, \infty\right)$.

Denotational semantics

Labeled DTSPNs

Definition 11 *A* labeled discrete time stochastic Petri net (LDTSPN) *is a tuple* $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$*:*

- P_N *and* T_N *are finite sets of* places *and* transitions *(*$P_N \cup T_N \neq \emptyset,\ P_N \cap T_N = \emptyset$*);*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to I\!N$ *is the* arc weight *function;*
- $\Omega_N : T_N \to (0;1)$ *is the* transition probability *function;*
- $L_N : T_N \to \mathcal{L}$ *is the* transition labeling *function;*
- $M_N \in I\!N_f^{P_N}$ *is the* initial marking*.*

Concurrent transition firings at discrete time moments.

LDTSPNs have *step* semantics.

Let M be a marking of a LDTSPN $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$. Then $t \in Ena(M)$ fires in the next time moment with probability $\Omega_N(t)$, if no other transition is enabled in M.

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U is ready for firing in M*:

$$PF(U, M) = \prod_{t \in U} \Omega_N(t) \cdot \prod_{u \in Ena(M) \setminus U} (1 - \Omega_N(u)).$$

In the case $U = \emptyset$ we define

$$PF(\emptyset, M) = \begin{cases} \prod_{u \in Ena(M)} (1 - \Omega_N(u)) & Ena(M) \neq \emptyset; \\ 1 & \text{otherwise}. \end{cases}$$

Let $U \subseteq Ena(M),\ U \neq \emptyset$ and ${}^{\bullet}U \subseteq M$. The *probability that the set of transitions U fires in M*:

$$PT(U, M) = \frac{PF(U, M)}{\sum_{\{V \mid {}^{\bullet}V \subseteq M\}} PF(V, M)}.$$

If $U = \emptyset$ then $M = \widetilde{M}$ and

$$PT(\emptyset, M) = \frac{PF(\emptyset, M)}{\sum_{\{V \mid {}^{\bullet}V \subseteq M\}} PF(V, M)}.$$

Firing of U changes marking M to $\widetilde{M} = M - {}^{\bullet}U + U^{\bullet}$, $M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT(U, M)$.

We write $M \stackrel{U}{\rightarrow} \widetilde{M}$ if $\exists \mathcal{P}\ M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \rightarrow \widetilde{M}$ if $\exists U\ M \stackrel{U}{\rightarrow} \widetilde{M}$.

For $U = \{t\}$ we write $M \stackrel{t}{\rightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \stackrel{t}{\rightarrow} \widetilde{M}$.

Definition 12 *Let N be an LDTSPN.*

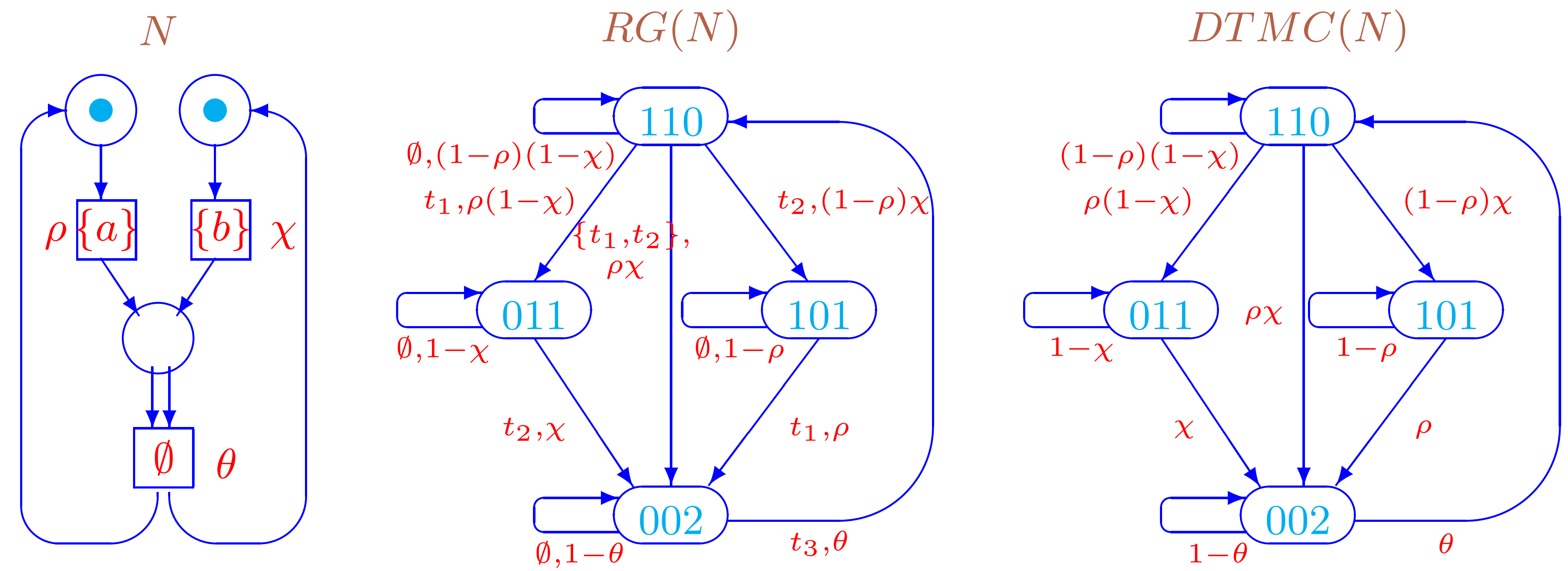
- *The* reachability set $RS(N)$ *is the minimal set of markings s.t.*
  - $M_N \in RS(N)$;
  - *if* $M \in RS(N)$ *and* $M \to \widetilde{M}$ *then* $\widetilde{M} \in RS(N)$.
- *The* reachability graph $RG(N)$ *is a directed labeled graph with*
  - *the set of nodes* $RS(N)$;
  - *an arc labeled by* $(U, \mathcal{P})$ *between nodes* M *and* $\widetilde{M}$ *if* $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$.
- *The* underlying Discrete Time Markov Chain (DTMC) $DTMC(N)$ *is a DTMC with*
  - *the state space* $RS(N)$;
  - *a transition* $M \rightarrow_{\mathcal{P}} \widetilde{M}$, *where* $\mathcal{P} = PM(M, \widetilde{M})$ *is the* probability to move from M to $\widetilde{M}$ by firing any set of transitions*:*

$$PM(M, \widetilde{M}) = \sum_{\{U \mid M \xrightarrow{U} \widetilde{M}\}} PT(U, M).$$

Let N be an LDTSPN and $M \in RS(N)$. The *average sojourn time in the marking M* is

$$SJ(M) = \frac{1}{1 - PM(M, M)}.$$

The *average sojourn time vector SJ* of N is that with the elements $SJ(M),\ M \in RS(N)$.



LDTSPN, its reachability graph and the underlying DTMC

The transitions are t_1 (labeled by $\{a\}$), t_2 (labeled by $\{b\}$) and t_3 (labeled by $\emptyset$).

The transition probabilities are $\rho = \Omega_N(t_1),\ \chi = \Omega_N(t_2),\ \theta = \Omega_N(t_3)$.

$RS(N)$ consists of $M_1 = (1,1,0),\ M_2 = (0,1,1),\ M_3 = (1,0,1),\ M_4 = (0,0,2)$.

The average sojourn time vector for $DTMC(N)$ is

$$SJ = \left(\frac{1}{\rho+\chi-\rho\chi}, \frac{1}{\chi}, \frac{1}{\rho}, \frac{1}{\theta}\right).$$

The elements $\mathcal{P}_{ij}(1 \le i, j \le 4)$ of (one-step) transition probability matrix (TPM) of $DTMC(N)$ are

$$\mathcal{P}_{ij} = \begin{cases} PM(s_i, s_j) & s_i \to s_j; \\ 0 & \text{otherwise}. \end{cases}$$

The (one-step) TPM is

$$\mathbf{P} = \begin{bmatrix} (1-\rho)(1-\chi) & \rho(1-\chi) & \chi(1-\rho) & \rho\chi \\ 0 & 1-\chi & 0 & \chi \\ 0 & 0 & 1-\rho & \rho \\ \theta & 0 & 0 & 1-\theta \end{bmatrix}$$

The steady-state PMF ψ is the solution of

$$\begin{cases} \psi(\mathbf{P}-\mathbf{E})=\mathbf{0} \\ \psi\mathbf{1}^T=1 \end{cases},$$

where **E** is the unitary matrix of dimension four and $\mathbf{0}=(0,0,0,0),\ \mathbf{1}=(1,1,1,1)$.

For $\rho=\chi=\theta$

$$\psi=\left(\frac{1}{5-3\rho},\frac{1-\rho}{5-3\rho},\frac{1-\rho}{5-3\rho},\frac{2-\rho}{5-3\rho}\right).$$

The inverse of the steady-state PMF is the mean recurrence time vector

$$RC=\left(5-3\rho,\frac{5-3\rho}{1-\rho},\frac{5-3\rho}{1-\rho},\frac{5-3\rho}{2-\rho}\right).$$

The average time to come back to the initial marking $M_N=M_1$ in the long-term behaviour is in $(2;5)$.

Algebra of dts-boxes

Definition 13 *A* discrete time stochastic Petri box (dts-box) *is* $N = (P_N, T_N, W_N, \Lambda_N)$, *where*

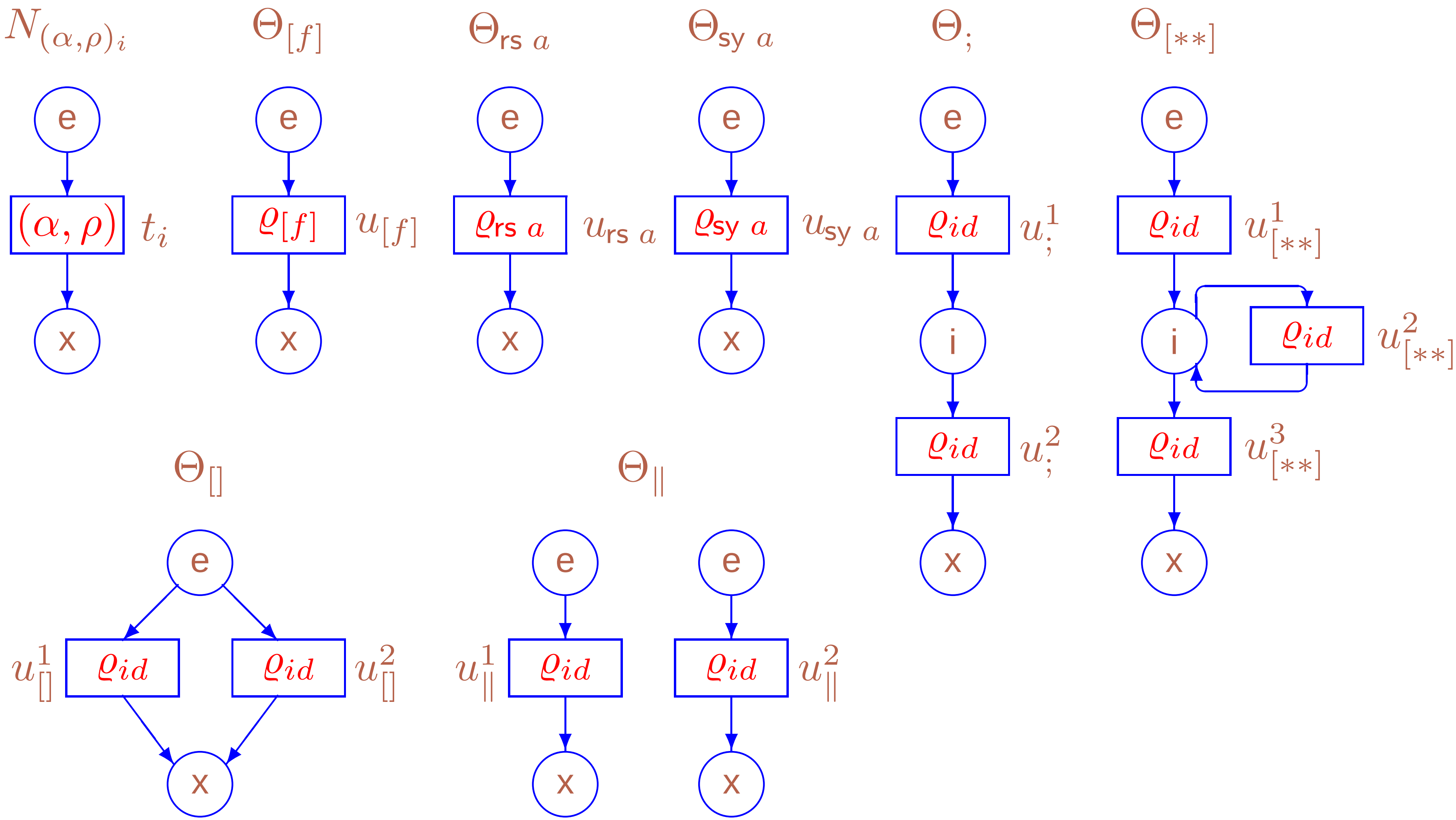
- P_N *and* T_N *are finite sets of* places *and* transitions, *respectively, s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$;
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to I\!N$ *is a function of the* weights of arcs *between places and transitions and vice versa;*
- Λ_N *is the* place and transition labeling *function s.t.*
  - $\Lambda_N|_{P_N} : P_N \to \{\mathsf{e}, \mathsf{i}, \mathsf{x}\}$ *(it specifies* entry, internal *and* exit *places);*
  - $\Lambda_N|_{T_N} : T_N \to \{\varrho \mid \varrho \subseteq I\!N_f^{\mathcal{SL}} \times \mathcal{SL}\}$ *(it associates transitions with the* relabeling relations *on activities).*

Moreover, $\forall t \in T_N\ {}^\bullet t \neq \emptyset \neq t^\bullet$.

For the set of entry *places of* N, ${}^\circ N = \{p \in P_N \mid \Lambda_N(p) = \mathsf{e}\}$, *and the set of* exit *places of* N, $N^\circ = \{p \in P_N \mid \Lambda_N(p) = \mathsf{x}\}$, *it holds:* ${}^\circ N \neq \emptyset \neq N^\circ$ *and* ${}^\bullet({}^\circ N) = \emptyset = (N^\circ)^\bullet$.

A dts-box is *plain* if $\forall t \in T_N\ \Lambda_N(t) \in \mathcal{SL}$, i.e., $\Lambda_N(t)$ is the constant relabeling.

A *marked plain dts-box* is a pair (N, M_N), where N is a plain dts-box and $M_N \in I\!N_f^{P_N}$ is the *initial marking*. Let $\overline{N} = (N, {}^\circ N)$ and $\underline{N} = (N, N^\circ)$.



The plain and operator dts-boxes

Definition 14 *Let* $(\alpha,\rho)\in\mathcal{SL},\ a\in Act$ *and* $E,F,K\in RegStatExpr$. *The* denotational semantics *of* $dtsPBC$ *is a mapping* Box_{dts} *from* $RegStatExpr$ *into plain dts-boxes:*

1. $Box_{dts}((\alpha,\rho)_i)=N_{(\alpha,\rho)_i}$;
2. $Box_{dts}(E\circ F)=\Theta_\circ(Box_{dts}(E),Box_{dts}(F)),\ \circ\in\{;,[],\|\}$;
3. $Box_{dts}(E[f])=\Theta_{[f]}(Box_{dts}(E))$;
4. $Box_{dts}(E\circ a)=\Theta_{\circ a}(Box_{dts}(E)),\ \circ\in\{\text{rs},\text{sy}\}$;
5. $Box_{dts}([E*F*K])=\Theta_{[**]}(Box_{dts}(E),Box_{dts}(F),Box_{dts}(K))$.

For $E\in RegStatExpr$, let $Box_{dts}(\overline{E})=\overline{Box_{dts}(E)}$ and $Box_{dts}(\underline{E})=\underline{Box_{dts}(E)}$.

We denote isomorphism of transition systems by $\simeq$,

and the same symbol denotes isomorphism of reachability graphs and DTMCs

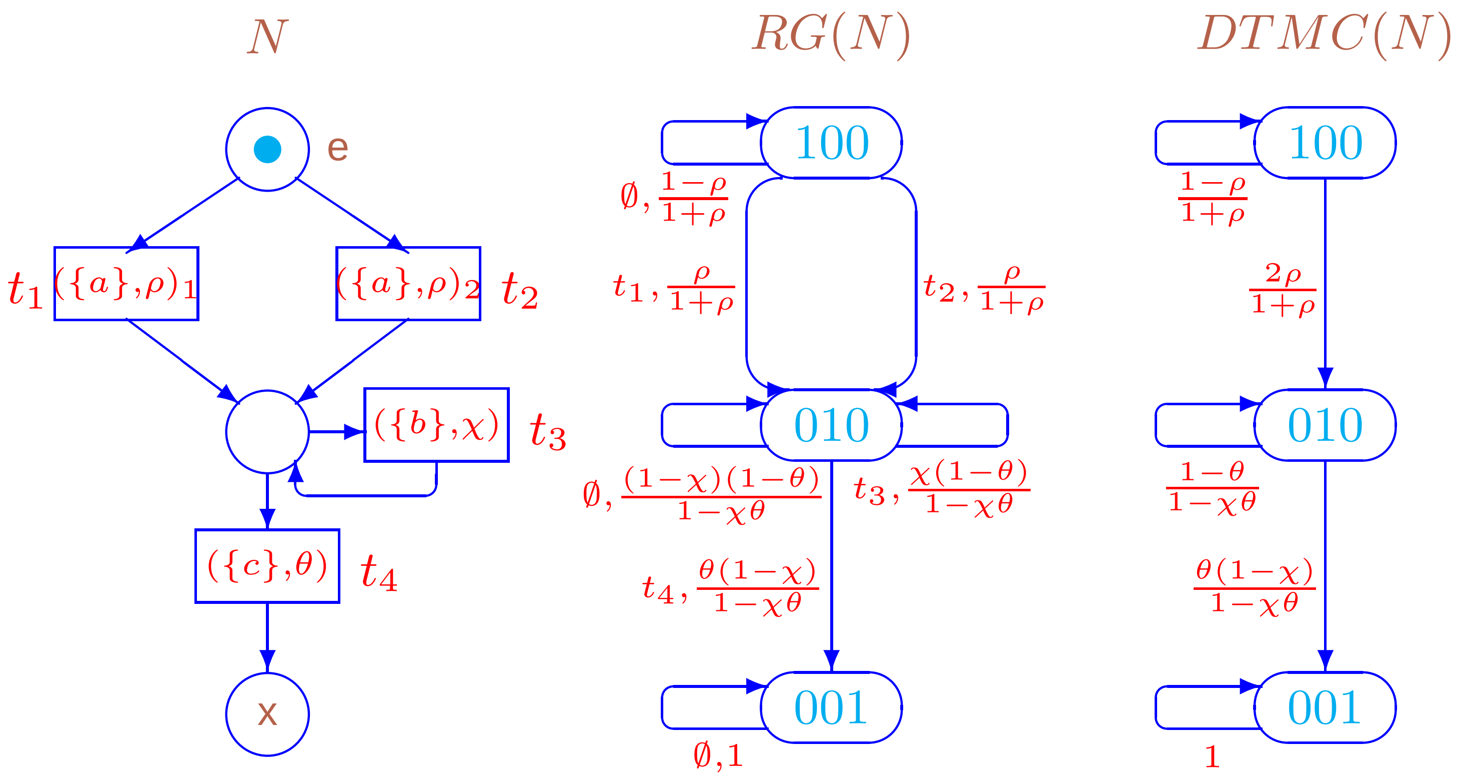
as well as isomorphism between transition systems and reachability graphs.

Theorem 1 *For any static expression* E

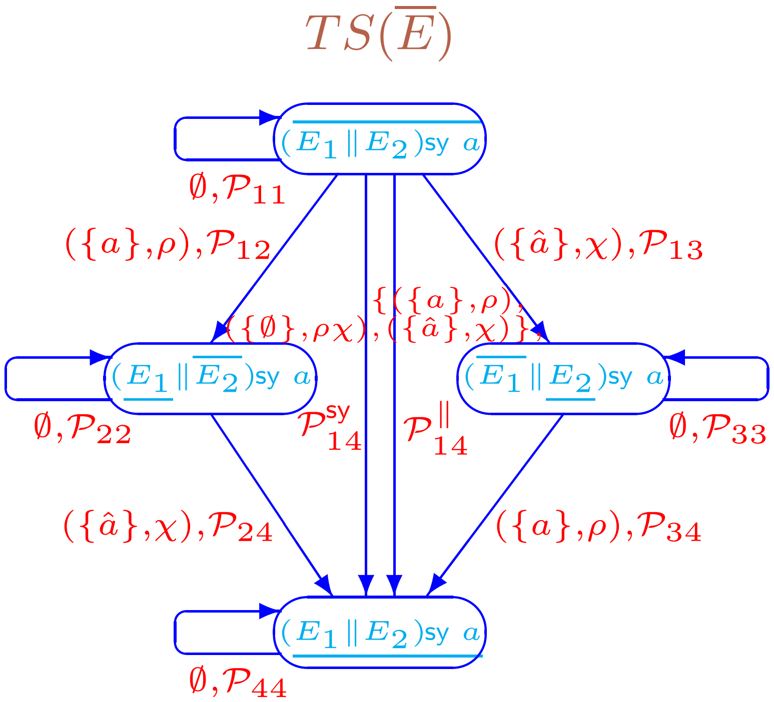
$$TS(\overline{E}) \simeq RG(Box_{dts}(\overline{E})).$$

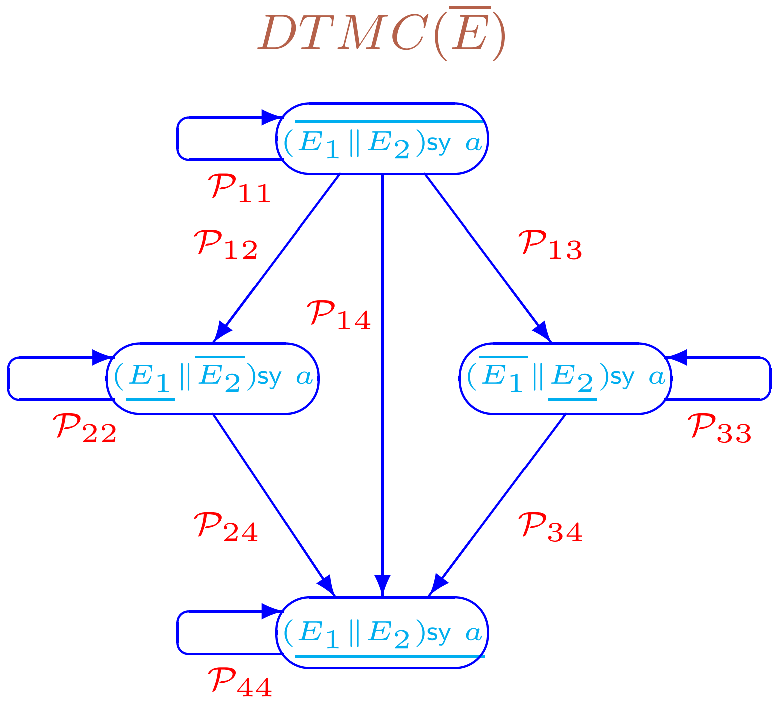
Proposition 1 *For any static expression* E

$$DTMC(\overline{E}) \simeq DTMC(Box_{dts}(\overline{E})).$$

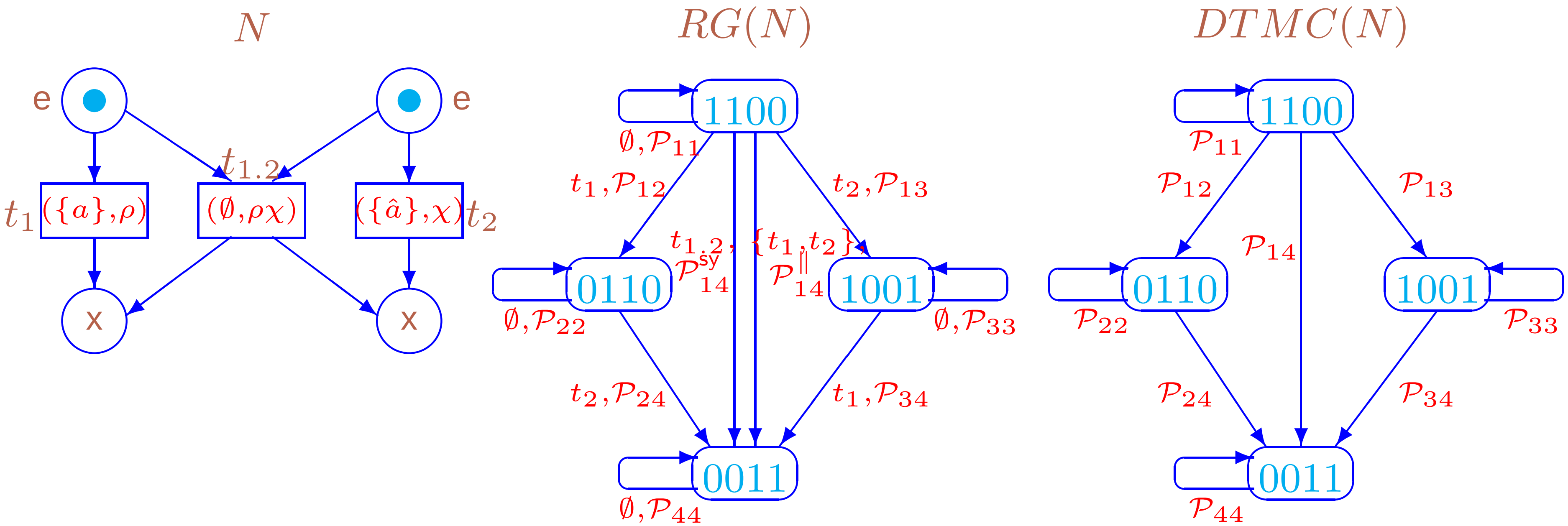


BOXIT:The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$, its reachability graph and the underlying DTMC





EXPR: The transition system and the underlying DTMC of $\overline{E}$ for $E = ((\{a\}, \rho) \| (\{\hat{a}\}, \chi))$ sy a



BOX: The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = ((\{a\},\rho)\|(\{\hat{a}\},\chi))$ sy a, its reachability graph and the underlying DTMC

The normalization factor $\mathcal{N} = \frac{1}{1-\rho^2\chi-\rho\chi^2+\rho^2\chi^2}$.

$$\mathcal{P}_{11} = \mathcal{N}(1-\rho)(1-\chi)(1-\rho\chi) \qquad \mathcal{P}_{12} = \mathcal{N}\rho(1-\chi)(1-\rho\chi)$$

$$\mathcal{P}_{13} = \mathcal{N}\chi(1-\rho)(1-\rho\chi) \qquad \mathcal{P}_{14}^{\mathrm{sy}} = \mathcal{N}\rho\chi(1-\rho)(1-\chi)$$

$$\mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(1-\rho\chi) \qquad \mathcal{P}_{22} = 1-\chi$$

$$\mathcal{P}_{24} = \chi \qquad \mathcal{P}_{33} = 1-\rho$$

$$\mathcal{P}_{34} = \rho \qquad \mathcal{P}_{44} = 1$$

$$\mathcal{P}_{14} = \mathcal{P}_{14}^{\mathrm{sy}} + \mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(2-\rho-\chi)$$

The case $\rho = \chi = \frac{1}{2}$:

$$\mathcal{P}_{11} = \mathcal{P}_{12} = \mathcal{P}_{13} = \mathcal{P}_{14}^{\|} = \frac{3}{13},\ \mathcal{P}_{14}^{\mathrm{sy}} = \frac{1}{13},$$

$$\mathcal{P}_{22} = \mathcal{P}_{24} = \mathcal{P}_{33} = \mathcal{P}_{34} = \frac{1}{2},\ \mathcal{P}_{44} = 1,\ \mathcal{P}_{14} = \frac{4}{13}.$$

Stochastic equivalences

Empty loops in transition systems

Let G be a dynamic expression and $s \in DR(G)$.

The *probability to stay in s due to k $(k \geq 1)$ empty loops* is $(PT(\emptyset, s))^k$.

Let $\Gamma \in Exec(s) \setminus \{\emptyset\}$. The *probability to execute the non-empty multiset of activities Γ in s after possible empty loops*:

$$PT^*(\Gamma, s) = PT(\Gamma, s) \sum_{k=0}^{\infty} (PT(\emptyset, s))^k = \frac{PT(\Gamma, s)}{1 - PT(\emptyset, s)} = EL(s)PT(\Gamma, s),$$

where $EL(s) = \frac{1}{1-PT(\emptyset,s)}$ is the *empty loops abstraction factor*.

The *empty loops abstraction vector* of G, EL, is that with the elements $EL(s),\ s \in DR(G)$.

Definition 15 *The* (labeled probabilistic) transition system without empty loops $TS^*(G)$ *has the state space* $DR(G)$ *and the transitions* $s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$, *if* $s \stackrel{\Gamma}{\rightarrow} \tilde{s},\ \Gamma \neq \emptyset$ *and* $\mathcal{P} = PT^*(\Gamma, s)$.

We write $s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$ if $\exists \mathcal{P}\ s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \twoheadrightarrow \tilde{s}$ if $\exists \Gamma\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$.

For $\Gamma = \{(\alpha, \rho)\}$ we write $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}$.

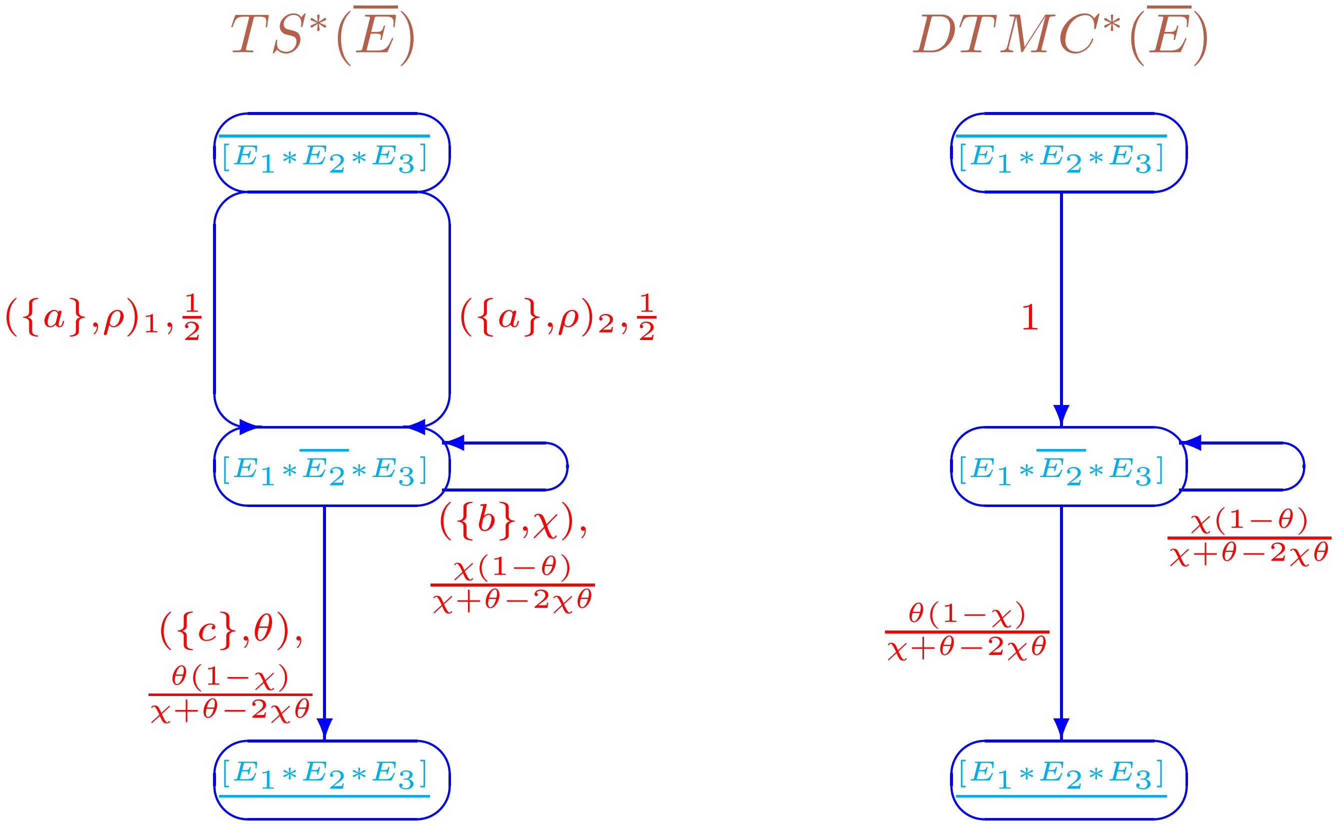
For $E \in RegStatExpr$, let $TS^*(E) = TS^*(\overline{E})$.

Definition 16 *G and G' are equivalent w.r.t. transition systems without empty loops, $G =_{ts*} G'$, if $TS^*(G) \simeq TS^*(G')$.*

Definition 17 *The underlying DTMC without empty loops $DTMC^*(G)$ has the state space $DR(G)$ and transitions $s \twoheadrightarrow_{\mathcal{P}} \tilde{s}$, if $s \twoheadrightarrow \tilde{s}$, where $\mathcal{P} = PM^*(s, \tilde{s})$ is the probability to move from s to $\tilde{s}$ by executing any non-empty multiset of activities after possible empty loops:*

$$PM^*(s,\tilde{s}) = \sum_{\{\Gamma \mid s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}} PT^*(\Gamma, s) = \begin{cases} EL(s)(PM(s,s) - PT(\emptyset, s)), & s = \tilde{s}; \\ EL(s)PM(s,\tilde{s}), & \text{otherwise}. \end{cases}$$

For $E \in RegStatExpr$, let $DTMC^*(E) = DTMC^*(\overline{E})$.



The transition system and the underlying DTMC without empty loops of $\overline{E}$ from Figure EXPRIT

Empty loops in reachability graphs

Let N be an LDTSPN and $M \in RS(N)$.

The *probability to stay in M due to k $(k \geq 1)$ empty loops* is $(PT(\emptyset, M))^k$.

Let $U \subseteq Ena(M),\ U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the non-empty set of transitions U fires in M after possible empty loops*:

$$PT^*(U, M) = PT(U, M) \sum_{k=0}^{\infty} (PT(\emptyset, M))^k = \frac{PT(U, M)}{1 - PT(\emptyset, M)} = EL(M)PT(U, M),$$

where $EL(M) = \frac{1}{1-PT(\emptyset,M)}$ is the *empty loops abstraction factor*.

The *empty loops abstraction vector* of N, EL, is that with the elements $EL(M),\ M \in RS(N)$.

Definition 18 *The* reachability graph without empty loops $RG^*(N)$ *with the set of nodes* $RS(N)$ *and the set of arcs corresponding to the transitions* $M \stackrel{U}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$, *if* $M \stackrel{U}{\rightarrow} \widetilde{M},\ U \neq \emptyset$ *and* $\mathcal{P} = PT^*(U, M)$.

We write $M \stackrel{U}{\twoheadrightarrow} \widetilde{M}$ if $\exists \mathcal{P}\ M \stackrel{U}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \twoheadrightarrow \widetilde{M}$ if $\exists U\ M \stackrel{U}{\twoheadrightarrow} \widetilde{M}$.

For $U = \{t\}$ we write $M \stackrel{t}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \stackrel{t}{\twoheadrightarrow} \widetilde{M}$.

Definition 19 *The* underlying DTMC without empty loops $DTMC^*(N)$ *has the state space* $RS(N)$ *and transitions* $M \twoheadrightarrow_{\mathcal{P}} \widetilde{M}$, *if* $M \twoheadrightarrow \widetilde{M}$, *where* $\mathcal{P} = PM^*(M, \widetilde{M})$ *is the* probability to move from M to $\widetilde{M}$ by firing any non-empty set of transitions after possible empty loops*:*

$$PM^*(M, \widetilde{M}) = \sum_{\{U \in Ena(M) \mid M \stackrel{U}{\twoheadrightarrow} \widetilde{M}\}} PT^*(U, M) = \begin{cases} EL(M)(PM(M, M) - PT(\emptyset, M)), & M = \widetilde{M}; \\ EL(M)PM(M, \widetilde{M}), & \text{otherwise}. \end{cases}$$

Let N be an LDTSPN, $M, \widetilde{M} \in RS(N)$ and $M \stackrel{t}{\twoheadrightarrow} \widetilde{M}$. We write $M \stackrel{t}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = pt^*(t, M)$ is the *probability that the transition* t *fires in* M *after possible empty loops when only firings of single transitions are allowed*:

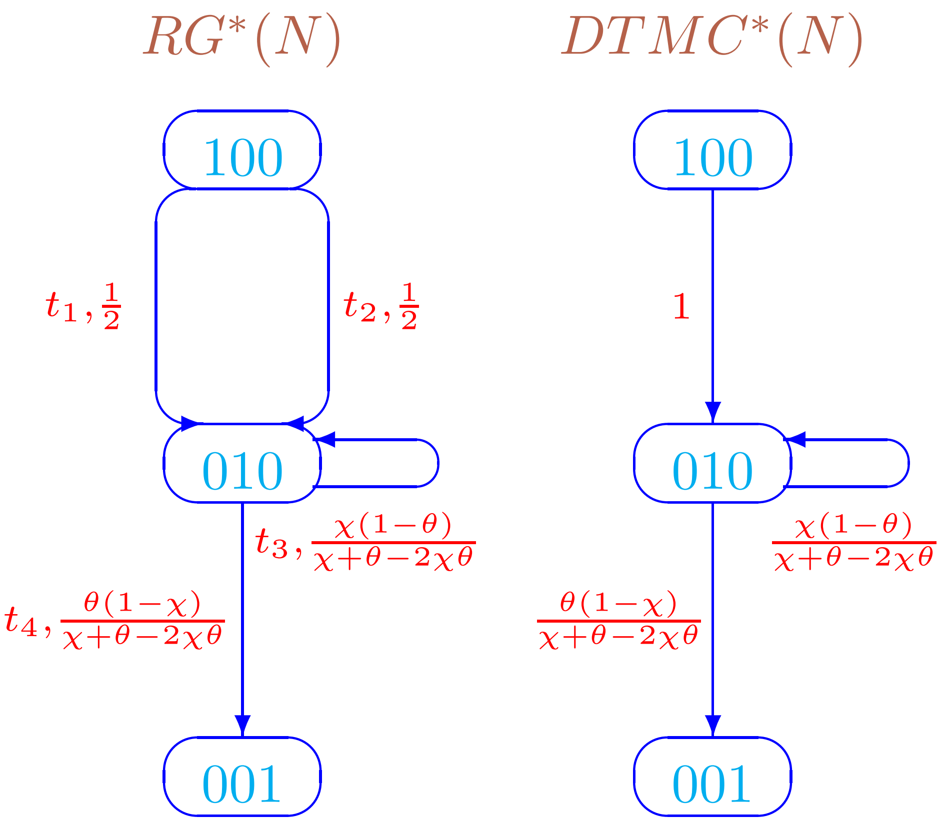
$$pt^*(t, M) = \frac{PT^*(\{t\}, M)}{\sum_{u \in Ena(M)} PT^*(\{u\}, M)}.$$

Theorem 2 *For any static expression* E

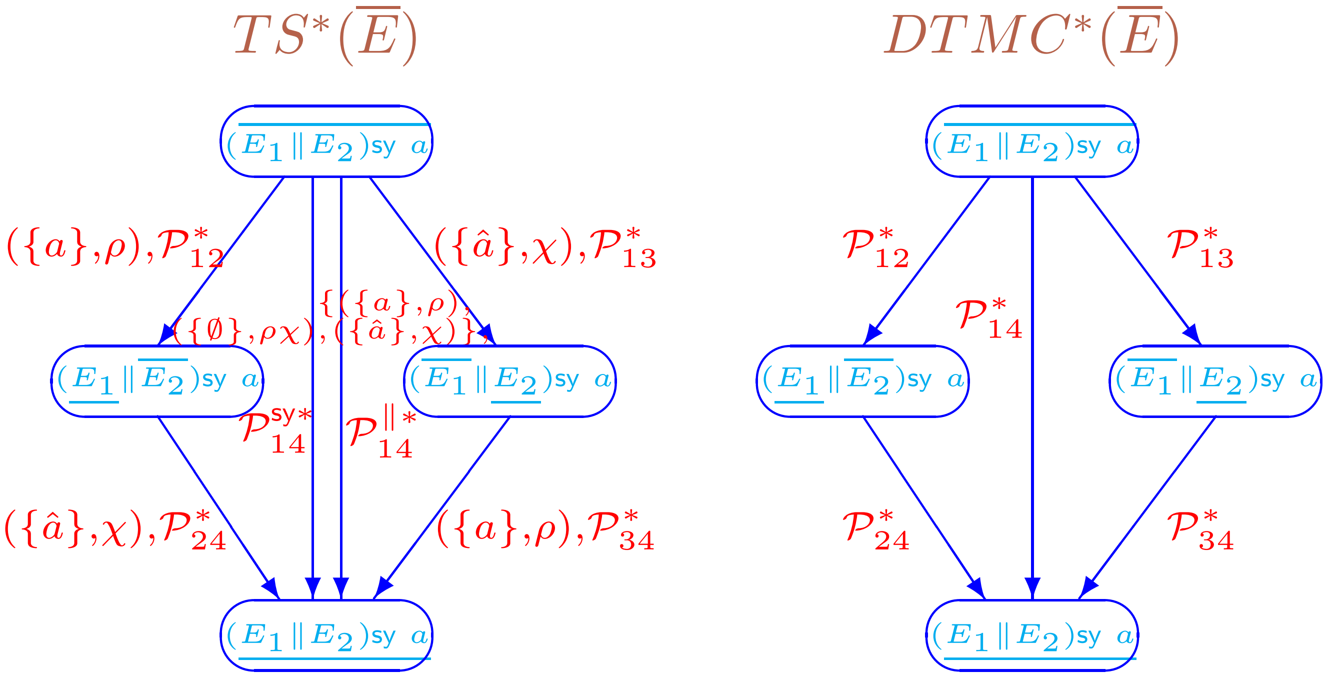
$$TS^*(\overline{E}) \simeq RG^*(Box_{dts}(\overline{E})).$$

Proposition 2 *For any static expression* E

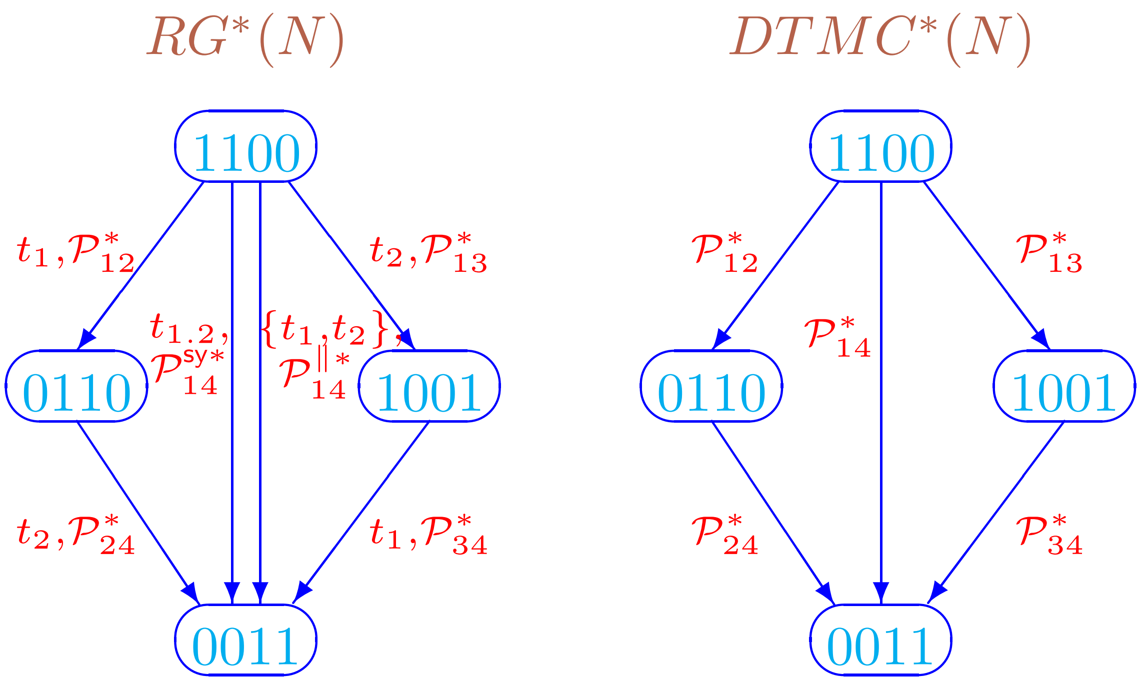
$$DTMC^*(\overline{E}) \simeq DTMC^*(Box_{dts}(\overline{E})).$$



The reachability graph and the underlying DTMC without empty loops of N from Figure BOXIT



The transition system and the underlying DTMC without empty loops of $\overline{E}$ from Figure EXPR



The reachability graph and the underlying DTMC without empty loops of N from Figure BOX

The normalization factor $\mathcal{N}^* = \frac{1}{\rho+\chi-2\rho^2\chi-2\rho\chi^2+2\rho^2\chi^2}$.

$$\mathcal{P}^*_{12} = \frac{\mathcal{P}_{12}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho(1-\chi)(1-\rho\chi)$$
$$\mathcal{P}^*_{13} = \frac{\mathcal{P}_{13}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\chi(1-\rho)(1-\rho\chi)$$
$$\mathcal{P}^{\mathrm{sy}*}_{14} = \frac{\mathcal{P}^{\mathrm{sy}}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho)(1-\chi)$$
$$\mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho\chi)$$
$$\mathcal{P}^*_{24} = \frac{\mathcal{P}_{24}}{1-\mathcal{P}_{22}} = 1$$
$$\mathcal{P}^*_{34} = \frac{\mathcal{P}_{34}}{1-\mathcal{P}_{33}} = 1$$
$$\mathcal{P}^*_{14} = \mathcal{P}^{\mathrm{sy}*}_{14} + \mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\mathrm{sy}}_{14}+\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(2-\rho-\chi)$$

The case $\rho = \chi = \frac{1}{2}$:

$$\mathcal{P}^*_{12} = \mathcal{P}^*_{13} = \mathcal{P}^{\|*}_{14} = \frac{3}{10},\ \mathcal{P}^{\mathrm{sy}*}_{14} = \frac{1}{10},\ \mathcal{P}^*_{24} = \mathcal{P}^*_{34} = 1,\ \mathcal{P}^*_{14} = \frac{2}{5}.$$

Stochastic trace equivalences

Let G be a dynamic expression, $s, \tilde{s} \in DR(G)$ and $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}$. We write $s \xrightarrow{(\alpha,\rho)}\!\!\!\!\twoheadrightarrow_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = pt^*((\alpha, \rho), s)$ is the *probability to execute the activity* (α, ρ) *in* s *after possible empty loops when only one-element steps are allowed*:

$$pt^*((\alpha, \rho), s) = \frac{PT^*(\{(\alpha, \rho)\}, s)}{\sum_{\{(\beta,\chi)\}\in Exec(s)} PT^*(\{(\beta, \chi)\}, s)}.$$

The *multiaction part* of $\Gamma \in I\!N_f^{\mathcal{SL}}$ is $\mathcal{L}(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma} \alpha$.

We consider $\mathcal{L}(\Gamma) \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}$, i.e., the non-empty multisets of multiactions.

Definition 20 *An* interleaving stochastic trace *of a dynamic expression* G *is a pair* $(\sigma, pt^*(\sigma))$, *where* $\sigma = \alpha_1 \cdots \alpha_n \in \mathcal{L}^*$ *and*

$$pt^*(\sigma) = \sum_{\{(\alpha_1,\rho_1),\ldots,(\alpha_n,\rho_n) \mid [G]_\approx = s_0 \stackrel{(\alpha_1,\rho_1)}{\twoheadrightarrow} s_1 \stackrel{(\alpha_2,\rho_2)}{\twoheadrightarrow} \cdots \stackrel{(\alpha_n,\rho_n)}{\twoheadrightarrow} s_n\}} \prod_{i=1}^{n} pt^*((\alpha_i, \rho_i), s_{i-1}).$$

We denote a set of all interleaving stochastic traces *of a dynamic expression* G *by* $IntStochTraces(G)$.

G *and* G' *are* interleaving stochastic trace equivalent, $G \equiv_{is} G'$, *if*

$$IntStochTraces(G) = IntStochTraces(G').$$

Definition 21 *A* step stochastic trace *of a dynamic expression* G *is a pair* $(\Sigma, PT^*(\Sigma))$, *where* $\Sigma = A_1 \cdots A_n \in (I\!N_f^{\mathcal{L}} \setminus \{\emptyset\})^*$ *and*

$$PT^*(\Sigma) = \sum_{\{\Gamma_1,\ldots,\Gamma_n | [G]_\approx = s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \le i \le n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}).$$

We denote a set of all step stochastic traces *of a dynamic expression* G *by* $StepStochTraces(G)$.

G *and* G' *are* step stochastic trace equivalent, $G \equiv_{ss} G'$, *if*

$$StepStochTraces(G) = StepStochTraces(G').$$

Stochastic bisimulation equivalences

Let G be a dynamic expression and $\mathcal{H} \subseteq DR(G)$. For $s \in DR(G)$ and $A \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}$ we write $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM_A^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via non-empty steps with the multiaction part A after possible empty loops*:

$$PM_A^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s},\ \mathcal{L}(\Gamma) = A\}} PT^*(\Gamma, s).$$

We write $s \stackrel{A}{\twoheadrightarrow} \mathcal{H}$ if $\exists \mathcal{Q}\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{Q}} \mathcal{H}$.

We write $s \twoheadrightarrow_{\mathcal{P}} \mathcal{H}$ if $\exists A\ s \stackrel{A}{\twoheadrightarrow} \mathcal{H}$, where $\mathcal{P} = PM^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via any non-empty steps after possible empty loops*:

$$PM^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}} PT^*(\Gamma, s).$$

We write $s \xrightarrow{\alpha}\!\!\!\!\rightarrow_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = pm^*_\alpha(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via steps with the multiaction part $\{\alpha\}$ after possible empty loops when only one-element steps are allowed*:

$$pm^*_\alpha(s, \mathcal{H}) = \sum_{\{(\alpha,\rho) \mid \exists \tilde{s} \in \mathcal{H}\ s \overset{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}\}} pt^*((\alpha, \rho), s).$$

Definition 22 *Let* G *and* G' *be dynamic expressions. An* equivalence *relation* $\mathcal{R} \subseteq (DR(G) \cup DR(G'))^2$ *is a* $\star$-stochastic bisimulation *between* G *and* G', $\star \in$ {interleaving, step}, $\mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$, *if:*

1. $([G]_\approx, [G']_\approx) \in \mathcal{R}$.
2. $(s_1, s_2) \in \mathcal{R} \Rightarrow \forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$
   - $\forall x \in \mathcal{L}$ *and* $\hookrightarrow = \rightharpoonup$, *if* $\star = i$;
   - $\forall x \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}$ *and* $\hookrightarrow = \twoheadrightarrow$, *if* $\star = s$;

$$s_1 \stackrel{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{H} \Leftrightarrow s_2 \stackrel{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{H}.$$

Two dynamic expressions G *and* G' *are* $\star$-stochastic bisimulation equivalent, $\star \in$ {interleaving, step}, $G \underline{\leftrightarrow}_{\star s} G'$, *if* $\exists \mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$.

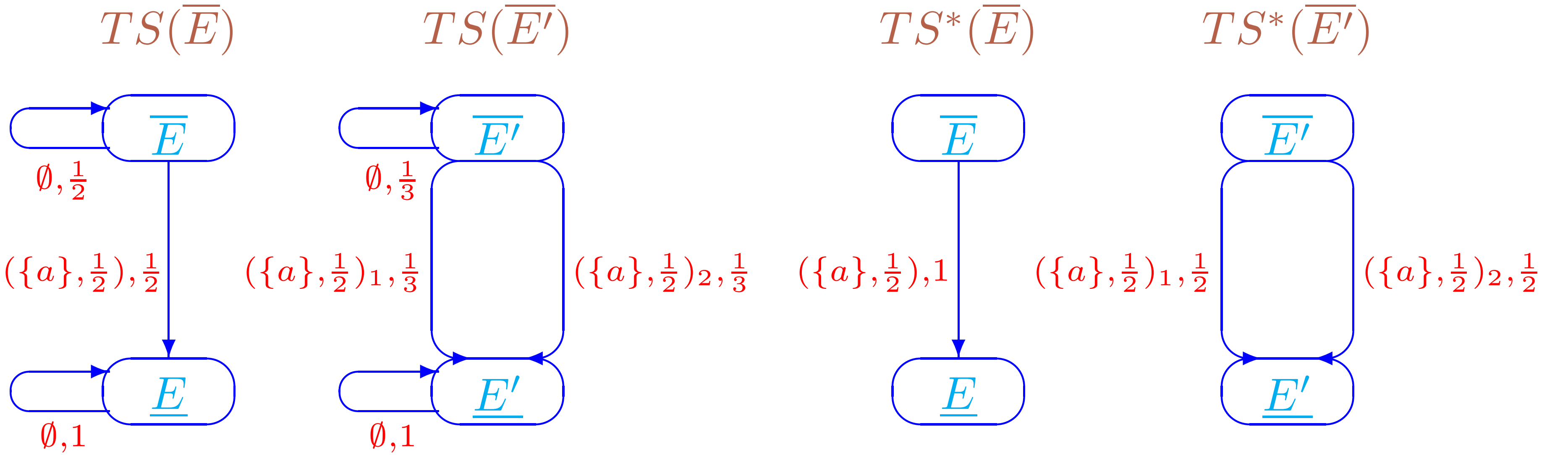
Stochastic isomorphism

Let G be a dynamic expression, $s, \tilde{s} \in DR(G)$ and $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \{\tilde{s}\}$. We write $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$.

Definition 23 *Let G, G' be dynamic expressions. A mapping $\beta : DR(G) \to DR(G')$ is a* stochastic isomorphism *between G and G', $\beta : G =_{sto} G'$, if*

1. *β is a bijection s.t. $\beta([G]_\approx) = [G']_\approx$;*
2. *$\forall s, \tilde{s} \in DR(G)\ \forall A \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \beta(\tilde{s})$.*

G and G' are stochastically isomorphic*, $G =_{sto} G'$, if $\exists \beta : G =_{sto} G'$.*



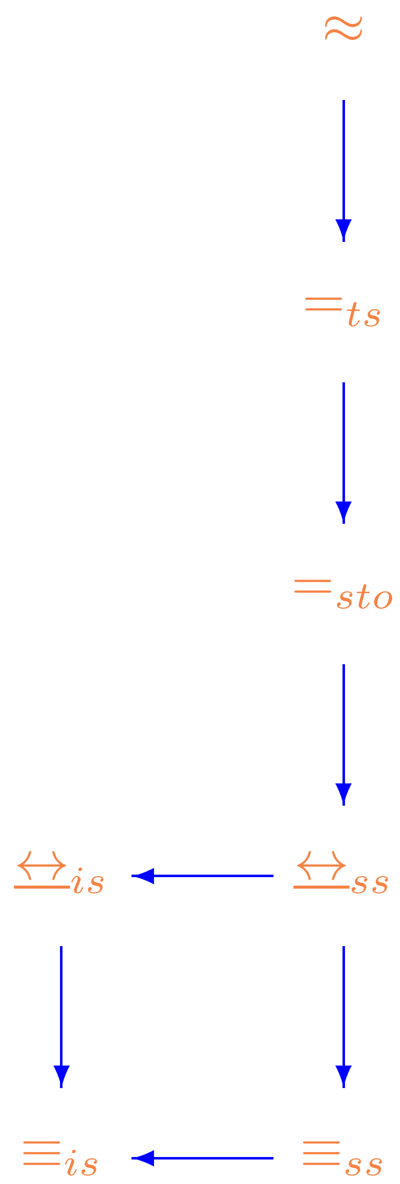
A problem with stochastic isomorphism based on transition systems with empty loops

Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2})_1[](\{a\}, \frac{1}{2})_2$.

Then $\overline{E} =_{sto} \overline{E'}$, but $\overline{E}$ is not equivalent to $\overline{E'}$ w.r.t. the stronger version of stochastic isomorphism, since the probability of the only non-empty transition in $TS(\overline{E})$ is $\frac{1}{2}$ whereas the probability of both non-empty transitions in $TS(\overline{E'})$ is $\frac{1}{3}$, and $\frac{1}{2} \neq \frac{1}{3} + \frac{1}{3}$.

The probability of the only non-empty transition in $TS^*(\overline{E})$ is 1, the probability of both non-empty transitions in $TS^*(\overline{E'})$ is $\frac{1}{2}$, and $1 = \frac{1}{2} + \frac{1}{2}$.

Interrelations of the stochastic equivalences



Interrelations of the stochastic equivalences

Proposition 3 *Let* $\star \in \{i, s\}$. *For dynamic expressions* G *and* G':

1. $G \underline{\leftrightarrow}_{\star s} G' \;\Rightarrow\; G \equiv_{\star s} G'$;
2. $G =_{ts\star} G' \;\Leftrightarrow\; G =_{ts} G'$.

Theorem 3 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, is, ss, sto, ts\}$. *For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.

Validity of the implications

- The implications $\leftrightarrow_{ss} \to \leftrightarrow_{is}, \leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$ are valid, since single activities are one-element multisets.
- The implications $\underline{\leftrightarrow}_{\star s} \to \equiv_{\star s}, \star \in \{i, s\}$, are valid by the proposition above.
- The implication $=_{sto} \to \underline{\leftrightarrow}_{ss}$ is proved as follows. Let $\beta : G{=_{sto}}G'$. Then $\mathcal{S} : G\underline{\leftrightarrow}_{ss}G'$, where $\mathcal{S} = \{(s, \beta(s)) \mid s \in DR(G)\}$.
- The implication $=_{ts} \to =_{sto}$ is valid, since stochastic isomorphism is that of transition systems without empty loops up to merging of transitions with labels having identical multiaction parts.
- The implication $\approx \to =_{ts}$ is valid, since the transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

- Let $E = (\{a\}, \frac{1}{2})\|(\{b\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{b\}, \frac{1}{2}); (\{a\}, \frac{1}{2}))$. Then $\overline{E}\underline{\leftrightarrow}_{is}\overline{E'}$, but $\overline{E}\not\equiv_{ss}\overline{E'}$, since only in $TS^*(\overline{E'})$ multiactions $\{a\}$ and $\{b\}$ cannot be executed concurrently.

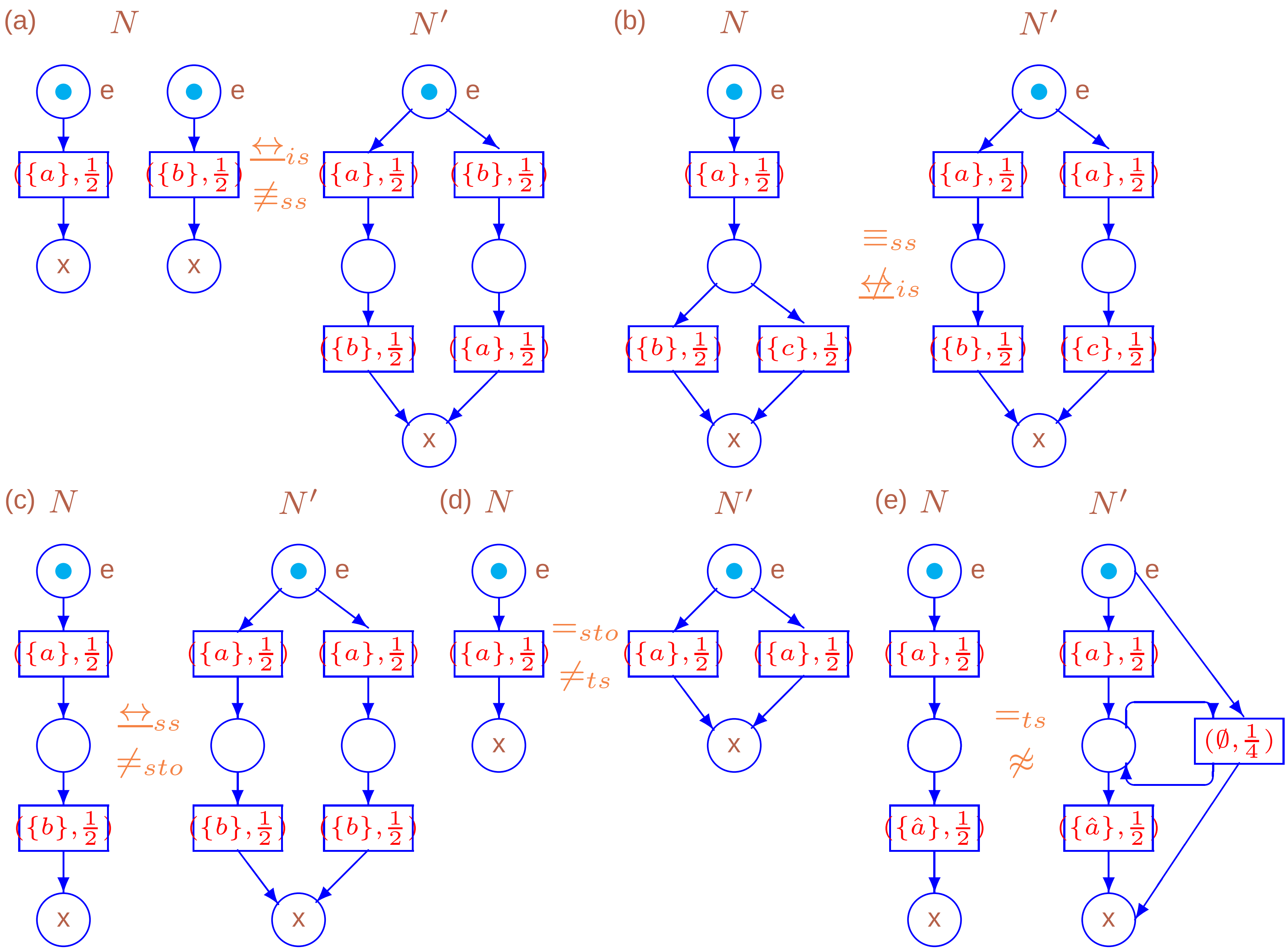
- Let $E = (\{a\}, \frac{1}{2}); ((\{b\}, \frac{1}{2})[](\{c\}, \frac{1}{2}))$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{a\}, \frac{1}{2}); (\{c\}, \frac{1}{2}))$. Then $\overline{E}\equiv_{ss}\overline{E'}$, but $\overline{E}\underline{\not\leftrightarrow}_{is}\overline{E'}$, since only in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.

- Let $E = (\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})[](\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})$. Then $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$, but$\overline{E}\neq_{sto}\overline{E'}$, since $TS^*(\overline{E'})$ has more states than $TS^*(\overline{E})$.

- Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2})_1[](\{a\}, \frac{1}{2})_2$. Then $\overline{E}=_{sto}\overline{E'}$, but $\overline{E}\neq_{ts}\overline{E'}$, since only $TS(\overline{E'})$ has two transitions.

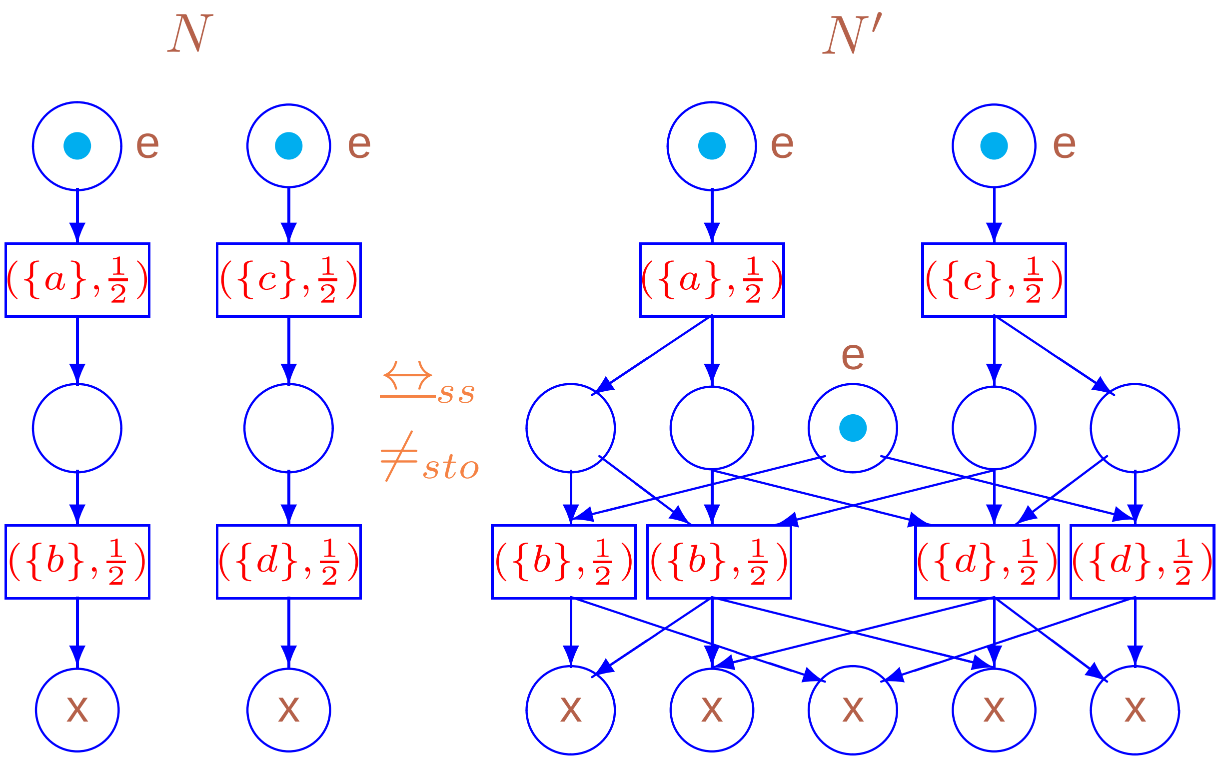
- Let $E = (\{a\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E}=_{ts}\overline{E'}$, but $\overline{E}\not\approx\overline{E'}$, since $\overline{E}$ and $\overline{E'}$ cannot be reached each from other by applying inaction rules.

In the figure below $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$ for each picture (a)–(e).



Dts-boxes of the dynamic expressions from equivalence examples of the theorem above

Reduction modulo equivalences



Reduction of a dts-box up to $\underline{\leftrightarrow}_{ss}$

Let $E = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))\|((\{c\}, \frac{1}{2}); (\{d\}, \frac{1}{2}))$ and $E' = ((((\{a, x\}, \frac{1}{2}); ((\{b, y_1\}, \frac{1}{2})[] (\{b, y_2\}, \frac{1}{2})))\|((\{a, \hat{x}\}, \frac{1}{2}); ((\{b, \widehat{y_2}, y_2'\}, \frac{1}{2})[](\{d, v_1\}, \frac{1}{2})))\|((\{c, z\}, \frac{1}{2}); ((\{b, \widehat{y_2'}\}, \frac{1}{2})[] (\{d, \widehat{v_1}, v_1'\}, \frac{1}{2})))\|((\{c, z\}, \frac{1}{2}); ((\{d, v_1\}, \frac{1}{2})[](\{d, \widehat{v_1'}\}, \frac{1}{2})))\|((\{b, \widehat{y_1}\}, \frac{1}{2})[](\{d, \widehat{v_2}\}, \frac{1}{2}))) \text{ sy } x \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_2' \text{ sy } z \text{ sy } v_1 \text{ sy } v_1' \text{ sy } v_2 \text{ rs } x \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_2' \text{ rs } z \text{ rs } v_1 \text{ rs } v_1' \text{ rs } v_2$.

We have $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$, but $\overline{E}\not=_{sto}\overline{E'}$, since $TS^*(\overline{E'})$ has more states than $TS^*(\overline{E})$.

Thus, E is a reduction of E' w.r.t. $\underline{\leftrightarrow}_{ss}$.

For $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'}),\ N$ is a reduction of N' w.r.t. the net version of $\underline{\leftrightarrow}_{ss}$.

An *autobisimulation (equivalence)* is a bisimulation (equivalence) between an expression and itself.

For a dynamic expression G and step stochastic autobisimulation equivalence $G\underline{\leftrightarrow}_{ss}G$, let $\mathcal{K} \in DR(G)/_{\underline{\leftrightarrow}_{ss}}$ and $s_1, s_2 \in \mathcal{K}$.

We have $\forall\widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}}\ \forall A \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}\ s_1 \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}} \Leftrightarrow s_2 \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

The equality is valid for all $s_1, s_2 \in \mathcal{K}$, hence, we can rewrite it as $\mathcal{K}\stackrel{A}{\twoheadrightarrow}_{\mathcal{P}}\widetilde{\mathcal{K}}$, where $\mathcal{P} = PM_A^*(\mathcal{K}, \widetilde{\mathcal{K}}) = PM_A^*(s_1, \widetilde{\mathcal{K}}) = PM_A^*(s_2, \widetilde{\mathcal{K}})$.

We write $\mathcal{K}\stackrel{A}{\twoheadrightarrow}\widetilde{\mathcal{K}}$ if $\exists\mathcal{P}\ \mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$ and $\mathcal{K}\twoheadrightarrow\widetilde{\mathcal{K}}$ if $\exists A\ \mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}$.

The similar arguments: we write $\mathcal{K}\twoheadrightarrow_{\mathcal{P}}\widetilde{\mathcal{K}}$, where $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}}) = PM^*(s_1, \widetilde{\mathcal{K}}) = PM^*(s_2, \widetilde{\mathcal{K}})$.

Definition 24 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) (labeled probabilistic) transition system without empty loops *of a dynamic expression* G *is* $TS^*_{\underline{\leftrightarrow}_{ss}}(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$*, where*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\underline{\leftrightarrow}_{ss}}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq (I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}) \times (0;1]$;
- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{K}, (A, PM_A^*(\mathcal{K}, \widetilde{\mathcal{K}})), \widetilde{\mathcal{K}}) \mid \mathcal{K} \in DR(G)/_{\underline{\leftrightarrow}_{ss}},\ \mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = \{[G]_\approx\}$.

The transition $(\mathcal{K}, (A, \mathcal{P}), \widetilde{\mathcal{K}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

For $E \in RegStatExpr$, let $TS^*_{\underline{\leftrightarrow}_{ss}}(E) = TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Definition 25 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) underlying DTMC without empty loops *of a dynamic expression* G*,* $DTMC^*_{\underline{\leftrightarrow}_{ss}}(G)$*, has the state space* $DR(G)/_{\underline{\leftrightarrow}_{ss}}$ *and the transitions* $\mathcal{K} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$*, where* $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}})$.

For $E \in RegStatExpr$, let $DTMC^*_{\underline{\leftrightarrow}_{ss}}(E) = DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Logical characterization

Logic iPML

Definition 26 $\top$ *is the truth,* $\alpha \in \mathcal{L}$, $\mathcal{P} \in (0; 1]$. *A* formula *of* $iPML$*:*

$$\Phi ::= \top \mid \neg\Phi \mid \Phi \wedge \Phi \mid \langle\alpha\rangle_{\mathcal{P}}\Phi$$

iPML is the set of *all formulas of the logic* $iPML$.

Definition 27 *Let* G *be a dynamic expression and* $s \in DR(G)$. *The* satisfaction relation $\models_G \subseteq DR(G) \times \mathbf{iPML}$*:*

1. $s \models_G \top$ *— always;*
2. $s \models_G \neg\Phi$*, if* $s \not\models_G \Phi$*;*
3. $s \models_G \Phi \wedge \Psi$*, if* $s \models_G \Phi$ *and* $s \models_G \Psi$*;*
4. $s \models_G \langle\alpha\rangle_{\mathcal{P}}\Phi$*, if* $\exists\mathcal{H} \subseteq DR(G)\ s \xrightarrow{\alpha}_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\tilde{s} \in \mathcal{H}\ \tilde{s} \models_G \Phi$.

$\langle\alpha\rangle\Phi = \exists\mathcal{P}\ \langle\alpha\rangle_{\mathcal{P}}\Phi$. $\langle\alpha\rangle_{\mathcal{Q}}\Phi$ implies $\langle\alpha\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

$G \models_G \Phi$, if $[G]_{\approx} \models_G \Phi$.

Definition 28 *G and G' are* logically equivalent *in* $iPML$, $G =_{iPML} G'$, *if* $\forall \Phi \in \mathbf{iPML}\ G \models_G \Phi \ \Leftrightarrow\ G' \models_{G'} \Phi$.

Let G be a dynamic expression and $s \in DR(G),\ \alpha \in \mathcal{L}$.

The set of states reached from s by execution of α, the *image set*, is

$Image(s, \alpha) = \{\tilde{s} \mid \exists\{(\alpha, \rho)\} \in Exec(s)\ s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}\}$.

A dynamic expression G is an *image-finite* one, if $\forall s \in DR(G)\ \forall \alpha \in \mathcal{L}\ |Image(s, \alpha)| < \infty$.

Theorem 4 *For image-finite dynamic expressions G and G'*

$$G \underline{\leftrightarrow}_{is} G' \ \Leftrightarrow\ G =_{iPML} G'.$$

Let $E = (\{a\}, \frac{1}{2}); ((\{b\}, \frac{1}{2})[](\{c\}, \frac{1}{2}))$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{a\}, \frac{1}{2}); (\{c\}, \frac{1}{2}))$.

Then $\overline{E} \neq_{iPML} \overline{E'}$, because for $\Phi = \langle\{a\}\rangle_1 \langle\{b\}\rangle_{\frac{1}{2}} \top$ we have $\overline{E} \models_{\overline{E}} \Phi$, but $\overline{E'} \not\models_{\overline{E'}} \Phi$, since in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.

Logic sPML

Definition 29 $\top$ *is the truth,* $A \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}$, $\mathcal{P} \in (0;1]$.

A formula *of* $sPML$:

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \langle A\rangle_{\mathcal{P}}\Phi$$

sPML is the set of *all formulas of the logic* $sPML$.

Definition 30 *Let* G *be a dynamic expression and* $s \in DR(G)$. *The* satisfaction relation $\models_G \subseteq DR(G) \times \mathbf{sPML}$*:*

1. $s \models_G \top$ *— always;*
2. $s \models_G \neg\Phi$, *if* $s \not\models_G \Phi$*;*
3. $s \models_G \Phi\wedge\Psi$, *if* $s \models_G \Phi$ *and* $s \models_G \Psi$*;*
4. $s \models_G \langle A\rangle_{\mathcal{P}}\Phi$, *if* $\exists\mathcal{H} \subseteq DR(G)\ s \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\tilde{s} \in \mathcal{H}\ \tilde{s} \models_G \Phi$.

$\langle A\rangle\Phi = \exists\mathcal{P}\ \langle A\rangle_{\mathcal{P}}\Phi$. $\langle A\rangle_{\mathcal{Q}}\Phi$ implies $\langle A\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

$G \models_G \Phi$, if $[G]_{\approx} \models_G \Phi$.

Definition 31 *G and G' are logically equivalent in $sPML$, $G =_{sPML} G'$, if*
$\forall \Phi \in \mathbf{sPML}\ G \models_G \Phi \ \Leftrightarrow\ G' \models_{G'} \Phi$.

Let G be a dynamic expression and $s \in DR(G),\ A \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}$.

The set of states reached from s by execution of A, the *image set*, is
$Image(s, A) = \{\tilde{s} \mid \exists \Gamma \in Exec(s)\ \mathcal{L}(\Gamma) = A,\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}$.

A dynamic expression G is an *image-finite* one, if
$\forall s \in DR(G)\ \forall A \in I\!N_f^{\mathcal{L}} \setminus \{\emptyset\}\ |Image(s, A)| < \infty$.

Theorem 5 *For image-finite dynamic expressions G and G'*

$$G \underline{\leftrightarrow}_{ss} G' \ \Leftrightarrow\ G =_{sPML} G'.$$

Let $E = (\{a\}, \frac{1}{2}) \| (\{b\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{b\}, \frac{1}{2}); (\{a\}, \frac{1}{2}))$. Then $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$ but $\overline{E} \neq_{sPML} \overline{E'}$, because for $\Phi = \langle \{a, b\} \rangle_{\frac{1}{3}} \top$ we have $\overline{E} \models_{\overline{E}} \Phi$, but $\overline{E'} \not\models_{\overline{E'}} \Phi$, since in $TS^*(\overline{E'})$ multiactions $\{a\}$ and $\{b\}$ cannot be executed concurrently.

Stationary behaviour

Theoretical background

The elements $\mathcal{P}^*_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of *(one-step) transition probability matrix (TPM)* $\mathbf{P}^*$ for $DTMC^*(G)$:

$$\mathcal{P}^*_{ij} = \begin{cases} PM^*(s_i, s_j), & s_i \twoheadrightarrow s_j; \\ 0, & \text{otherwise}. \end{cases}$$

The *transient (k-step, $k \in I\!N$) probability mass function (PMF)* $\psi^*[k] = (\psi^*_1[k], \ldots, \psi^*_n[k])$ for $DTMC^*(N)$ is the solution of

$$\psi^*[k] = \psi^*[0](\mathbf{P}^*)^k,$$

where $\psi^*[0] = (\psi_1^*[0], \ldots, \psi_n^*[0])$ is the *initial PMF*:

$$\psi_i^*[0] = \begin{cases} 1, & s_i = [G]_\approx; \\ 0, & \text{otherwise}. \end{cases}$$

We have $\psi^*[k+1] = \psi^*[k]\mathbf{P}^*,\ k \in I\!N$.

The *steady-state PMF* $\psi^* = (\psi_1^*, \ldots, \psi_n^*)$ for $DTMC^*(G)$ is the solution of

$$\begin{cases} \psi^*(\mathbf{P}^* - \mathbf{E}) = \mathbf{0} \\ \psi^*\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{0}$ is a vector with n values $0,\ \mathbf{1}$ is that with n values 1.

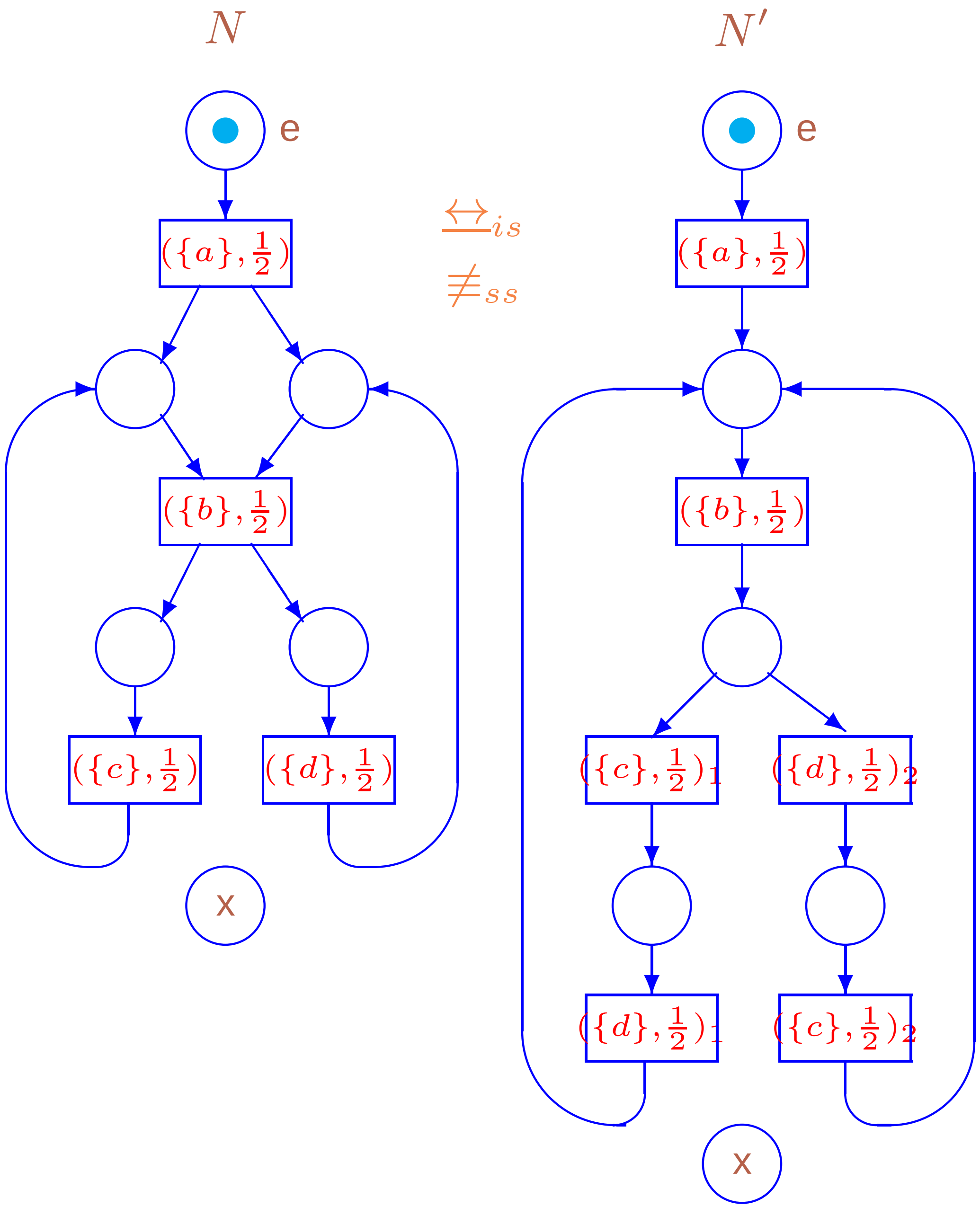
When $DTMC^*(G)$ has the single steady state, $\psi^* = \lim_{k\to\infty} \psi^*[k]$.

Steady state and equivalences

Proposition 4 *Let* G, G' *be dynamic expressions with* $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$. *Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \psi'^*(s').$$

$\mathsf{Stop} = (\{c\}, \frac{1}{2})$ rs c is the process that performs empty loops with probability 1 and never terminates.



$\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady-state probabilities to come in an equivalence class

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]$ and
$E' = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]$.
We have $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$.

$DR(\overline{E})$ consists of

$s_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]]_{\approx}$,
$s_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]]_{\approx}$,
$s_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (\overline{(\{c\}, \frac{1}{2})} \| \overline{(\{d\}, \frac{1}{2})})) * \mathsf{Stop}]]_{\approx}$,
$s_4 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (\underline{(\{c\}, \frac{1}{2})} \| \overline{(\{d\}, \frac{1}{2})})) * \mathsf{Stop}]]_{\approx}$,
$s_5 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (\overline{(\{c\}, \frac{1}{2})} \| \underline{(\{d\}, \frac{1}{2})})) * \mathsf{Stop}]]_{\approx}$.

$DR(\overline{E'})$ consists of

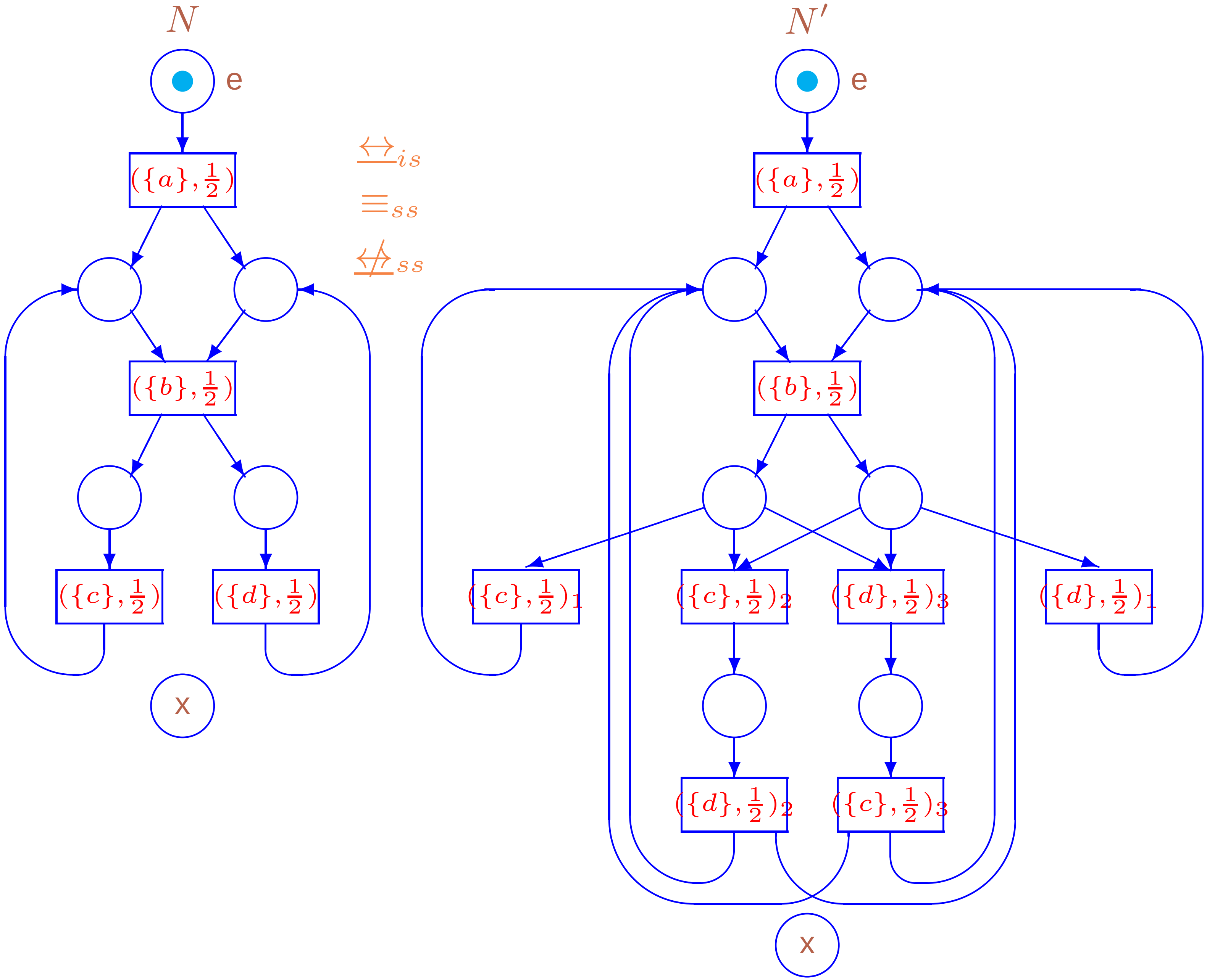
$s'_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]]_{\approx}$,
$s'_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]]_{\approx}$,
$s'_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); \overline{(((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))}) * \mathsf{Stop}]]_{\approx}$,
$s'_4 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; \overline{(\{d\}, \frac{1}{2})_1}) [] ((\{d\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))) * \mathsf{Stop}]]_{\approx}$,
$s'_5 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1; (\{d\}, \frac{1}{2})_1) [] ((\{d\}, \frac{1}{2})_2; \overline{(\{c\}, \frac{1}{2})_2}))) * \mathsf{Stop}]]_{\approx}$.

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ \psi'^* = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right).$$

Consider $\mathcal{H} = \{s_3, s_3'\}$. We have $\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$ whereas $\sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'^*(s') = \psi'^*(s_3') = \frac{1}{3}$. Thus, $\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady-state probabilities to come in an equivalence class.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.



The intersection of $\underline{\leftrightarrow}_{is}$ and $\equiv_{ss}$ does not guarantee a coincidence of steady-state probabilities to come in an equivalence class

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]$ and
$E' = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1)) [] (((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2) []$
$((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]$.

We have $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$ and $\overline{E} \equiv_{ss} \overline{E'}$.

$DR(\overline{E})$ is as in the previous example.

$DR(\overline{E'})$ consists of

$s_1' = [\![\overline{(\{a\},\frac{1}{2})} * ((\{b\},\frac{1}{2});((((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[\!]((((\{c\},\frac{1}{2})_2;(\{d\},\frac{1}{2})_2)[\!]$
$((\{d\},\frac{1}{2})_3;(\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]\!]_\approx$,

$s_2' = [\![(\{a\},\frac{1}{2}) * (\overline{(\{b\},\frac{1}{2})};((((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[\!]((((\{c\},\frac{1}{2})_2;(\{d\},\frac{1}{2})_2)[\!]$
$((\{d\},\frac{1}{2})_3;(\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]\!]_\approx$,

$s_3' = [\![(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});(\overline{(((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[\!]((((\{c\},\frac{1}{2})_2;(\{d\},\frac{1}{2})_2)[\!]}$
$\overline{((\{d\},\frac{1}{2})_3;(\{c\},\frac{1}{2})_3)))})) * \mathsf{Stop}]\!]_\approx$,

$s_4' = [\![(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});((\underline{(\{c\},\frac{1}{2})_1}\|\overline{(\{d\},\frac{1}{2})_1}))[\!]((((\{c\},\frac{1}{2})_2;(\{d\},\frac{1}{2})_2)[\!]$
$((\{d\},\frac{1}{2})_3;(\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]\!]_\approx$,

$s_5' = [\![(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});((\overline{(\{c\},\frac{1}{2})_1}\|\underline{(\{d\},\frac{1}{2})_1}))[\!]((((\{c\},\frac{1}{2})_2;(\{d\},\frac{1}{2})_2)[\!]$
$((\{d\},\frac{1}{2})_3;(\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]\!]_\approx$,

$s_6' = [\![(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});((((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[\!]((((\{c\},\frac{1}{2})_2;\overline{(\{d\},\frac{1}{2})_2})[\!]$
$((\{d\},\frac{1}{2})_3;(\{c\},\frac{1}{2})_3))))) * \mathsf{Stop}]\!]_\approx$,

$s_7' = [\![(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});((((\{c\},\frac{1}{2})_1\|(\{d\},\frac{1}{2})_1))[\!]((((\{c\},\frac{1}{2})_2;(\{d\},\frac{1}{2})_2)[\!]$
$((\{d\},\frac{1}{2})_3;\overline{(\{c\},\frac{1}{2})_3}))))) * \mathsf{Stop}]\!]_\approx$.

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ \psi'^* = \left(0, \frac{13}{38}, \frac{13}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}\right).$$

Consider $\mathcal{H} = \{s_3, s_3'\}$. We have $\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$ whereas $\sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'^*(s') = \psi'^*(s_3') = \frac{13}{38}$. Thus, $\underline{\leftrightarrow}_{is}$ plus $\equiv_{ss}$ do not guarantee a coincidence of steady-state probabilities to come in an equivalence class.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

Definition 32 *A* step trace *of a dynamic expression* G *is* $\Sigma = A_1 \cdots A_n \in (\mathbb{N}_f^{\mathcal{L}} \setminus \{\emptyset\})^*$ *where* $\exists s \in DR(G)\ s \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \leq i \leq n)$.

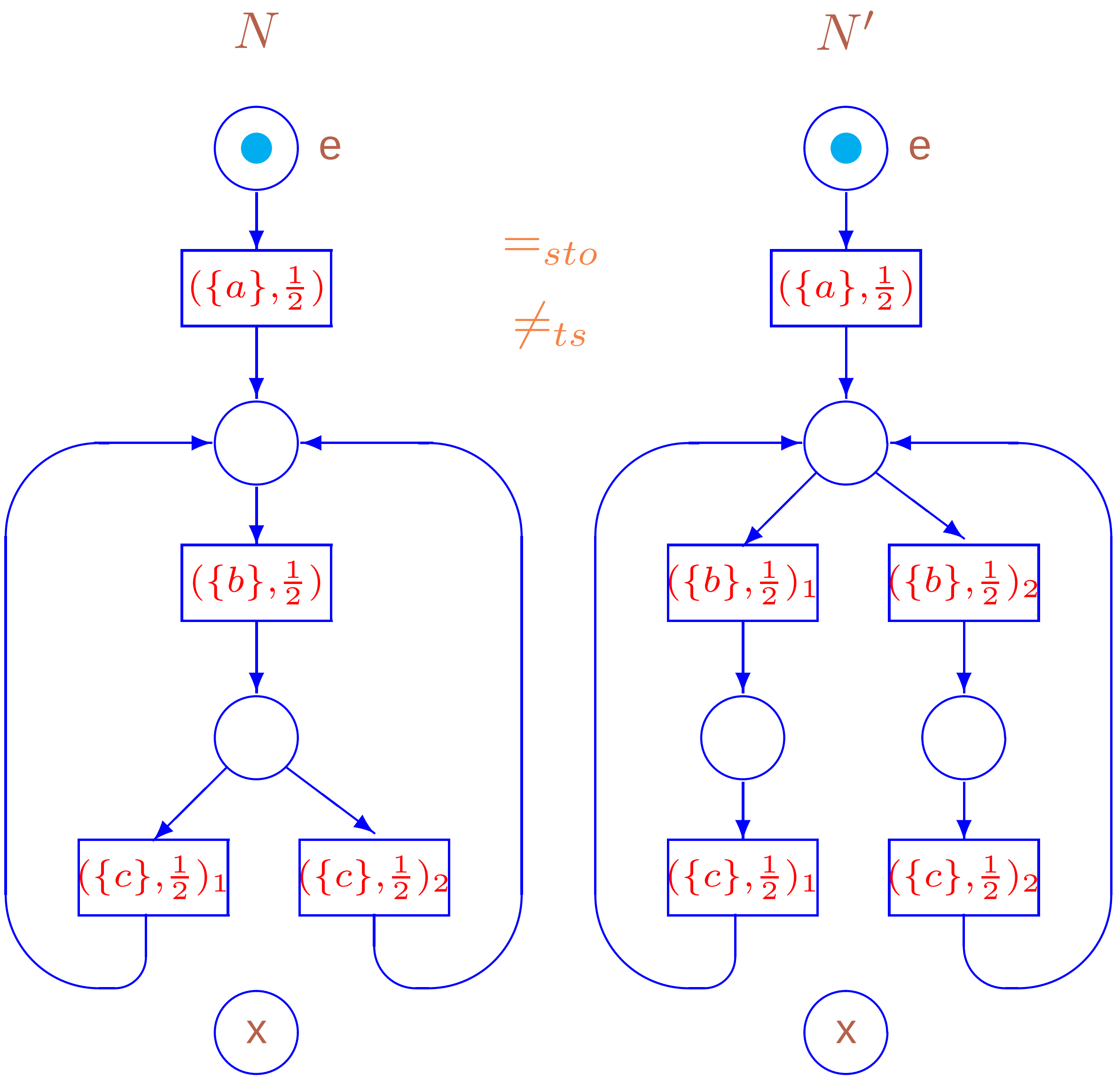
The probability to execute the step trace Σ in s:

$$PT^*(\Sigma, s) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid s=s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\leq i\leq n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}).$$

Theorem 6 *Let* G, G' *be dynamic expressions with* $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ *and* Σ *be a step trace. Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) PT^*(\Sigma, s) = \sum_{s' \in \mathcal{H} \cap DR(G')} {\psi'}^*(s') PT^*(\Sigma, s').$$

The result of the theorem above is valid if we replace steady-state probabilities with transient ones.



$\underline{\leftrightarrow}_{ss}$ implies coincidence of step trace probabilities

Let $E = [(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2)) * \mathsf{Stop}]$ and
$E' = [(\{a\},\frac{1}{2}) * (((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2)) * \mathsf{Stop}]$.
We have $\overline{E} =_{sto} \overline{E'}$, hence, $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$.

$DR(\overline{E})$ consists of

$s_1 = [[\overline{(\{a\},\frac{1}{2})} * ((\{b\},\frac{1}{2});((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s_2 = [[(\{a\},\frac{1}{2}) * (\overline{(\{b\},\frac{1}{2})};((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s_3 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});\overline{((\{c\},\frac{1}{2})_1[](\{c\},\frac{1}{2})_2)}) * \mathsf{Stop}]]_\approx$.

$DR(\overline{E'})$ consists of

$s'_1 = [[\overline{(\{a\},\frac{1}{2})} * (((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s'_2 = [[(\{a\},\frac{1}{2}) * \overline{(((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2))} * \mathsf{Stop}]]_\approx$,
$s'_3 = [[(\{a\},\frac{1}{2}) * (((\{b\},\frac{1}{2})_1;\overline{(\{c\},\frac{1}{2})_1})[]((\{b\},\frac{1}{2})_2;(\{c\},\frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s'_4 = [[(\{a\},\frac{1}{2}) * (((\{b\},\frac{1}{2})_1;(\{c\},\frac{1}{2})_1)[]((\{b\},\frac{1}{2})_2;\overline{(\{c\},\frac{1}{2})_2})) * \mathsf{Stop}]]_\approx$.

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{1}{2}, \frac{1}{2}\right),\ \psi'^* = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Consider $\mathcal{H} = \{s_3, s_3', s_4'\}$. The steady-state probabilities for $\mathcal{H}$ coincide:
$\sum_{s\in\mathcal{H}\cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \psi'^*(s_3') + \psi'^*(s_4') = \sum_{s'\in\mathcal{H}\cap DR(\overline{E'})} \psi'^*(s')$.
Let $\Sigma = \{\{c\}\}$. The steady-state probabilities to come in the equivalence class $\mathcal{H}$ and start the step trace Σ from it coincide as well: $\psi^*(s_3)(PT^*(\{(\{c\}, \frac{1}{2})_1\}, s_3) + PT^*(\{(\{c\}, \frac{1}{2})_2\}, s_3)) =$
$\frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2} = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 =$
$\psi'^*(s_3')PT^*(\{(\{c\}, \frac{1}{2})_1\}, s_3') + \psi'^*(s_4')PT^*(\{(\{c\}, \frac{1}{2})_2\}, s_4')$.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

The method of performance analysis simplification.

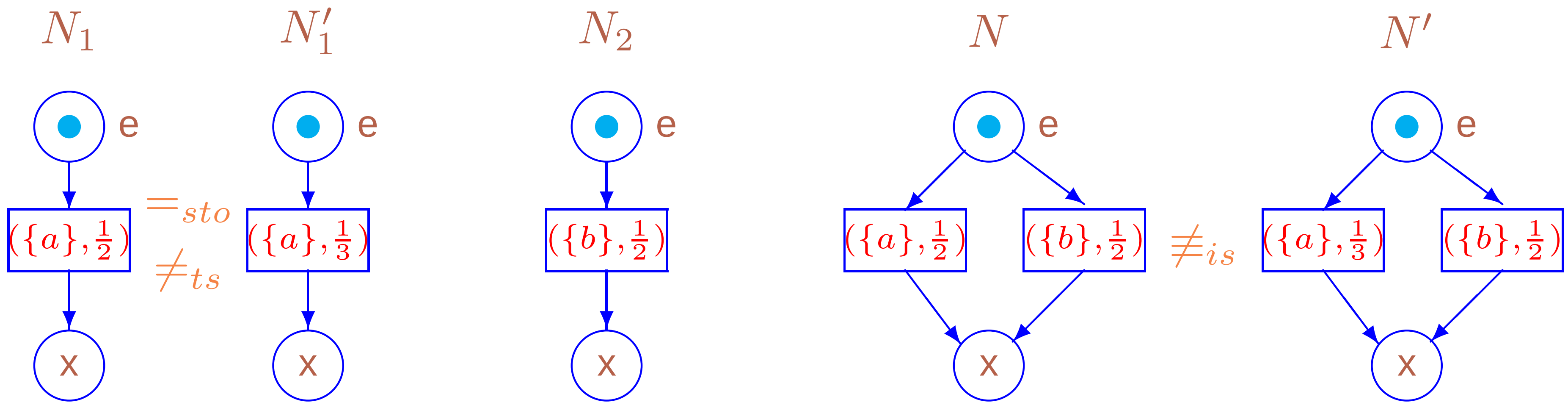
1. The system under investigation is specified by a static expression of $dtsPBC$.
2. The transition system without empty loops of the expression is constructed.
3. After examining this transition system for self-similarity and symmetry,
   a step stochastic autobisimulation equivalence for the expression is determined.
4. The quotient underlying DTMC without empty loops of the expression is constructed.
5. The steady-state probabilities and performance indices based on this DTMC are calculated.

Preservation by algebraic operations

Definition 33 *Let $\leftrightarrow$ be an equivalence of dynamic expressions. Static expressions E and E' are* equivalent w.r.t. $\leftrightarrow$, $E \leftrightarrow E'$, *if* $\overline{E} \leftrightarrow \overline{E'}$.

Proposition 5 *Let* $\star \in \{i, s\}$, $\star\star \in \{sto, ts\}$. *The equivalences* $\equiv_\star$, $\underline{\leftrightarrow}_\star$, $=_{\star\star}$ *are not preserved by algebraic operations.*

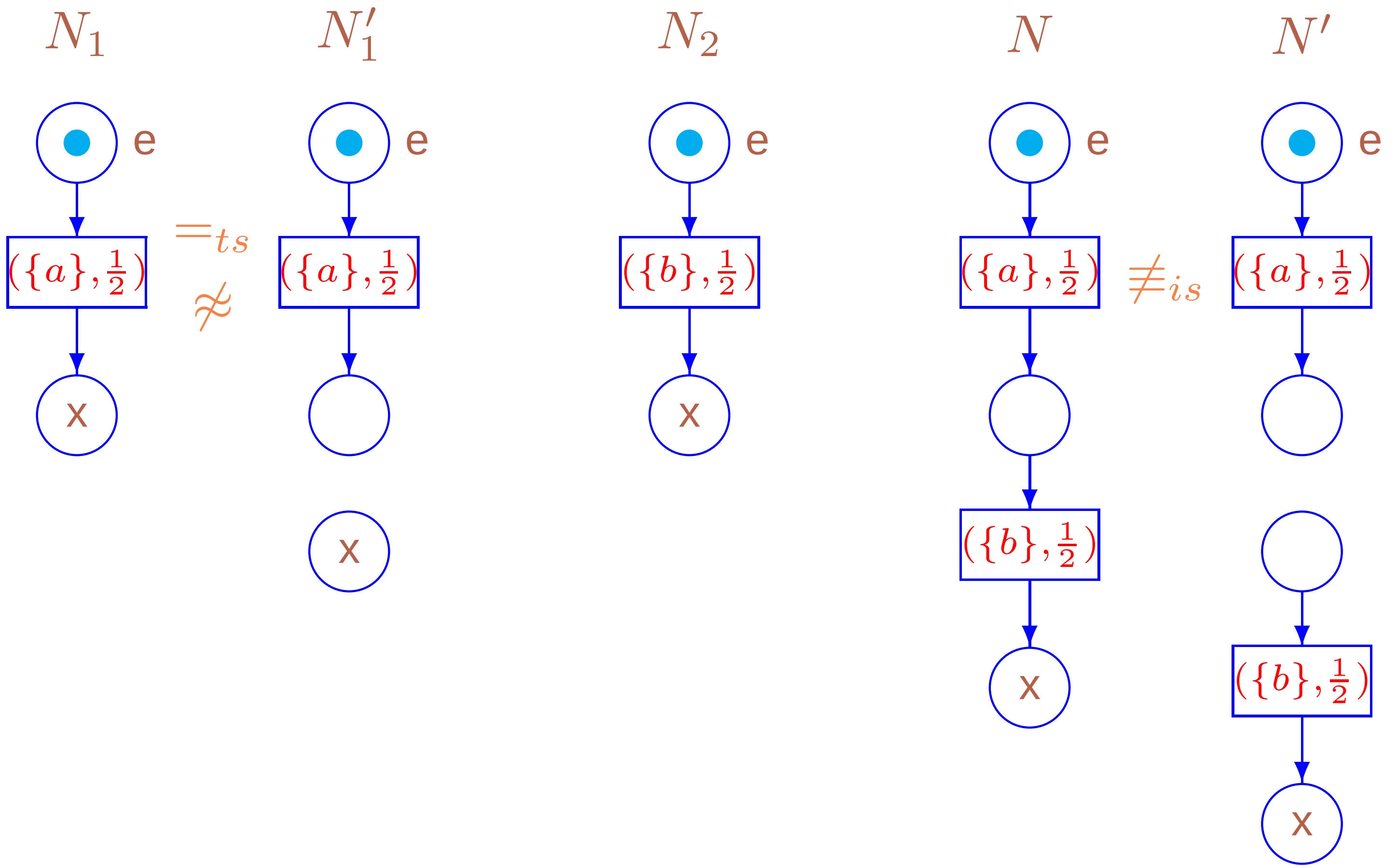
Proposition 6 *The equivalence $\approx$ is preserved by algebraic operations.*



SC1: The equivalences between $\equiv_{is}$ and $=_{sto}$ are not congruences

Let $E = (\{a\}, \frac{1}{2}),\ E' = (\{a\}, \frac{1}{3})$ and $F = (\{b\}, \frac{1}{2})$. We have $\overline{E} =_{sto} \overline{E'}$, since $TS^*(\overline{E})$ and $TS^*(\overline{E'})$ have the transitions with the multiaction part of labels $\{a\}$ and probability 1. $\overline{E[]F} \not\equiv_{is} \overline{E'[]F}$, since only in $TS^*(\overline{E'[]F})$ the probabilities of the transitions with the multiaction parts of labels $\{a\}$ and $\{b\}$ are different ($\frac{1}{3}$ and $\frac{2}{3}$, respectively). Thus, no equivalence between $\equiv_{is}$ and $=_{sto}$ is a congruence.

In the figure above $N_1 = Box_{dts}(\overline{E}),\ N_1' = Box_{dts}(\overline{E'}),\ N_2 = Box_{dts}(\overline{F})$ and $N = Box_{dts}(\overline{E[]F}),\ N' = Box_{dts}(\overline{E'[]F})$.



SC2: The equivalences between $\equiv_{is}$ and $=_{ts}$ are not congruences

Let $E = (\{a\}, \frac{1}{2})$, $E' = (\{a\}, \frac{1}{2})$; Stop and $F = (\{b\}, \frac{1}{2})$. We have $\overline{E} =_{ts} \overline{E'}$, since both $TS(\overline{E})$ and $TS(\overline{E'})$ have the transitions with the multiaction part of labels $\{a\}$ and probability $\frac{1}{2}$. $\overline{E;F} \neq_{is} \overline{E';F}$, since only in $TS^*(\overline{E';F})$ no other transition can fire after the transition with the multiaction part of label $\{a\}$. Thus, no equivalence between $\equiv_{is}$ and $=_{ts}$ is a congruence. In the figure above $N_1 = Box_{dts}(\overline{E})$, $N_1' = Box_{dts}(\overline{E'})$, $N_2 = Box_{dts}(\overline{F})$ and $N = Box_{dts}(\overline{E;F})$, $N' = Box_{dts}(\overline{E';F})$.

For an analogue of $=_{ts}$ to be a congruence, we have to equip transition systems with two extra transitions skip and redo as in [MVC02].

The equivalences between $\equiv_{is}$ and $=_{sto}$ defined on the basis of the enriched transition systems will still be non-congruences by Example SC1.

Let $E \in RegStatExpr$.

$$\overline{E} \stackrel{\mathsf{skip}}{\rightarrow} \underline{E} \qquad\qquad \underline{E} \stackrel{\mathsf{redo}}{\rightarrow} \overline{E}$$

Definition 34 *Let E be a static expression and $TS(\overline{E}) = (S, L, \mathcal{T}, s)$. The* (labeled probabilistic) sr-transition system *of $\overline{E}$ is a quadruple $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$:*

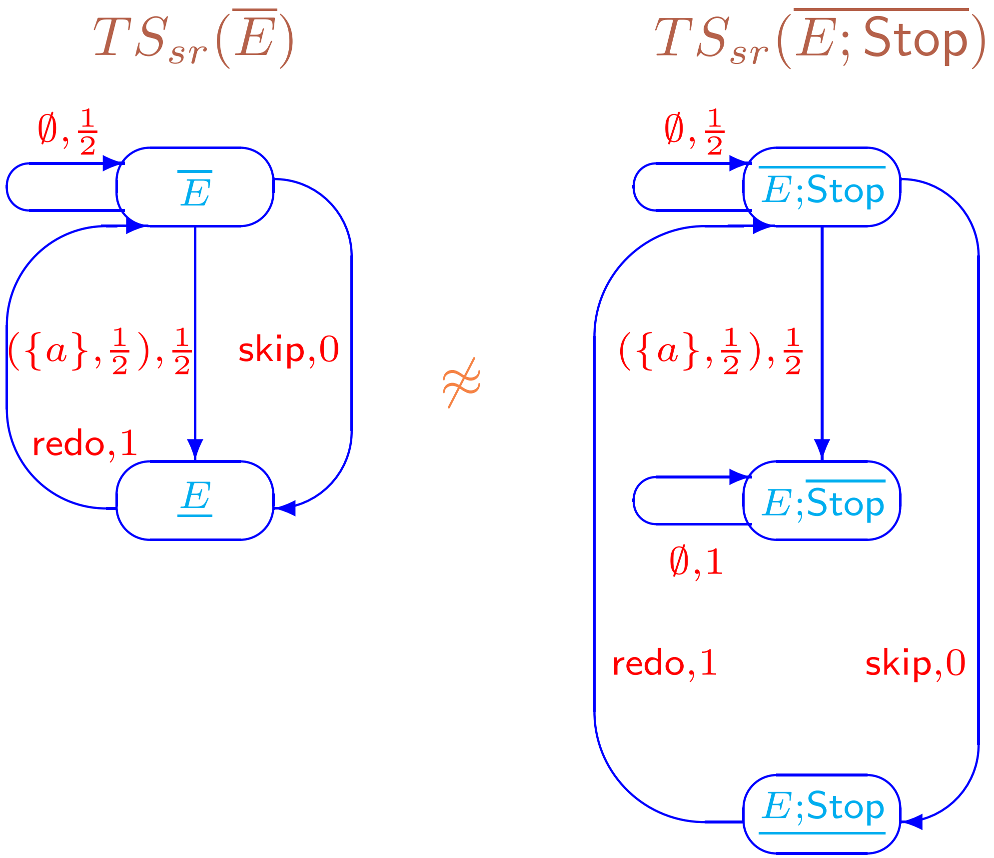
- $S_{sr} = S \cup \{[\underline{E}]_\approx\}$;
- $L_{sr} \subseteq (\mathbb{N}_f^{\mathcal{SL}} \times (0;1]) \cup \{(\mathsf{skip}, 0), (\mathsf{redo}, 1)\}$;
- $\mathcal{T}_{sr} = \mathcal{T} \setminus \{([\underline{E}]_\approx, (\emptyset, 1), [\underline{E}]_\approx)\} \cup \{([\overline{E}]_\approx, (\mathsf{skip}, 0), [\underline{E}]_\approx), ([\underline{E}]_\approx, (\mathsf{redo}, 1), [\overline{E}]_\approx)\}$;
- $s_{sr} = s$.

Definition 35 *Let* E, E' *be static expressions and* $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$, $TS_{sr}(\overline{E'}) = (S'_{sr}, L'_{sr}, \mathcal{T}'_{sr}, s'_{sr})$ *be their* sr*-transition systems. A mapping* $\beta : S_{sr} \to S'_{sr}$ *is an* isomorphism *between* $TS_{sr}(\overline{E})$ *and* $TS_{sr}(\overline{E'})$, $\beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, *if*

1. β *is a bijection s.t.* $\beta(s_{sr}) = s'_{sr}$ *and* $\beta([\underline{E}]_{\approx}) = [\underline{E'}]_{\approx}$;
2. $\forall s, \tilde{s} \in S_{sr}\ \forall \Gamma\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s} \ \Leftrightarrow\ \beta(s) \xrightarrow{\Gamma}_{\mathcal{P}} \beta(\tilde{s})$.

Two sr*-transition systems* $TS_{sr}(\overline{E})$ *and* $TS_{sr}(\overline{E'})$ *are* isomorphic, $TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, *if* $\exists \beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$.

For $E \in RegStatExpr$, let $TS_{sr}(E) = TS_{sr}(\overline{E})$.



TSSR: The sr-transition systems of $\overline{E}$ and $\overline{E;\mathsf{Stop}}$ for $E = (\{a\}, \frac{1}{2})$

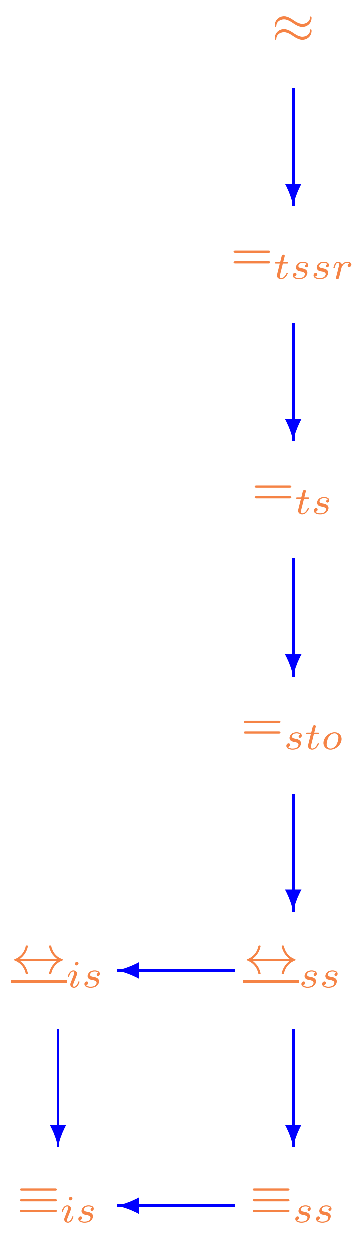
Let $E = (\{a\}, \frac{1}{2})$. In the figure above the transition systems $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E;\mathsf{Stop}})$ are presented.

In the latter sr-transition system the final state can be reached by the transition $(\mathsf{skip}, 0)$ only from the initial state .

Definition 36 $\overline{E}$ *and* $\overline{E'}$ *are* isomorphic w.r.t. sr-transition systems, $\overline{E} =_{tssr} \overline{E'}$, *if* $TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$.

sr*-transition systems without empty loops* can be defined and the equivalence $=_{tssr*}$ based on them.

The coincidence of $=_{tssr}$ and $=_{tssr*}$ can be proved as for $=_{ts}$ and $=_{ts*}$.



Interrelations of the stochastic equivalences and the new congruence

Theorem 7 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, is, ss, sto, ts, tssr\}$*. For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph in figure above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.

Validity of the implications

- The implication $=_{tssr} \rightarrow =_{ts}$ is valid, since sr-transition systems have more states and transitions than usual ones.
- The implication $\approx \rightarrow =_{tssr}$ is valid, since the sr-transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

- Let $E=(\{a\},\frac{1}{2})$ and $E'=(\{a\},\frac{1}{2});\mathsf{Stop}$. We have $\overline{E}=_{ts}\overline{E'}$, see example with Figure SC2. On the other hand, $\overline{E}\neq_{tssr}\overline{E'}$, since only in $TS_{sr}(\overline{E'})$ after the transition with multiaction part of label $\{a\}$ we do not reach the final state, see Figure TSSR.
- Let $E=(\{a\},\frac{1}{2})$ and $E'=((\{a\},\frac{1}{2});(\{\hat{a}\},\frac{1}{2}))$ sy a. Then $\overline{E}=_{tssr}\overline{E'}$, since $\overline{E}=_{ts}\overline{E'}$ by the last example from the equivalence interrelations theorem, and the final states of both $TS_{sr}(\overline{E'})$ and $TS_{sr}(\overline{E'})$ are reachable from the others with “normal” transitions (not with skip only). On the other hand, $\overline{E}\not\approx\overline{E'}$.

Theorem 8 *Let* $a \in Act$ *and* $E, E', F \in RegStatExpr$. *If* $\overline{E} =_{tssr} \overline{E'}$ *then*

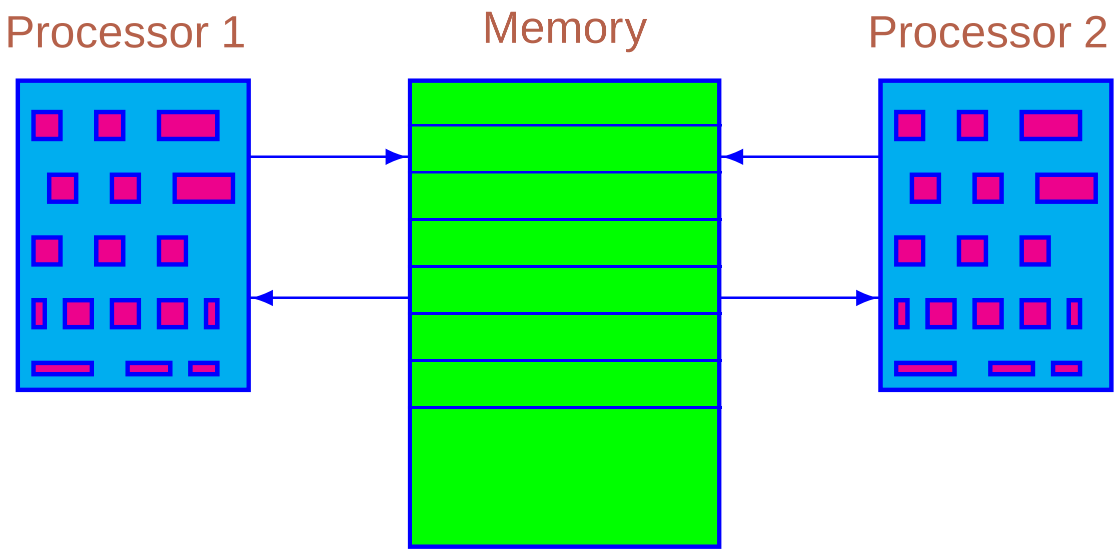
1. $\overline{E \circ F} =_{tssr} \overline{E' \circ F}$, $\overline{F \circ E} =_{tssr} \overline{F \circ E'}$, $\circ \in \{;, [], \|\}$;
2. $\overline{E[f]} =_{tssr} \overline{E'[f]}$;
3. $\overline{E \circ a} =_{tssr} \overline{E' \circ a}$, $\circ \in \{\mathsf{rs}, \mathsf{sy}\}$;
4. $\overline{[E * F * K]} =_{tssr} \overline{[E' * F * K]}$, $\overline{[F * E * K]} =_{tssr} \overline{[F * E' * K]}$, $\overline{[F * K * E]} =_{tssr} \overline{[F * K * E']}$.

Case studies

Shared memory system

The standard system

A model of two processors accessing a common shared memory [MBCDF95]



The diagram of the shared memory system

After activation of the system, two processors are active, and the common memory is available. Each processor can request an access to the memory.

When a processor starts an acquisition of the memory, another processor waits until the former one ends its operations, and the system returns to the state with both active processors and the available memory.

a corresponds to the system activation.

r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i.

b_i and e_i correspond to the beginning and the end of the common memory access of processor i.

The other actions are used for communication purpose only.

The static expression of the first processor is

$E_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r_1\}, \frac{1}{2}); (\{b_1, y_1\}, \frac{1}{2}); (\{e_1, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$E_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r_2\}, \frac{1}{2}); (\{b_2, y_2\}, \frac{1}{2}); (\{e_2, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$E_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the shared memory system with two processors is

$E = (E_1 \| E_2 \| E_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{E})$ consists of

$$s_1=[([\overline{(\{x_1\},\tfrac{1}{2})}*((\{r_1\},\tfrac{1}{2});(\{b_1,y_1\},\tfrac{1}{2});(\{e_1,z_1\},\tfrac{1}{2}))*\mathsf{Stop}]$$
$$\|[\overline{(\{x_2\},\tfrac{1}{2})}*((\{r_2\},\tfrac{1}{2});(\{b_2,y_2\},\tfrac{1}{2});(\{e_2,z_2\},\tfrac{1}{2}))*\mathsf{Stop}]$$
$$\|[\overline{(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2})}*(((\{\widehat{y_1}\},\tfrac{1}{2});(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2})))*\mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_2=[([(\{x_1\},\tfrac{1}{2})*(\overline{(\{r_1\},\tfrac{1}{2})};(\{b_1,y_1\},\tfrac{1}{2});(\{e_1,z_1\},\tfrac{1}{2}))*\mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2})*(\overline{(\{r_2\},\tfrac{1}{2})};(\{b_2,y_2\},\tfrac{1}{2});(\{e_2,z_2\},\tfrac{1}{2}))*\mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2})*(((\{\widehat{y_1}\},\tfrac{1}{2});(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2})))*\mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_3=[([(\{x_1\},\tfrac{1}{2})*((\{r_1\},\tfrac{1}{2});\overline{(\{b_1,y_1\},\tfrac{1}{2})};(\{e_1,z_1\},\tfrac{1}{2}))*\mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2})*(\overline{(\{r_2\},\tfrac{1}{2})};(\{b_2,y_2\},\tfrac{1}{2});(\{e_2,z_2\},\tfrac{1}{2}))*\mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2})*((\overline{(\{\widehat{y_1}\},\tfrac{1}{2})};(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},\tfrac{1}{2});(\{\widehat{z_2}\},\tfrac{1}{2})))*\mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_4 = [([(\{x_1\}, \tfrac{1}{2}) * ((\overline{(\{r_1\}, \tfrac{1}{2})}; (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{b_2, y_2\}, \tfrac{1}{2})}; (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[](\overline{(\{\widehat{y_2}\}, \tfrac{1}{2})}; (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_5 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); \overline{(\{e_1, z_1\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (\overline{(\{r_2\}, \tfrac{1}{2})}; (\{b_2, y_2\}, \tfrac{1}{2}); (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_1}\}, \tfrac{1}{2})})[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

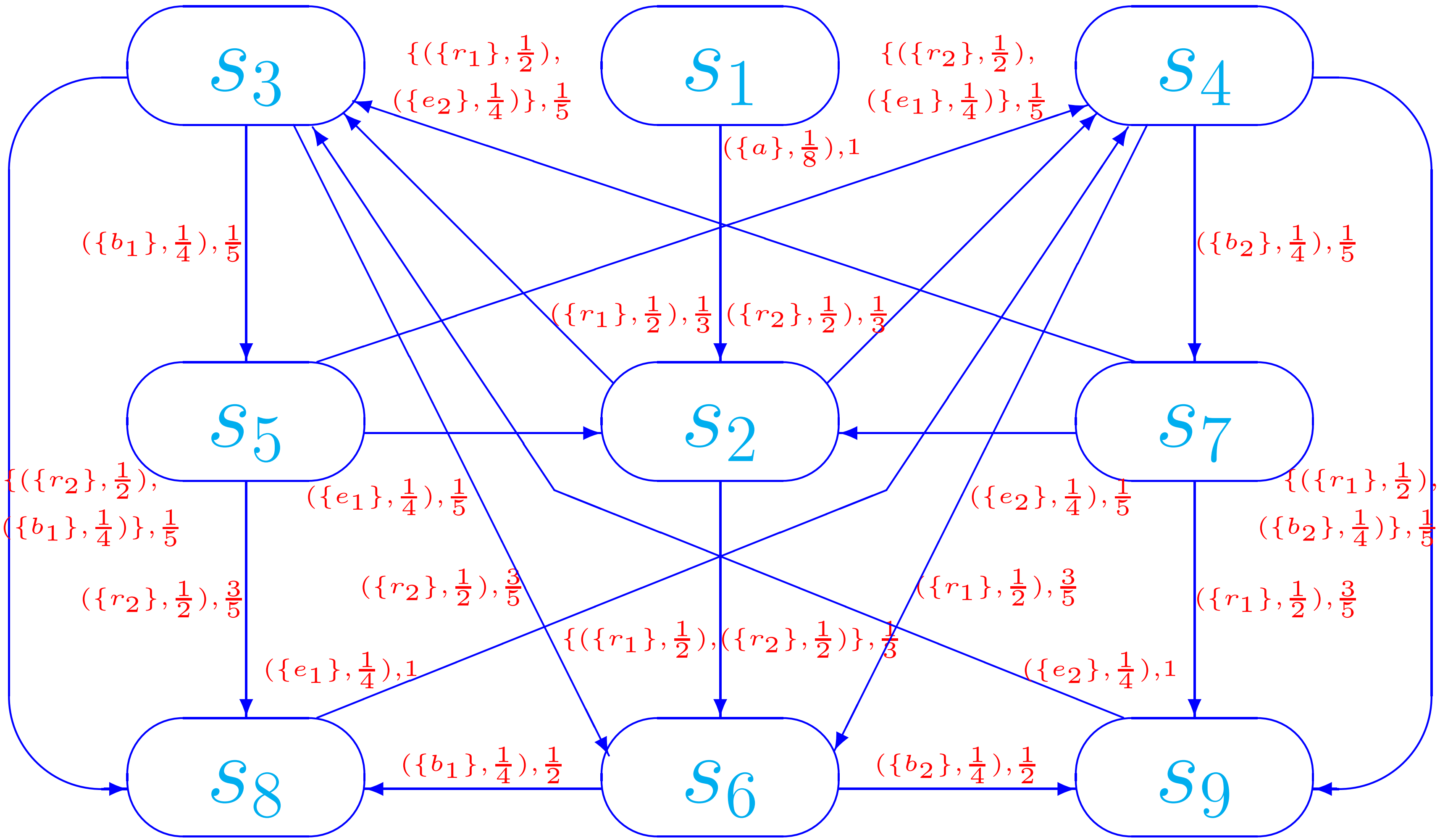
$$s_6 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); \overline{(\{b_1, y_1\}, \tfrac{1}{2})}; (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{b_2, y_2\}, \tfrac{1}{2})}; (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((\overline{(\{\widehat{y_1}\}, \tfrac{1}{2})}; (\{\widehat{z_1}\}, \tfrac{1}{2}))[](\overline{(\{\widehat{y_2}\}, \tfrac{1}{2})}; (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_7 = [([(\{x_1\}, \tfrac{1}{2}) * (\overline{(\{r_1\}, \tfrac{1}{2})}; (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); \overline{(\{e_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_8 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); \overline{(\{e_1, z_1\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_1}\}, \tfrac{1}{2})})[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

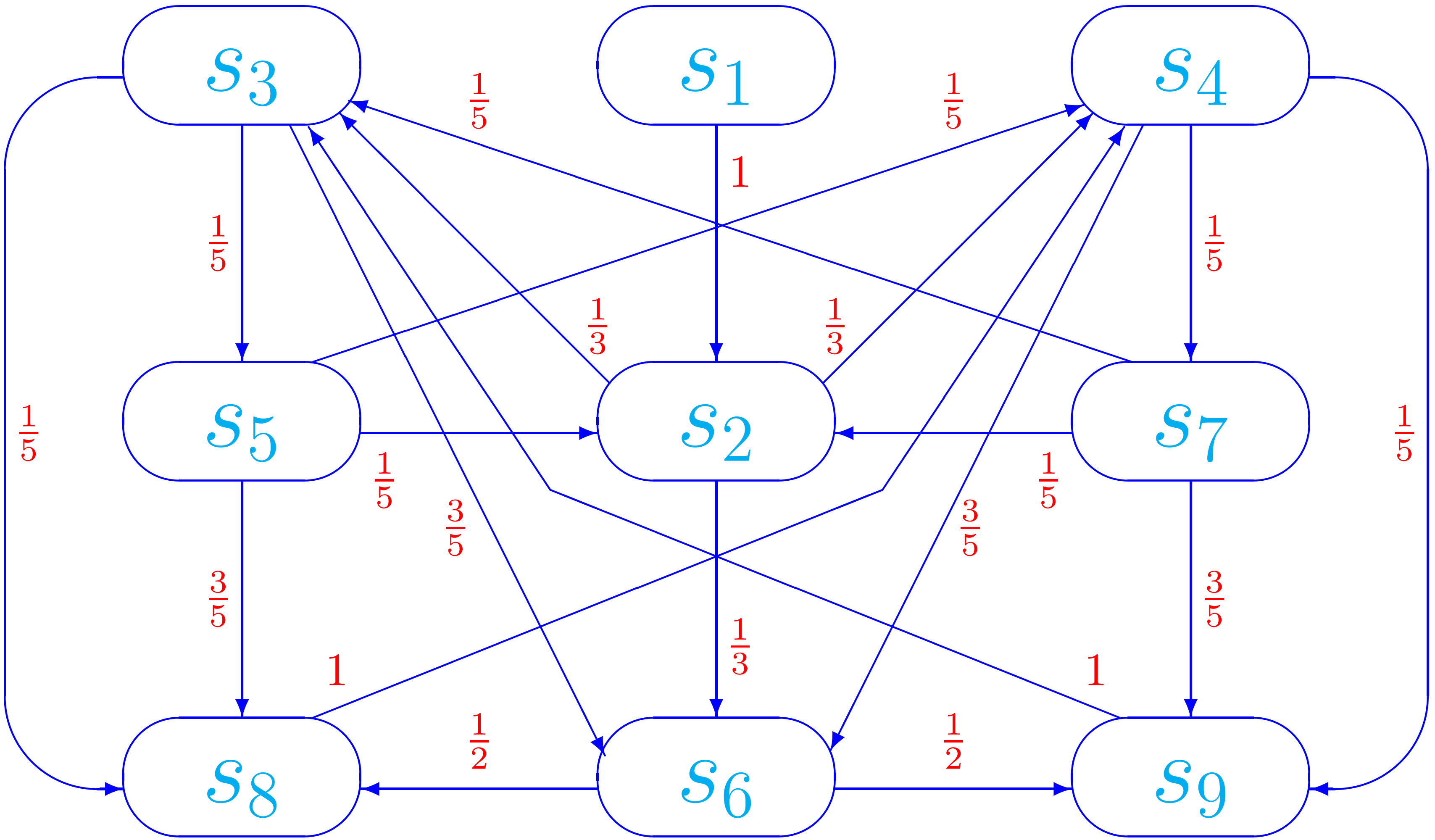
$$s_9 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); \overline{(\{e_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx.$$

$TS^*(\overline{E})$



The transition system without empty loops of the shared memory system

$DTMC^*(\overline{E})$



The underlying DTMC without empty loops of the shared memory system

The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

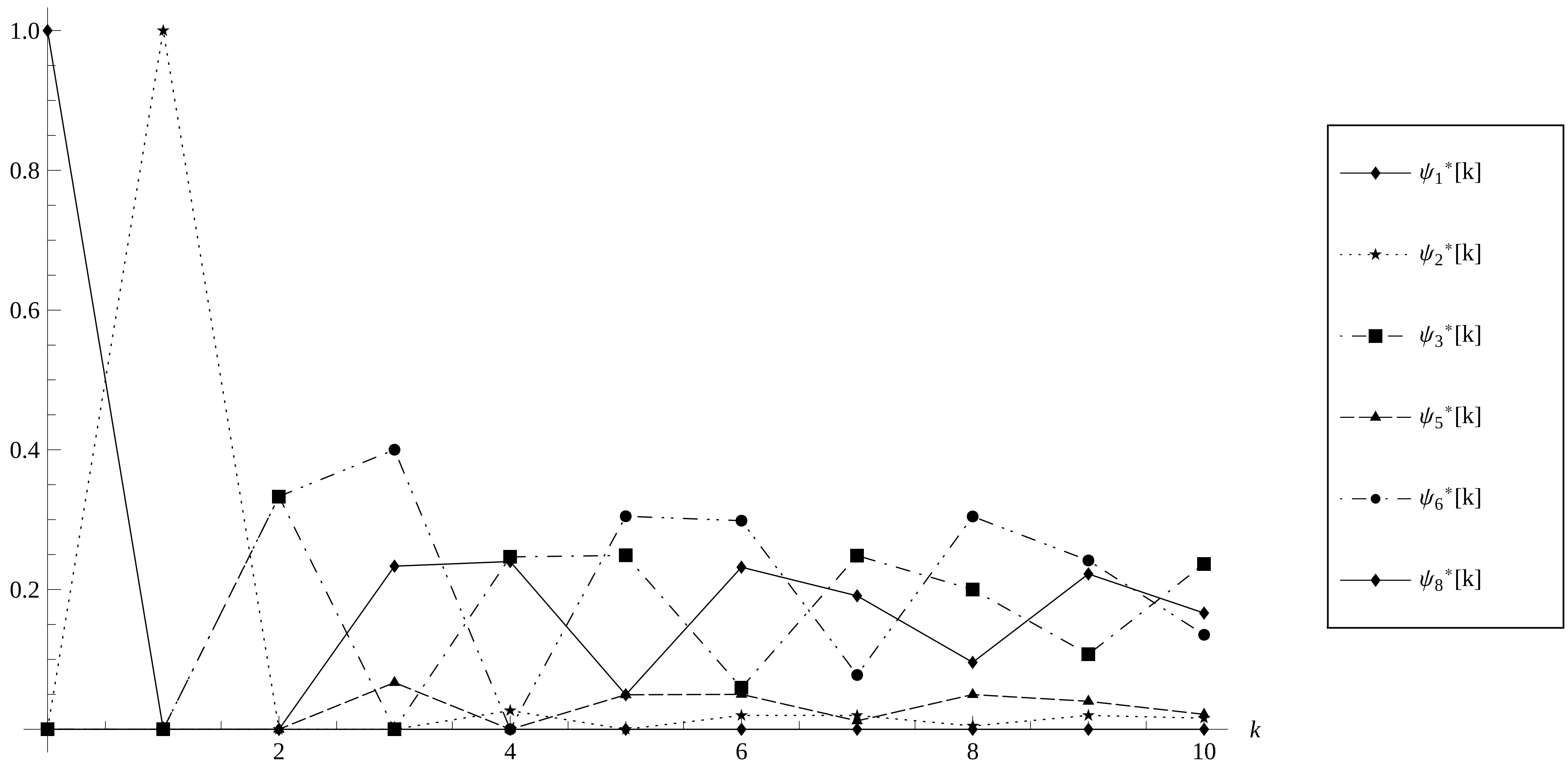
The steady-state PMF for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{3}{209}, \frac{75}{418}, \frac{75}{418}, \frac{15}{418}, \frac{46}{209}, \frac{15}{418}, \frac{35}{209}, \frac{35}{209}\right).$$

Transient and steady-state probabilities of the shared memory system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2^*[k]$	0	1	0	0	0.0267	0	0.0197	0.0199	0.0047	0.0199	0.0160	0.0144
$\psi_3^*[k]$	0	0	0.3333	0	0.2467	0.2489	0.0592	0.2484	0.2000	0.1071	0.2368	0.1794
$\psi_5^*[k]$	0	0	0	0.0667	0	0.0493	0.0498	0.0118	0.0497	0.0400	0.0214	0.0359
$\psi_6^*[k]$	0	0	0.3333	0.4000	0	0.3049	0.2987	0.0776	0.3047	0.2416	0.1351	0.2201
$\psi_8^*[k]$	0	0	0	0.2333	0.2400	0.0493	0.2318	0.1910	0.0956	0.2221	0.1662	0.1675

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.



Transient probabilities alteration diagram of the shared memory system

Performance indices

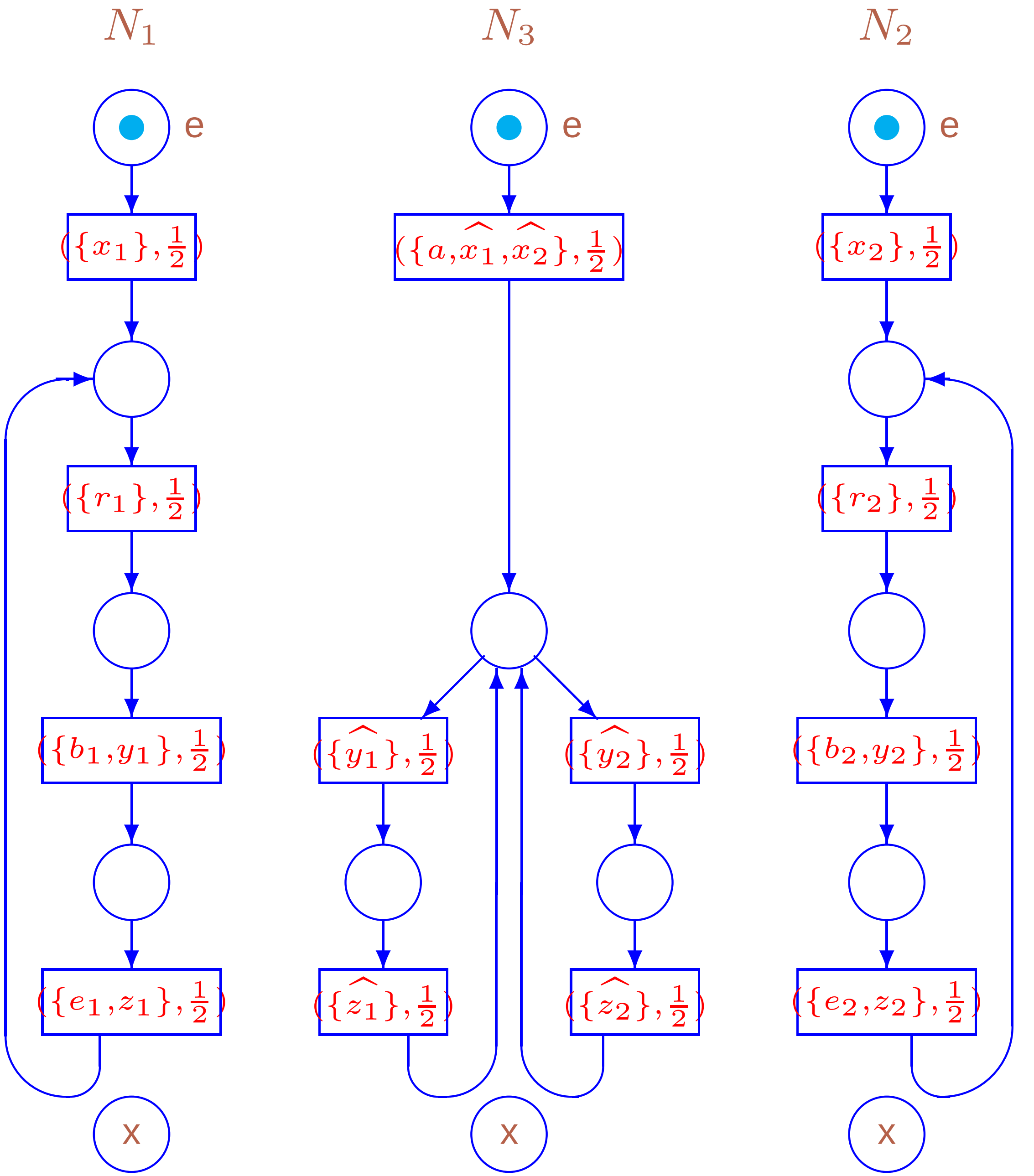
- The average recurrence time in the state s_2, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{209}{3} = 69\frac{2}{3}$.
- The common memory is available in the states s_2, s_3, s_4, s_6 only.

  The steady-state probability that the memory is available is $\psi_2^* + \psi_3^* + \psi_4^* + \psi_6^* = \frac{124}{209}$.

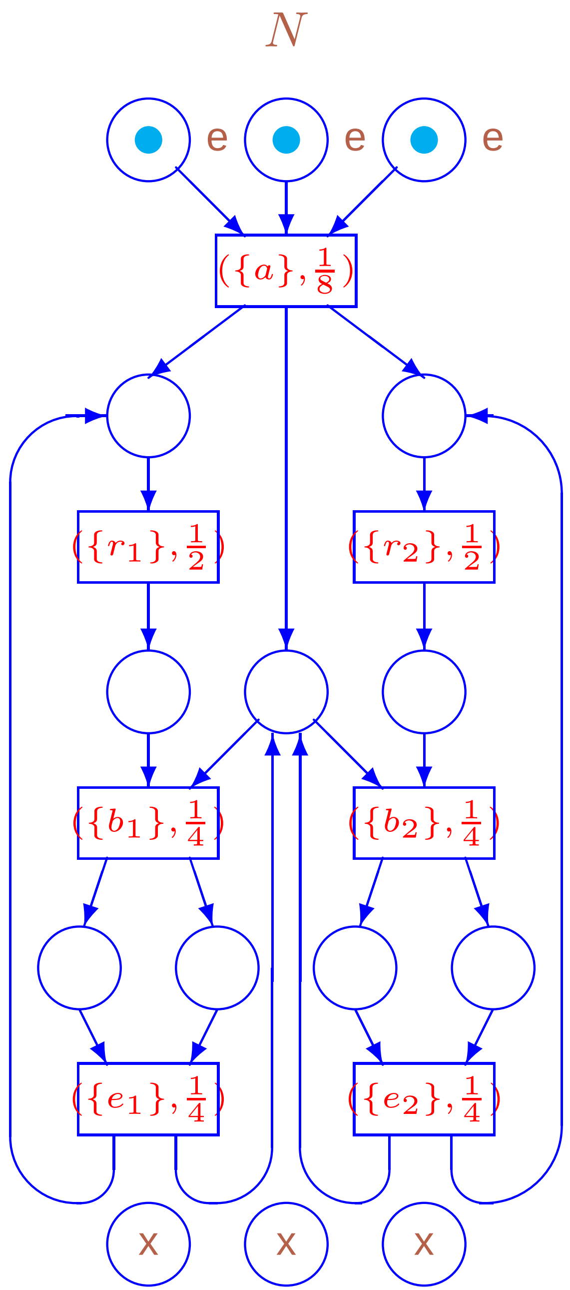
  The steady-state probability that the memory is used, the *shared memory utilization*, is $1 - \frac{124}{209} = \frac{85}{209}$.
- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is only possible from the states s_2, s_4, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from the first processor* is $\psi_2^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_4^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_7^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) = \frac{3}{209}\left(\frac{1}{3} + \frac{1}{3}\right) + \frac{75}{418}\left(\frac{3}{5} + \frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5} + \frac{1}{5}\right) = \frac{38}{209}$.



The marked dts-boxes of two processors and shared memory



The marked dts-box of the shared memory system

The abstract system

The static expression of the first processor is

$F_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_1\}, \frac{1}{2}); (\{e, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$F_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_2\}, \frac{1}{2}); (\{e, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$F_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract shared memory system with two processors is

$F = (F_1 \| F_2 \| F_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$.

$DTMC^*(\overline{F}) = DTMC^*(\overline{E})$, thus, the TPM and the steady-state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

Performance indices

The first and second performance indices are the same for the standard and abstract systems.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \frac{1}{2})$ is only possible from the states s_2, s_3, s_4, s_5, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from the first processor* is

$$\psi_2^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_3) +$$
$$\psi_4^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_5^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_5) +$$
$$\psi_7^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) =$$
$$\frac{3}{209}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) + \frac{75}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{75}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5}+\frac{1}{5}\right) = \frac{75}{209}.$$

The quotients for the abstract system

$DR(\overline{F})/_{\underline{\leftrightarrow}_{ss}} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6\}$, where

$\mathcal{K}_1 = \{s_1\}$ (the initial state),

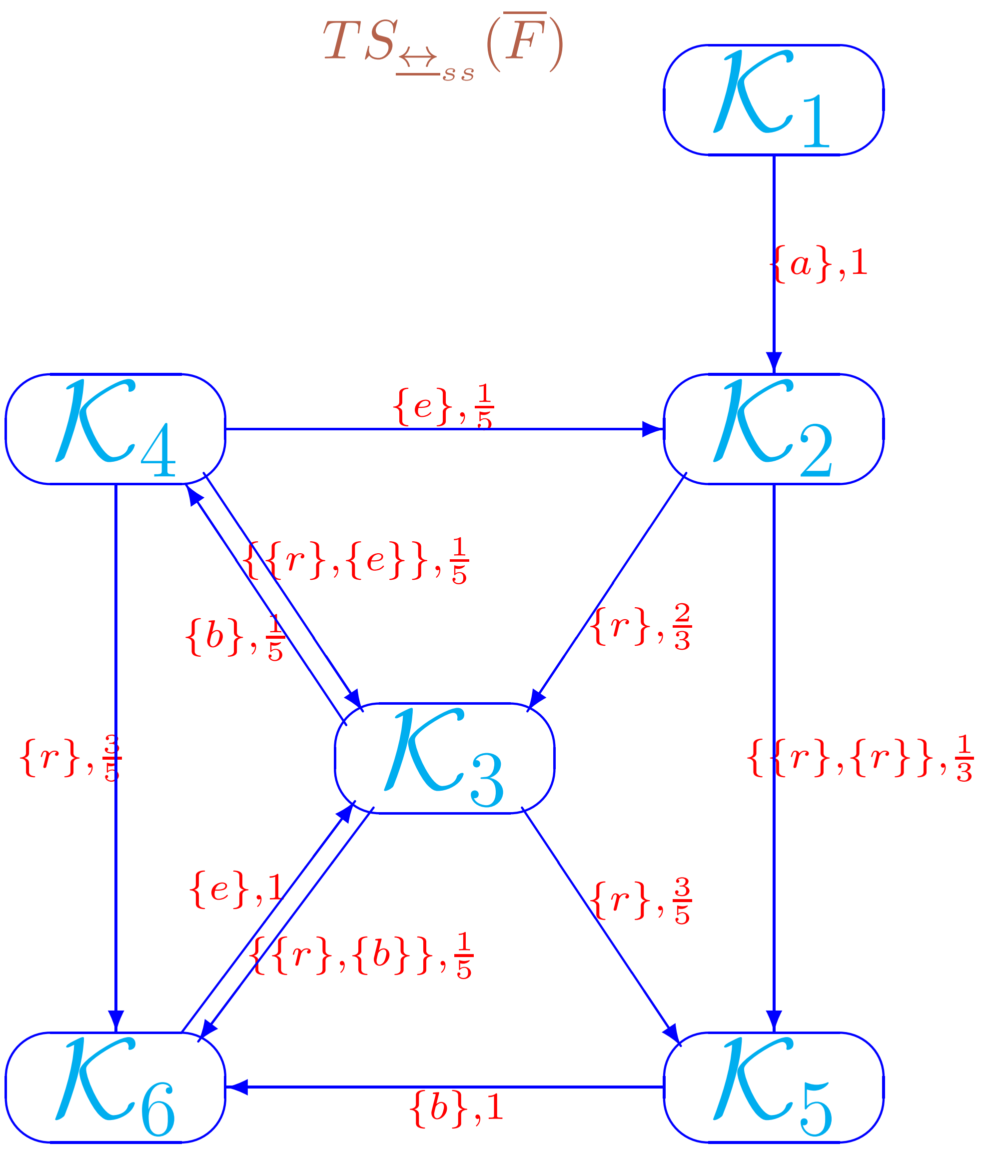
$\mathcal{K}_2 = \{s_2\}$ (the system is activated and the memory is not requested),

$\mathcal{K}_3 = \{s_3, s_4\}$ (the memory is requested by one processor),

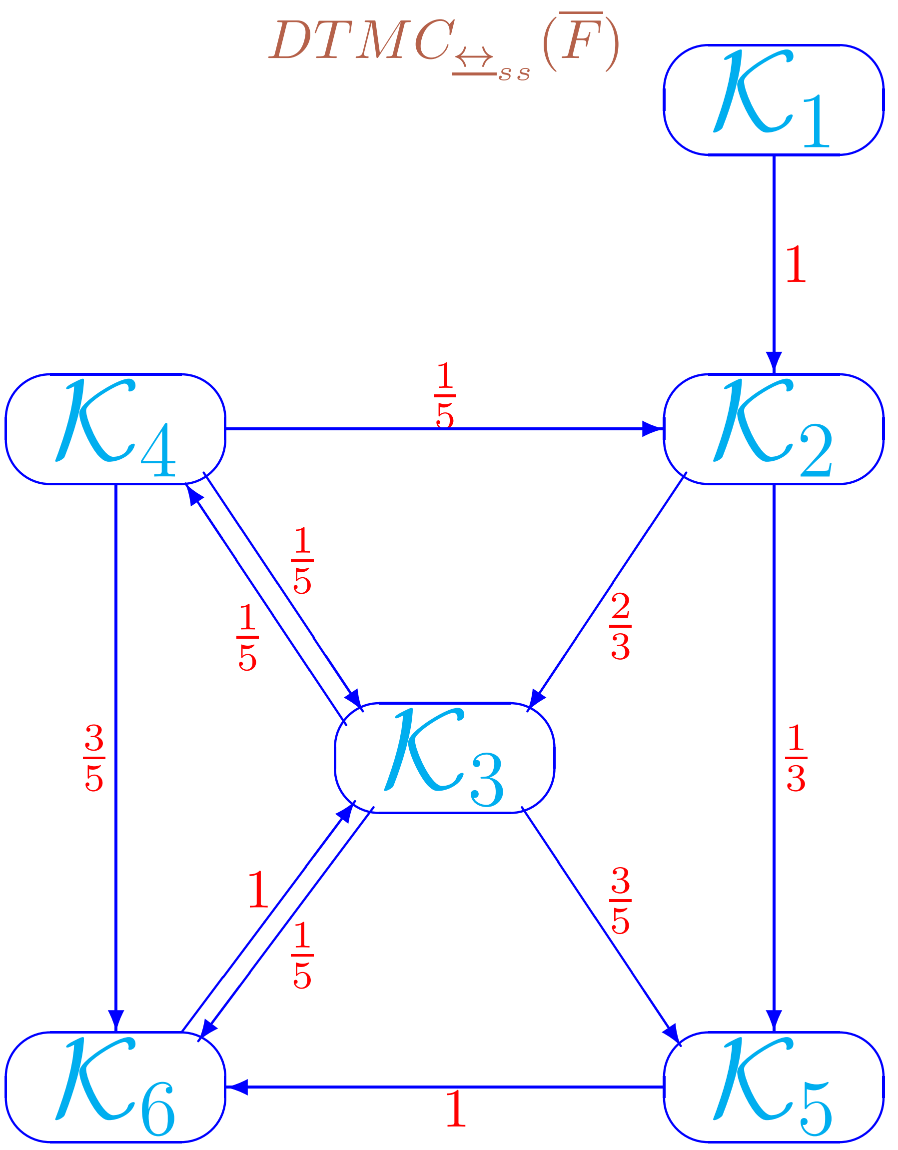
$\mathcal{K}_4 = \{s_5, s_7\}$ (the memory is allocated to a processor),

$\mathcal{K}_5 = \{s_6\}$ (the memory is requested by two processors),

$\mathcal{K}_6 = \{s_8, s_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).



The quotient transition system without empty loops of the abstract shared memory system



The quotient underlying DTMC without empty loops of the abstract shared memory system

The TPM for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

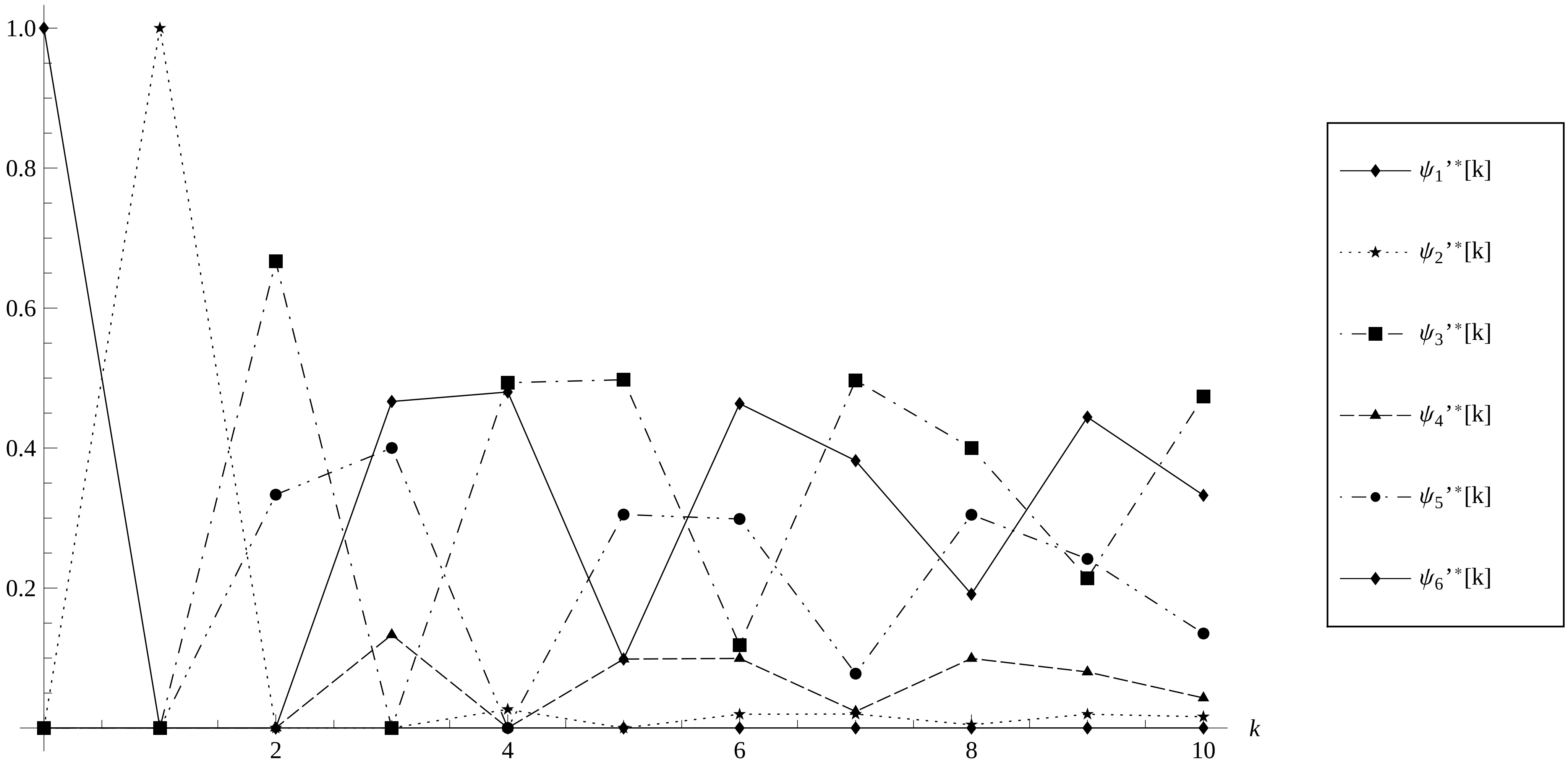
$$\mathbf{P}'^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The steady-state PMF for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\psi'^* = \left(0, \frac{3}{209}, \frac{75}{209}, \frac{15}{418}, \frac{46}{209}, \frac{70}{209}\right).$$

Transient and steady-state probabilities of the quotient abstract shared memory system

k	0	1	2	3	4	5	6	7	8	9	10	∞
${\psi'_1}^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
${\psi'_2}^*[k]$	0	1	0	0	0.0267	0	0.0197	0.0199	0.0047	0.0199	0.0160	0.0144
${\psi'_3}^*[k]$	0	0	0.6667	0	0.4933	0.4978	0.1184	0.4967	0.4001	0.2142	0.4735	0.3589
${\psi'_4}^*[k]$	0	0	0	0.1333	0	0.0987	0.0996	0.0237	0.0993	0.0800	0.0428	0.0718
${\psi'_5}^*[k]$	0	0	0.3333	0.4000	0	0.3049	0.2987	0.0776	0.3047	0.2416	0.1351	0.2201
${\psi'_6}^*[k]$	0	0	0	0.4667	0.4800	0.0987	0.4636	0.3821	0.1912	0.4443	0.3325	0.3349



Transient probabilities alteration diagram of the quotient abstract shared memory system

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{{\psi'_2}^*} = \frac{209}{3} = 69\frac{2}{3}$.
- The common memory is available in the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5$ only.

  The steady-state probability that the memory is available is
  ${\psi'_2}^* + {\psi'_3}^* + {\psi'_5}^* = \frac{3}{209} + \frac{75}{209} + \frac{46}{209} = \frac{124}{209}$.

  The steady-state probability that the memory is used (i.e., not available) called the *shared memory utilization* is $1 - \frac{124}{209} = \frac{85}{209}$.
- The common memory request of a processor $\{r\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$.

  The request probability in each of the states is a sum of execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is

$$
{\psi'_2}^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \mathcal{K}_2 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\mathcal{K}_2, \widetilde{\mathcal{K}}) +
{\psi'_3}^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \mathcal{K}_3 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\mathcal{K}, \widetilde{\mathcal{K}}) +
{\psi'_4}^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{r\}\in A,\ \mathcal{K}_4 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\mathcal{K}, \widetilde{\mathcal{K}}) =
\frac{3}{209}\left(\frac{2}{3} + \frac{1}{3}\right) + \frac{75}{209}\left(\frac{3}{5} + \frac{1}{5}\right) + \frac{15}{209}\left(\frac{3}{5} + \frac{1}{5}\right) = \frac{75}{209}.
$$

The performance indices are the same for the complete and the quotient abstract shared memory systems.

The coincidence of the first and second performance indices illustrates the result of proposition about steady-state probabilities.

The coincidence of the third performance index theorem about step traces from steady states:

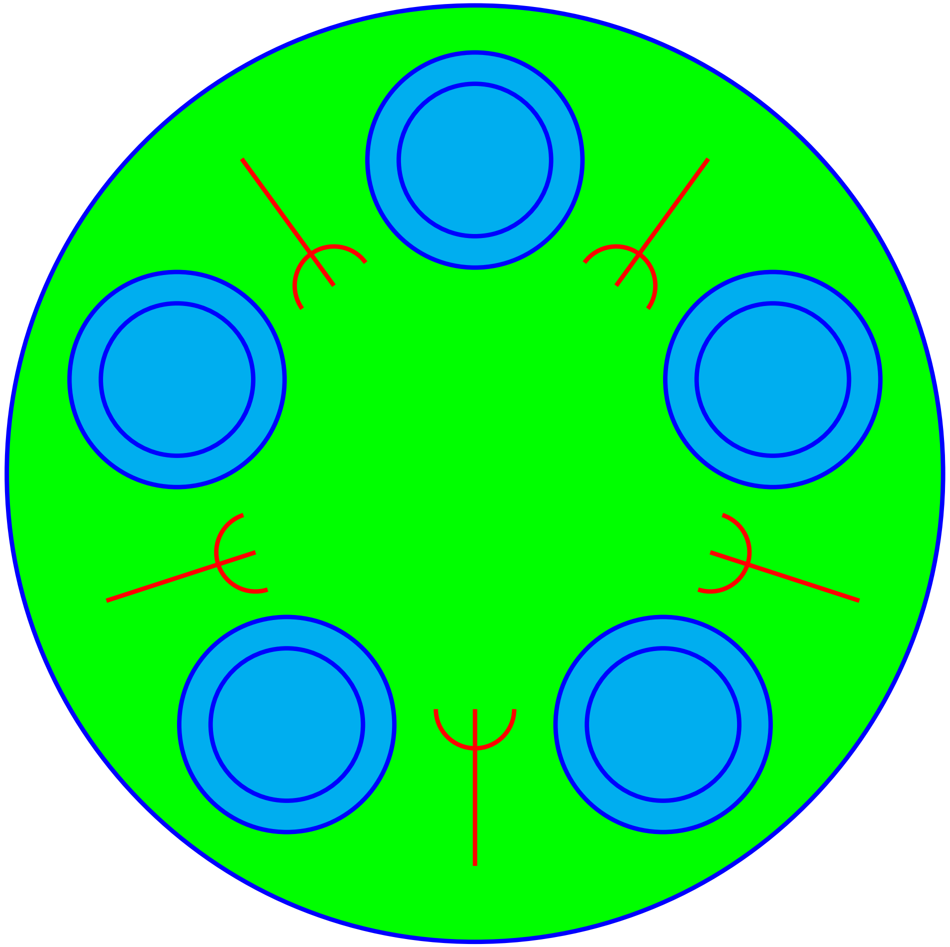
one should apply its result to the step traces $\{\{r\}\}, \{\{r\},\{r\}\}, \{\{r\},\{b\}\}, \{\{r\},\{e\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

Dining philosophers system

The standard system

A model of five dining philosophers [P81]



The diagram of the dining philosophers system

After activation of the system, five forks appear on the table.

If the left and right forks available for a philosopher, he takes them simultaneously and begins eating.

At the end of eating, the philosopher places both his forks simultaneously back on the table.

a corresponds to the system activation.

b_i and e_i correspond to the beginning and the end of eating of philosopher i $(1 \leq i \leq 5)$.

The other actions are used for communication purpose only.

The expression of each philosopher includes two alternative subexpressions:

the second one specifies a resource (fork) sharing with the right neighbor.

The static expression of the philosopher $i\ (1\le i\le 4)$ is

$E_i=[(\{x_i\},\frac{1}{2})*((((\{b_i,\widehat{y_i}\},\frac{1}{2});(\{e_i,\widehat{z_i}\},\frac{1}{2}))[]((\{y_{i+1}\},\frac{1}{2});(\{z_{i+1}\},\frac{1}{2})))*\mathsf{Stop}]$.

The static expression of the philosopher 5 is

$E_5=[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2})*((((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[]((\{y_1\},\frac{1}{2});(\{z_1\},\frac{1}{2})))*\mathsf{Stop}]$.

The static expression of the dining philosophers system is

$E=(E_1\|E_2\|E_3\|E_4\|E_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

$DR(\overline{E})$ consists of

$$s_1 = [([\overline{(\{x_1\},\tfrac{1}{2})} * (((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_2\},\tfrac{1}{2})} * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_3\},\tfrac{1}{2})} * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_4\},\tfrac{1}{2})} * (((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[\overline{(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2})} * (((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2});(\{z_1\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_\approx,$$

$$s_2 = [([(\{x_1\},\tfrac{1}{2}) * \overline{(((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * \overline{(((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_3\},\tfrac{1}{2}) * \overline{(((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_4\},\tfrac{1}{2}) * \overline{(((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * \overline{(((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2});(\{z_1\},\tfrac{1}{2})))} * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_\approx,$$

$$s_3 = [([(\{x_1\}, \tfrac{1}{2}) * ((((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{e_1, \widehat{z_1}\}, \tfrac{1}{2})})[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[](\overline{(\{y_3\}, \tfrac{1}{2})}; (\{z_3\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (\overline{((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))}) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * ((\overline{(\{b_4, \widehat{y_4}\}, \tfrac{1}{2})}; (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); \overline{(\{z_1\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_4 = [([(\{x_1\}, \tfrac{1}{2}) * ((((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{e_1, \widehat{z_1}\}, \tfrac{1}{2})})[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (\overline{((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))}) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); \overline{(\{z_4\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); \overline{(\{e_4, \widehat{z_4}\}, \tfrac{1}{2})})[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); \overline{(\{z_1\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_5 = [([(\{x_1\},\tfrac{1}{2}) * (((\{b_1,\widehat{y_1}\},\tfrac{1}{2});\overline{(\{e_1,\widehat{z_1}\},\tfrac{1}{2})})[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_2\},\tfrac{1}{2}) * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});\overline{(\{z_3\},\tfrac{1}{2})})) * \mathsf{Stop}]$$

$$\|[(\{x_3\},\tfrac{1}{2}) * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});\overline{(\{e_3,\widehat{z_3}\},\tfrac{1}{2})})[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_4\},\tfrac{1}{2}) * (\overline{((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2}))}) * \mathsf{Stop}]$$

$$\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * (((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2});\overline{(\{z_1\},\tfrac{1}{2})})) * \mathsf{Stop}])$$

$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$

$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_6 = [([(\{x_1\},\tfrac{1}{2}) * (\overline{((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2}))}) * \mathsf{Stop}]$$

$$\|[(\{x_2\},\tfrac{1}{2}) * ((\overline{(\{b_2,\widehat{y_2}\},\tfrac{1}{2})};(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_3\},\tfrac{1}{2}) * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});\overline{(\{z_4\},\tfrac{1}{2})})) * \mathsf{Stop}]$$

$$\|[(\{x_4\},\tfrac{1}{2}) * (((\{b_4,\widehat{y_4}\},\tfrac{1}{2});\overline{(\{e_4,\widehat{z_4}\},\tfrac{1}{2})})[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * (((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[](\overline{(\{y_1\},\tfrac{1}{2})};(\{z_1\},\tfrac{1}{2}))) * \mathsf{Stop}])$$

$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$

$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_7 = [([(\{x_1\},\tfrac{1}{2}) * ((\overline{(\{b_1,\widehat{y_1}\},\tfrac{1}{2})};(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_2\},\tfrac{1}{2}) * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});\overline{(\{z_3\},\tfrac{1}{2})})) * \mathsf{Stop}]$$

$$\|[(\{x_3\},\tfrac{1}{2}) * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});\overline{(\{e_3,\widehat{z_3}\},\tfrac{1}{2})})[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_4\},\tfrac{1}{2}) * (((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[](\overline{(\{y_5\},\tfrac{1}{2})};(\{z_5\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * \overline{(((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2});(\{z_1\},\tfrac{1}{2})))} * \mathsf{Stop}])$$

$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$

$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_8 = [([(\{x_1\},\tfrac{1}{2}) * (((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});\overline{(\{z_2\},\tfrac{1}{2})})) * \mathsf{Stop}]$$

$$\|[(\{x_2\},\tfrac{1}{2}) * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});\overline{(\{e_2,\widehat{z_2}\},\tfrac{1}{2})})[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_3\},\tfrac{1}{2}) * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});\overline{(\{z_4\},\tfrac{1}{2})})) * \mathsf{Stop}]$$

$$\|[(\{x_4\},\tfrac{1}{2}) * (((\{b_4,\widehat{y_4}\},\tfrac{1}{2});\overline{(\{e_4,\widehat{z_4}\},\tfrac{1}{2})})[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * \overline{(((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2});(\{z_1\},\tfrac{1}{2})))} * \mathsf{Stop}])$$

$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$

$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_9 = [([(\{x_1\}, \tfrac{1}{2}) * \overline{(((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); (\{z_2\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); \overline{(\{z_3\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); \overline{(\{e_3, \widehat{z_3}\}, \tfrac{1}{2})})[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); \overline{(\{z_5\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); \overline{(\{e_5, \widehat{z_5}\}, \tfrac{1}{2})})[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_{10} = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); \overline{(\{z_2\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{e_2, \widehat{z_2}\}, \tfrac{1}{2})})[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_3\}, \tfrac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[](\overline{(\{y_4\}, \tfrac{1}{2})}; (\{z_4\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$
$$\|[(\{x_4\}, \tfrac{1}{2}) * \overline{(((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); (\{z_5\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * ((\overline{(\{b_5, \widehat{y_5}\}, \tfrac{1}{2})}; (\{e_5, \widehat{z_5}\}, \tfrac{1}{2}))[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$
$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_{11} = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[](\overline{(\{y_2\}, \tfrac{1}{2})}; (\{z_2\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_2\}, \tfrac{1}{2}) * \overline{(((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); (\{e_2, \widehat{z_2}\}, \tfrac{1}{2}))[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$

$$\|[(\{x_3\}, \tfrac{1}{2}) * ((\overline{(\{b_3, \widehat{y_3}\}, \tfrac{1}{2})}; (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$

$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); \overline{(\{z_5\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$

$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); \overline{(\{e_5, \widehat{z_5}\}, \tfrac{1}{2})})[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$

$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$

$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx},$$

$$s_{12} = [([(\{x_1\}, \tfrac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \tfrac{1}{2}); (\{e_1, \widehat{z_1}\}, \tfrac{1}{2}))[]((\{y_2\}, \tfrac{1}{2}); \overline{(\{z_2\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$

$$\|[(\{x_2\}, \tfrac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{e_2, \widehat{z_2}\}, \tfrac{1}{2})})[]((\{y_3\}, \tfrac{1}{2}); (\{z_3\}, \tfrac{1}{2}))) * \mathsf{Stop}]$$

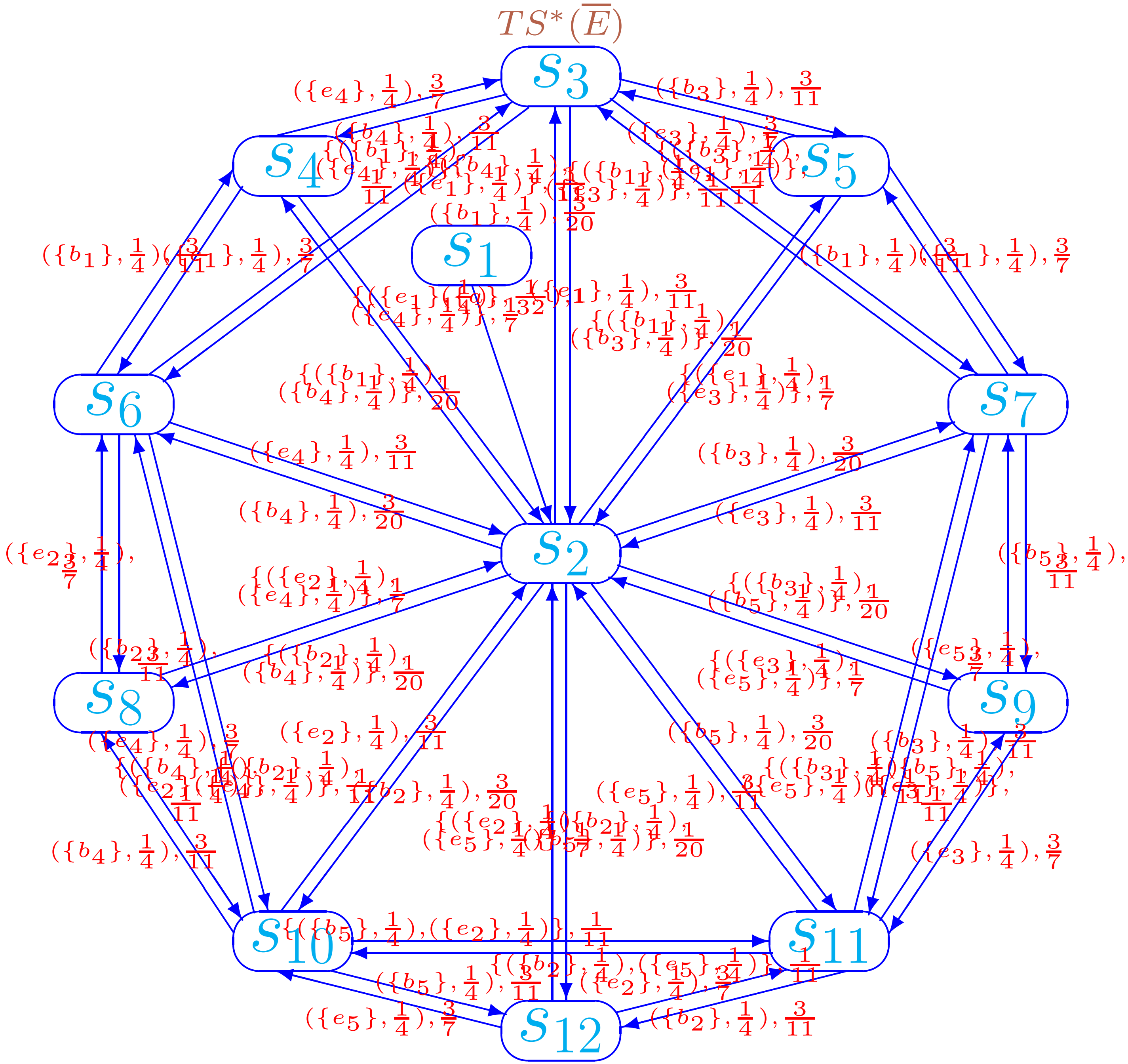
$$\|[(\{x_3\}, \tfrac{1}{2}) * \overline{(((\{b_3, \widehat{y_3}\}, \tfrac{1}{2}); (\{e_3, \widehat{z_3}\}, \tfrac{1}{2}))[]((\{y_4\}, \tfrac{1}{2}); (\{z_4\}, \tfrac{1}{2})))} * \mathsf{Stop}]$$

$$\|[(\{x_4\}, \tfrac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \tfrac{1}{2}); (\{e_4, \widehat{z_4}\}, \tfrac{1}{2}))[]((\{y_5\}, \tfrac{1}{2}); \overline{(\{z_5\}, \tfrac{1}{2})})) * \mathsf{Stop}]$$

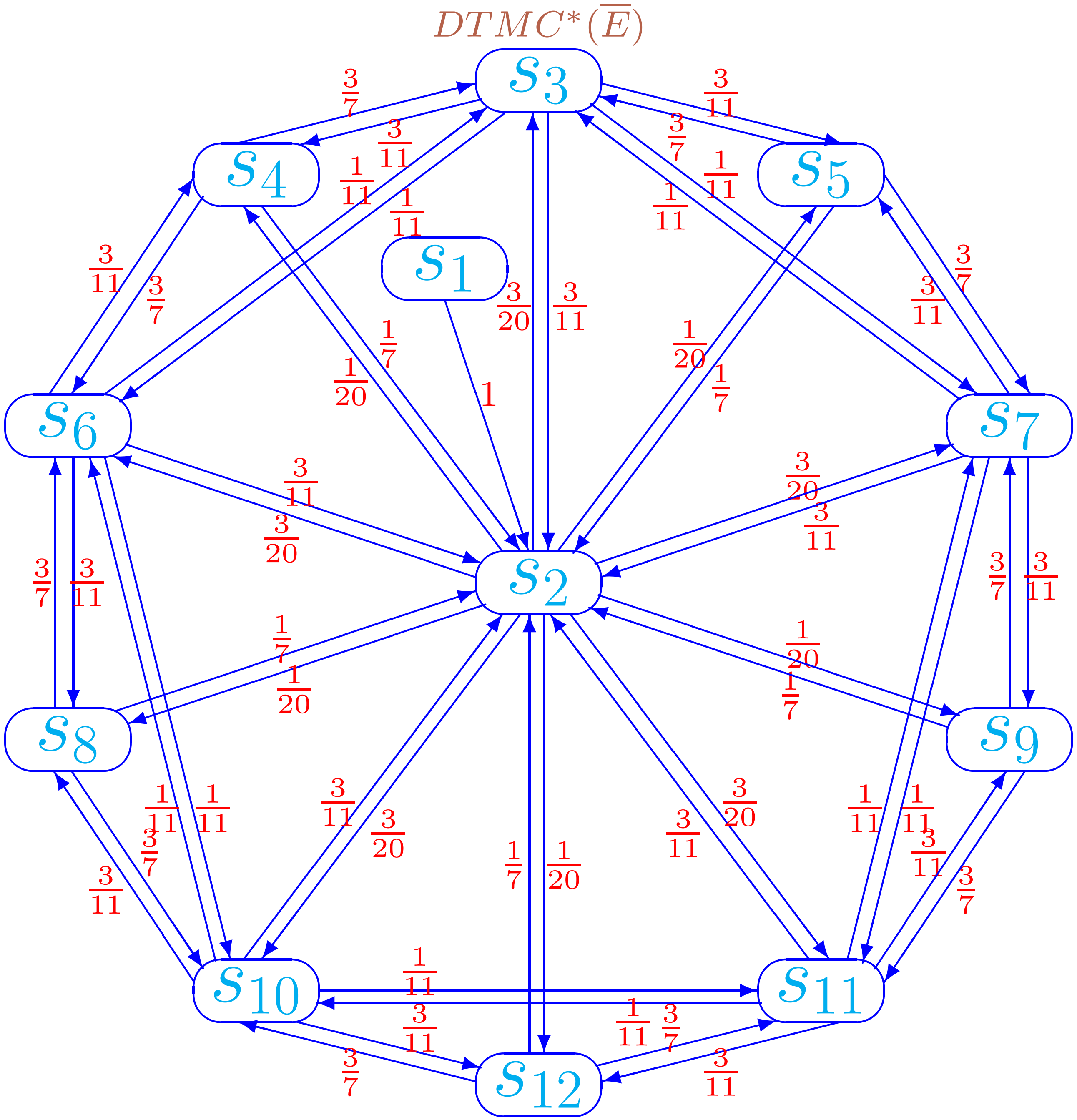
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \tfrac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \tfrac{1}{2}); \overline{(\{e_5, \widehat{z_5}\}, \tfrac{1}{2})})[]((\{y_1\}, \tfrac{1}{2}); (\{z_1\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$

$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2$$

$$\mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4\ \mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_{\approx}.$$



The transition system without empty loops of the dining philosophers system



The underlying DTMC without empty loops of the dining philosophers system

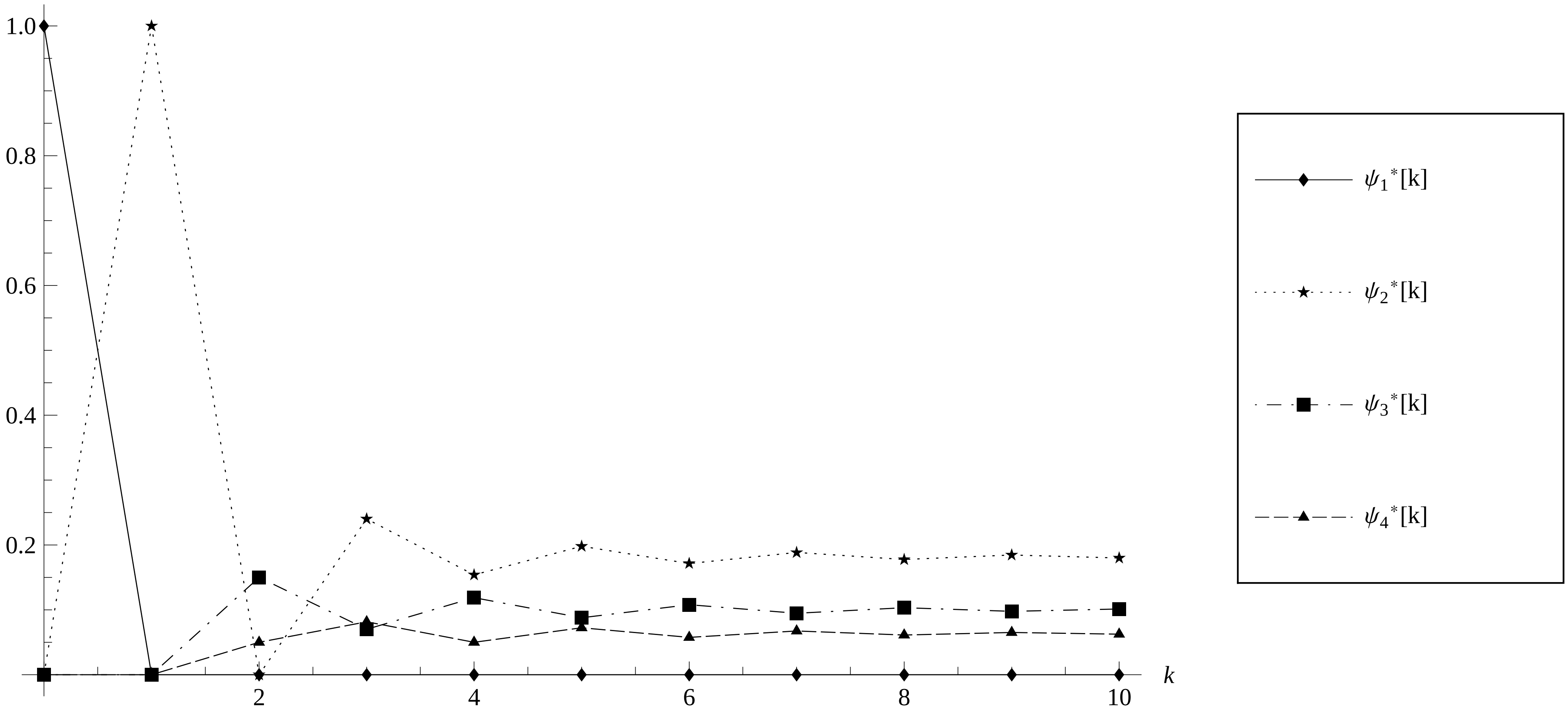
The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^* = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\
0 & \frac{3}{11} & 0 & \frac{3}{11} & \frac{3}{11} & \frac{1}{11} & \frac{1}{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 \\
0 & \frac{3}{11} & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & \frac{1}{11} & \frac{3}{11} \\
0 & \frac{3}{11} & 0 & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{7} & \frac{3}{7} & 0
\end{bmatrix}.$$

Transient and steady-state probabilities of the dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
$\psi_3^*[k]$	0	0	0.1500	0.0701	0.1189	0.0878	0.1079	0.0949	0.1033	0.0979	0.1014	0.1000
$\psi_4^*[k]$	0	0	0.0500	0.0818	0.0503	0.0726	0.0578	0.0674	0.0612	0.0652	0.0626	0.0636

We depict the probabilities for the states $s_1, \ldots, s_4$ only, since the corresponding values coincide for the states $s_3, s_6, s_7, s_{10}, s_{11}$ as well as for $s_4, s_5, s_8, s_9, s_{12}$.



Transient probabilities alteration diagram of the dining philosophers system

The steady-state PMF for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{2}{11}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}\right).$$

Performance indices

- The average recurrence time in the state s_2, where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{11}{2} = 5\frac{1}{2}$.

- Nobody eats in the state s_2. The *fraction of time when no philosophers dine* is $\psi_2^* = \frac{2}{11}$.

  Only one philosopher eats in the states $s_3, s_6, s_7, s_{10}, s_{11}$. The *fraction of time when only one philosopher dines* is $\psi_3^* + \psi_6^* + \psi_7^* + \psi_{10}^* + \psi_{11}^* = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}$.

  Two philosophers eat together in the states $s_4, s_5, s_8, s_9, s_{12}$. The *fraction of time when two philosophers dine* is $\psi_4^* + \psi_5^* + \psi_8^* + \psi_9^* + \psi_{12}^* = \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} = \frac{7}{22}$.

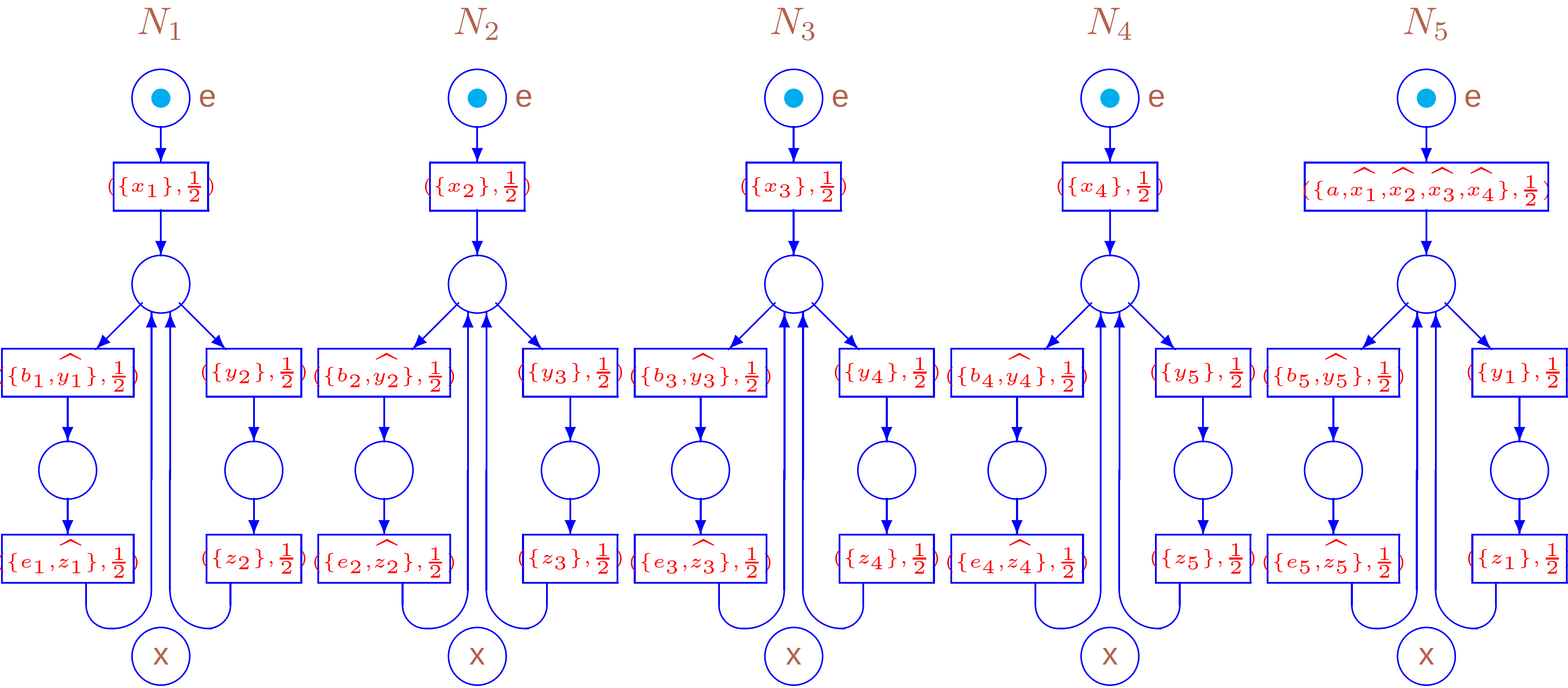
  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of first philosopher $(\{b_1\}, \frac{1}{4})$ is only possible from the states s_2, s_6, s_7.

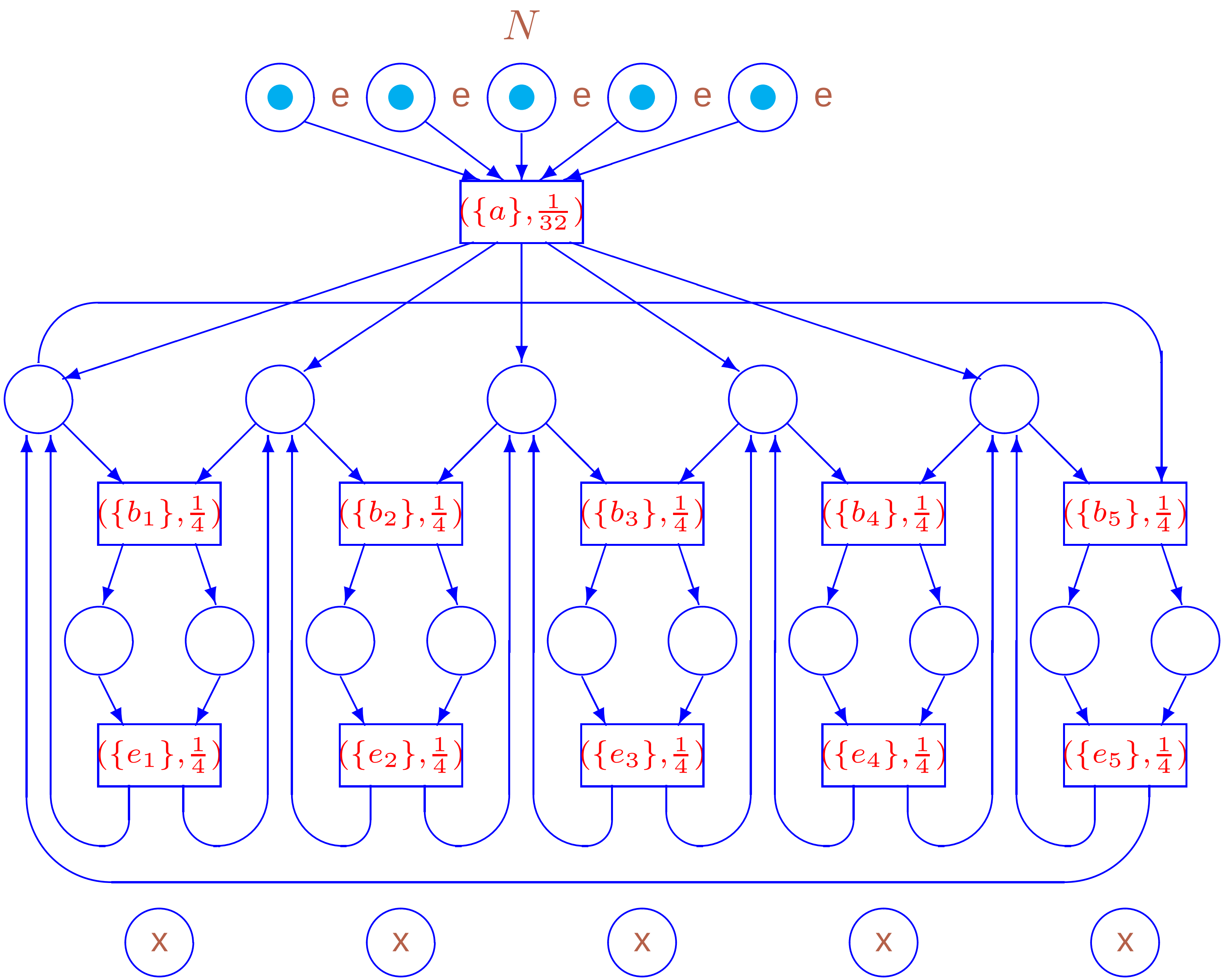
  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b_1\}, \frac{1}{4})$.

  The *steady-state probability of the beginning of eating of first philosopher* is

  $\psi_2^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_6^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) +$
  $\psi_7^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) =$
  $\frac{2}{11}\left(\frac{3}{20} + \frac{1}{20} + \frac{1}{20}\right) + \frac{1}{10}\left(\frac{3}{11} + \frac{1}{11}\right) + \frac{1}{10}\left(\frac{3}{11} + \frac{1}{11}\right) = \frac{13}{110}$.



The marked dts-boxes of the dining philosophers



The marked dts-box of the dining philosophers system

The abstract system

The static expression of the philosopher i $(1 \leq i \leq 4)$ is

$F_i = [(\{x_i\}, \frac{1}{2}) * ((((\{b, \widehat{y_i}\}, \frac{1}{2}); (\{e, \widehat{z_i}\}, \frac{1}{2}))[]((\{y_{i+1}\}, \frac{1}{2}); (\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is

$F_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * ((((\{b, \widehat{y_5}\}, \frac{1}{2}); (\{e, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2}); (\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract dining philosophers system is

$F = (F_1 \| F_2 \| F_3 \| F_4 \| F_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$.

$DTMC^*(\overline{F}) = DTMC^*(\overline{E})$, thus, TPM and the steady-state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

Performance indices

The first performance index and the second group of the indices are the same for the standard and abstract systems.

The following performance index: non-personalized viewpoint to the philosophers.

- The beginning of eating of a philosopher $(\{b\}, \frac{1}{4})$ is only possible from the states $s_2, s_3, s_6, s_7, s_{10}, s_{11}$.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{1}{4})$.

  The *steady-state probability of the beginning of eating of a philosopher* is

$$\psi_2^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_3) + \psi_6^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) + \psi_7^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) + \psi_{10}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{10}) + \psi_{11}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{11}) =$$

$$\frac{2}{11}\left(\frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) = \frac{6}{11}.$$

The reduction of the abstract system

The static expression of the philosopher 1 is $F_1' = [(\{x\},\frac{1}{2}) * ((\{b\},\frac{2}{5});(\{e\},\frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the philosopher 2 is $F_2' = [(\{a,\hat{x}\},\frac{1}{16}) * ((\{b\},\frac{2}{5});(\{e\},\frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the reduced abstract dining philosophers system is $F' = (F_1' \| F_2')\ \mathsf{sy}\ x\ \mathsf{rs}\ x$.

$DR(\overline{F'})$ consists of

$s_1' = [([\overline{(\{x\},\frac{1}{2})} * ((\{b\},\frac{2}{5})_1;(\{e\},\frac{1}{4})_1) * \mathsf{Stop}] \|$
$[\overline{(\{a,\hat{x}\},\frac{1}{16})} * ((\{b\},\frac{2}{5})_2;(\{e\},\frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\approx$,

$s_2' = [([(\{x\},\frac{1}{2}) * (\overline{(\{b\},\frac{2}{5})_1};(\{e\},\frac{1}{4})_1) * \mathsf{Stop}] \|$
$[(\{a,\hat{x}\},\frac{1}{16}) * (\overline{(\{b\},\frac{2}{5})_2};(\{e\},\frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\approx$,

$s_3' = [([(\{x\},\frac{1}{2}) * ((\{b\},\frac{2}{5})_1;\overline{(\{e\},\frac{1}{4})_1}) * \mathsf{Stop}] \|$
$[(\{a,\hat{x}\},\frac{1}{16}) * (\overline{(\{b\},\frac{2}{5})_2};(\{e\},\frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\approx$,

$s_4' = [([(\{x\},\frac{1}{2}) * (\overline{(\{b\},\frac{2}{5})_1};(\{e\},\frac{1}{4})_1) * \mathsf{Stop}] \|$
$[(\{a,\hat{x}\},\frac{1}{16}) * ((\{b\},\frac{2}{5})_2;\overline{(\{e\},\frac{1}{4})_2}) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\approx$,

$s_5' = [([(\{x\},\frac{1}{2}) * ((\{b\},\frac{2}{5})_1;\overline{(\{e\},\frac{1}{4})_1}) * \mathsf{Stop}] \|$
$[(\{a,\hat{x}\},\frac{1}{16}) * ((\{b\},\frac{2}{5})_2;\overline{(\{e\},\frac{1}{4})_2}) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\approx$.

$\overline{F}\underline{\leftrightarrow}_{ss}\overline{F'}$ with $(DR(\overline{F}) \cup DR(\overline{F'}))/_{\underline{\leftrightarrow}_{ss}} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$, where

$\mathcal{H}_1 = \{s_1, s'_1\}$ (the initial state),

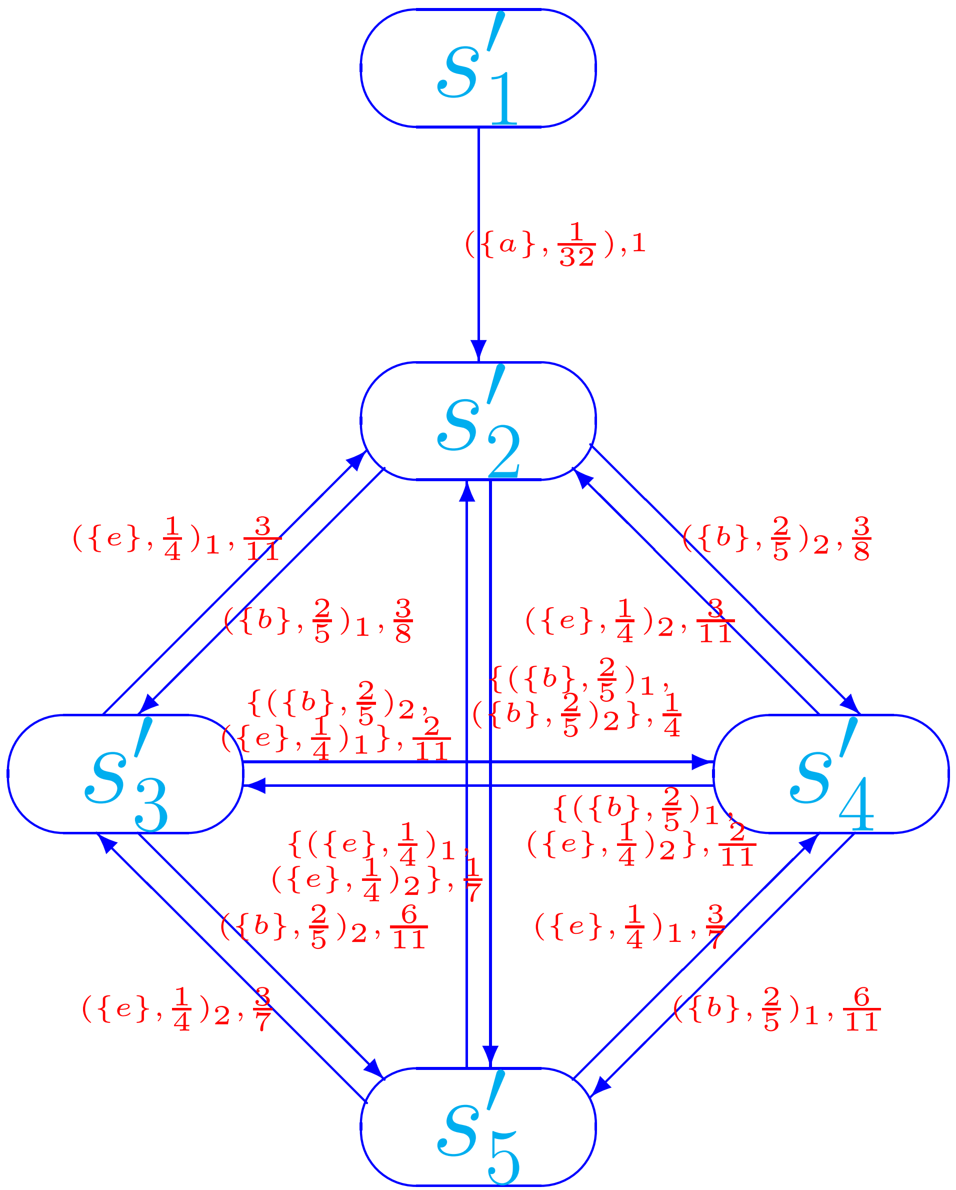
$\mathcal{H}_2 = \{s_2, s'_2\}$ (the system is activated and no philosophers dine),

$\mathcal{H}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}, s'_3, s'_4\}$ (one philosopher dines),

$\mathcal{H}_4 = \{s_4, s_5, s_8, s_9, s_{12}, s'_5\}$ (two philosophers dine).

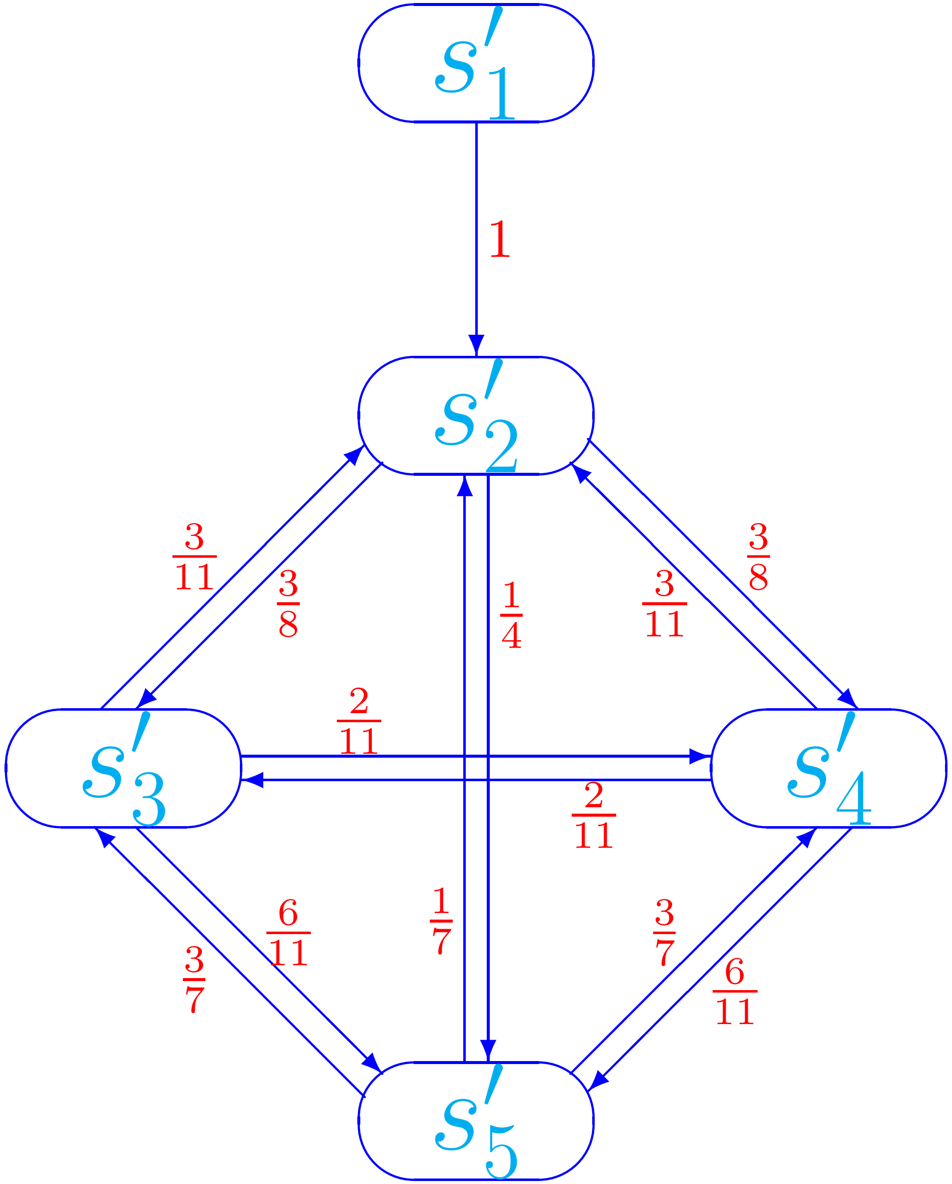
F' is a reduction of F w.r.t. $\underline{\leftrightarrow}_{ss}$.

$TS^*(\overline{F'})$



The transition system without empty loops of the reduced abstract dining philosophers system

$DTMC^*(\overline{F'})$



The underlying DTMC without empty loops of the reduced abstract dining philosophers system

The TPM for $DTMC^*(\overline{F'})$ is

$$\mathbf{P}'^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ 0 & \frac{3}{11} & 0 & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{3}{11} & \frac{2}{11} & 0 & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{3}{7} & \frac{3}{7} & 0 \end{bmatrix}.$$

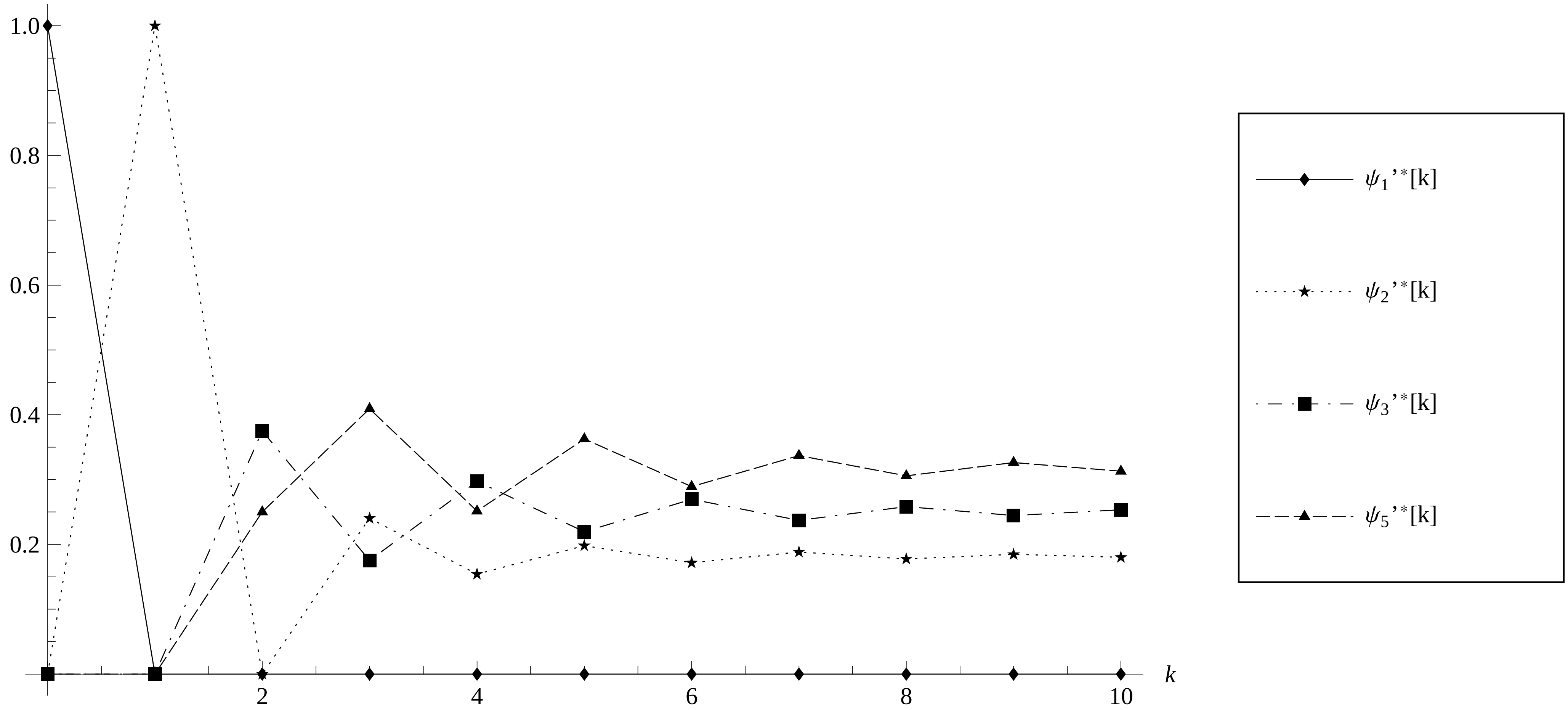
The steady-state PMF for $DTMC^*(\overline{F'})$ is

$$\psi'^* = \left(0, \frac{2}{11}, \frac{1}{4}, \frac{1}{4}, \frac{7}{22}\right).$$

Transient and steady-state probabilities of the reduced abstract dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
${\psi'_1}^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
${\psi'_2}^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
${\psi'_3}^*[k]$	0	0	0.3750	0.1753	0.2973	0.2195	0.2697	0.2372	0.2583	0.2446	0.2535	0.2500
${\psi'_5}^*[k]$	0	0	0.2500	0.4091	0.2513	0.3628	0.2890	0.3371	0.3059	0.3261	0.3130	0.3182

We depict the probabilities for the states s'_1, s'_2, s'_3, s'_5 only, since the corresponding values coincide for s'_3, s'_4.



Transient probabilities alteration diagram of the reduced abstract dining philosophers system

Performance indices

- The average recurrence time in the state s'_2, where all the forks are available, the *average system run-through*, is $\frac{1}{{\psi'_2}^*} = \frac{11}{2} = 5\frac{1}{2}$.

- Nobody eats in the state s'_2. The *fraction of time when no philosophers dine* is ${\psi'_2}^* = \frac{2}{11}$.

  Only one philosopher eats in the states s'_3, s'_4. The *fraction of time when only one philosopher dines* is ${\psi'_3}^* + {\psi'_4}^* = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

  Two philosophers eat together in the state s'_5. The *fraction of time when two philosophers dine* is ${\psi'_5}^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of a philosopher $(\{b\}, \frac{2}{5})$ is only possible from the states s'_2, s'_3, s'_4.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{2}{5})$.

  The *steady-state probability of the beginning of eating of a philosopher* is

$$
{\psi'_2}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_2) + {\psi'_3}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_3) + {\psi'_4}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_4) = \frac{2}{11}\left(\frac{3}{8} + \frac{3}{8} + \frac{1}{4}\right) + \frac{1}{4}\left(\frac{6}{11} + \frac{2}{11}\right) + \frac{1}{4}\left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}.
$$

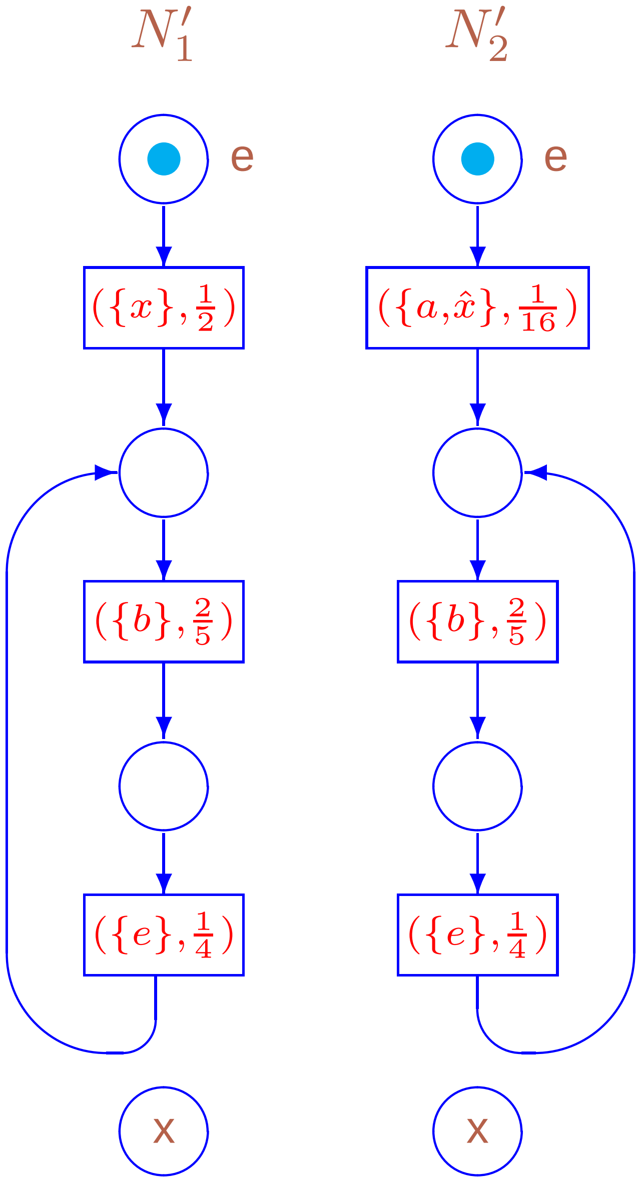
The performance indices are the same for the complete and the reduced abstract dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates the result of proposition about steady-state probabilities.

The coincidence of the third performance index is due to the theorem about step traces from steady states:

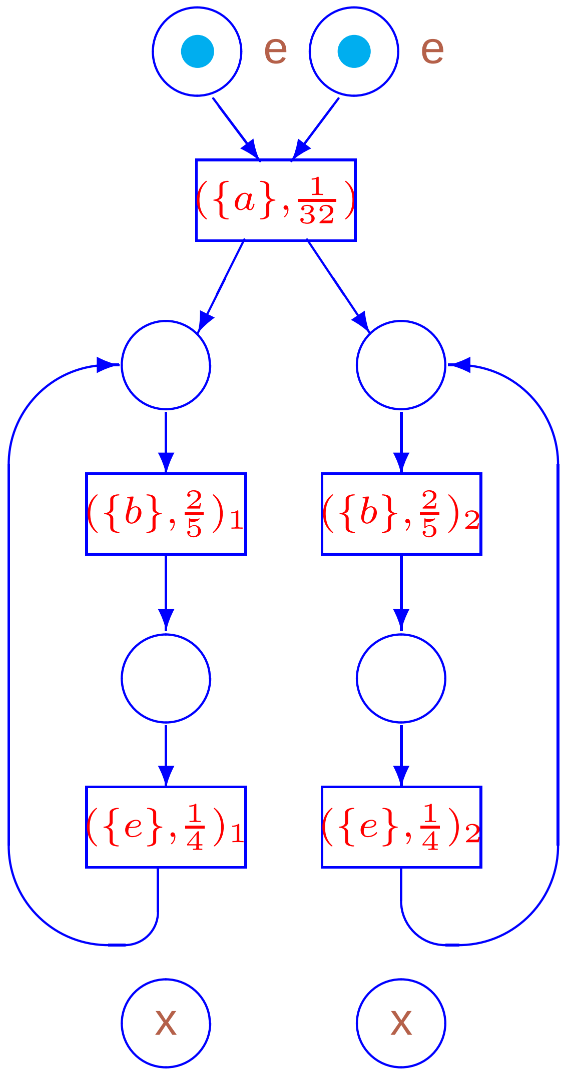
one should apply its result to the step traces $\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{F}$ and $\overline{F'}$,

and sum the left and right parts of the three resulting equalities.



The marked dts-boxes of the reduced abstract dining philosophers

N'



The marked dts-box of the reduced abstract dining philosophers system

The quotients for the abstract system

$DR(\overline{F})/_{\underline{\leftrightarrow}_{ss}} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$, where

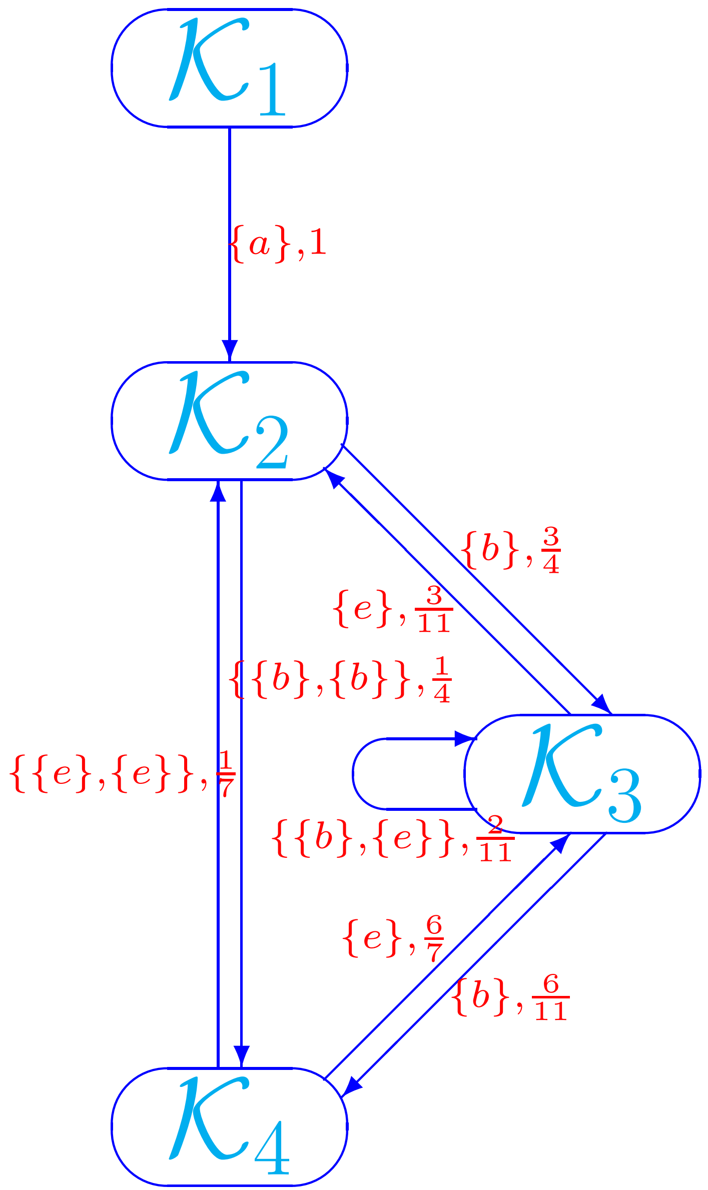
$\mathcal{K}_1 = \{s_1\}$ (the initial state),

$\mathcal{K}_2 = \{s_2\}$ (the system is activated and no philosophers dine),

$\mathcal{K}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}\}$ (one philosopher dines),

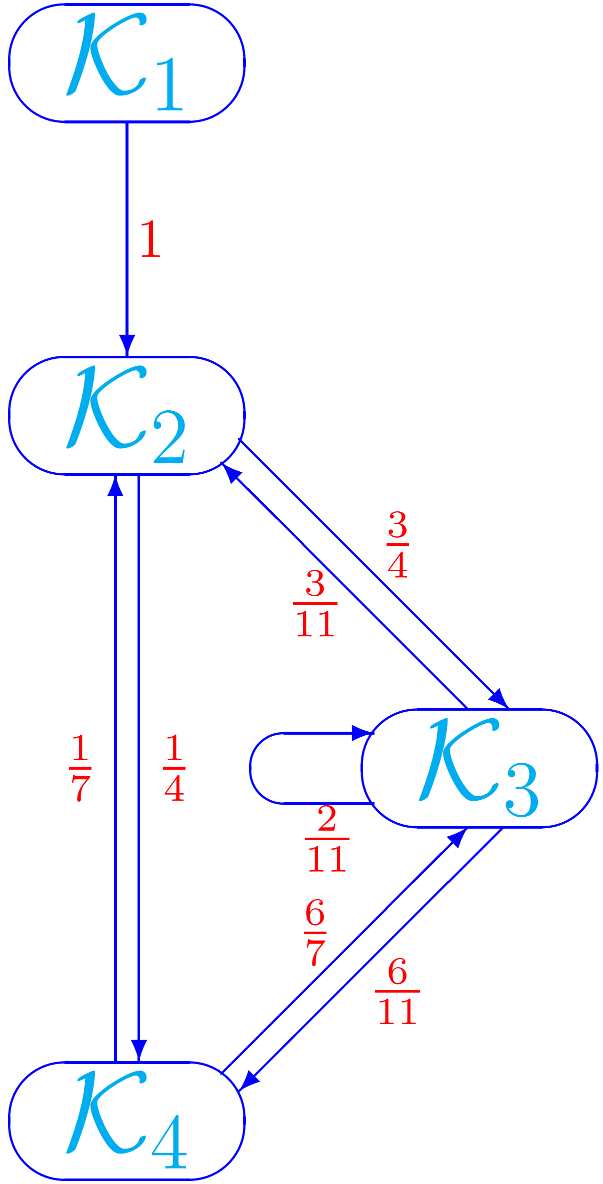
$\mathcal{K}_4 = \{s_4, s_5, s_8, s_9, s_{12}\}$ (two philosophers dine).

$TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



The quotient transition system without empty loops of the abstract dining philosophers system

$DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



The quotient underlying DTMC without empty loops of the abstract dining philosophers system

The TPM for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

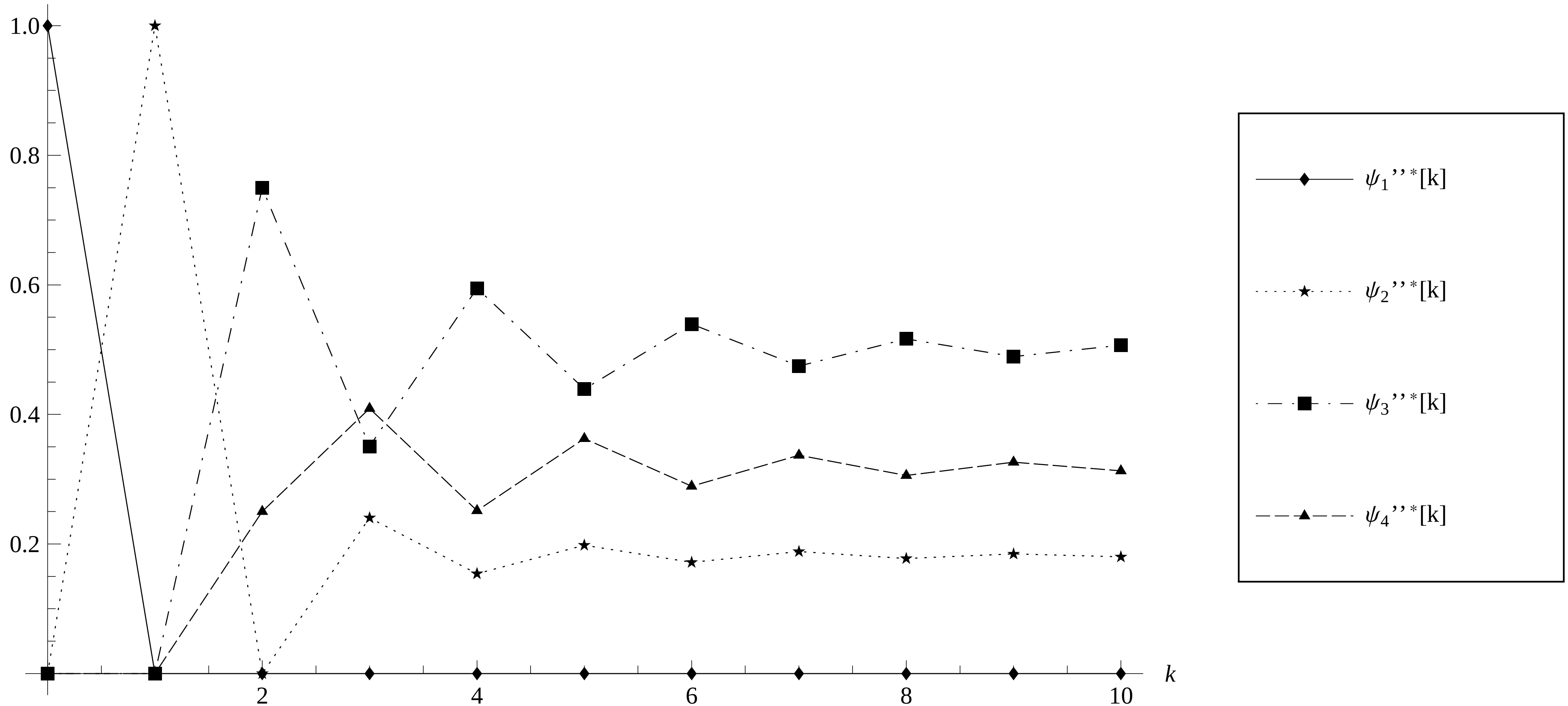
$$\mathbf{P}''^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{11} & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{6}{7} & 0 \end{bmatrix}.$$

The steady-state PMF for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\psi''^* = \left(0, \frac{2}{11}, \frac{1}{2}, \frac{7}{22}\right).$$

Transient and steady-state probabilities of the quotient abstract dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1''^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2''^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
$\psi_3''^*[k]$	0	0	0.7500	0.3506	0.5946	0.4391	0.5394	0.4745	0.5165	0.4893	0.5069	0.5000
$\psi_4''^*[k]$	0	0	0.2500	0.4091	0.2513	0.3628	0.2890	0.3371	0.3059	0.3261	0.3130	0.3182



Transient probabilities alteration diagram of the quotient abstract dining philosophers system

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2''^*} = \frac{11}{2} = 5\frac{1}{2}$.
- Nobody eats in the state $\mathcal{K}_2$. The *fraction of time when no philosophers dine* is $\psi_2''^* = \frac{2}{11}$.

  Only one philosopher eats in the state $\mathcal{K}_3$. The *fraction of time when only one philosopher dines* is $\psi_3''^* = \frac{1}{2}$.

  Two philosophers eat together in the state $\mathcal{K}_4$. The *fraction of time when two philosophers dine* is $\psi_4''^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.
- The beginning of eating of a philosopher $\{b\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_3$.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of multiactions containing $\{b\}$.

  The *steady-state probability of the beginning of eating of a philosopher* is
  $\psi_2''^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{b\} \in A,\ \mathcal{K}_2 \xrightarrow{A} \widetilde{\mathcal{K}}\}} PM_A^*(\mathcal{K}_2, \widetilde{\mathcal{K}}) + \psi_3''^* \sum_{\{A,\widetilde{\mathcal{K}} \mid \{b\} \in A,\ \mathcal{K}_3 \xrightarrow{A} \widetilde{\mathcal{K}}\}} PM_A^*(\mathcal{K}_3, \widetilde{\mathcal{K}}) = \frac{2}{11}\left(\frac{3}{4} + \frac{1}{4}\right) + \frac{1}{2}\left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}$.

The performance indices are the same for the complete and quotient abstract dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates the result of proposition about steady-state probabilities.

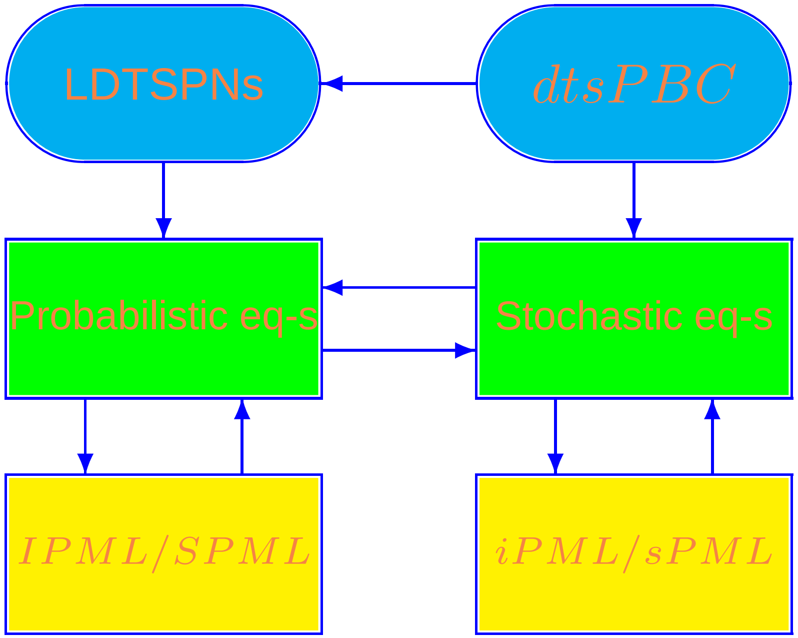
The coincidence of the third performance index is due to the theorem about step traces from steady states:

one should apply its result to the step traces $\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

Overview and open questions

The results obtained



Stochastic formalisms and equivalences

- A discrete time stochastic extension $dtsPBC$ of finite PBC enriched with iteration.
- The step operational semantics based on labeled probabilistic transition systems.
- The denotational semantics in terms of a subclass of LDTSPNs.

- The stochastic algebraic equivalences which have natural net analogues on LDTSPNs.
- The transition systems reduction modulo stochastic equivalences.
- A logical characterization of stochastic bisimulation equivalences via probabilistic modal logics.
- An application of the equivalences to comparison of stationary behaviour.
- A preservation w.r.t. algebraic operations and the congruence relation.
- The case studies of performance analysis.

Further research

- Abstracting from silent activities in definitions of the equivalences.
- Introducing the immediate activities with zero delay.
- Extending the syntax with recursion operator.

References

[And99] ANDOVA S. *Process algebra with probabilistic choice.* *LNCS* **1601**, p. 111–129, 1999.

[BDH92] BEST E., DEVILLERS R., HALL J.G. *The box calculus: a new causal algebra with multi-label communication.* *LNCS* **609**, p. 21–69, 1992.

[BGo98] BERNARDO M., GORRIERI R. *A tutorial on EMPA: a theory of concurrent processes with nondeterminism, priorities, probabilities and time.* *TCS* **202**, p. 1–54, July 1998.

[BHe97] BAIER C., HERMANNS H. *Weak bisimulation for fully probabilistic processes.* *LNCS* **1254**, p. 119–130, 1997.

[BKe01] BUCHHOLZ P., KEMPER P. *Quantifying the dynamic behavior of process algebras.* *LNCS* **2165**, p. 184–199, 2001.

[BM89] BLOOM B., MEYER A. *A remark on bisimulation between probabilistic processes.* *LNCS* **363**, p. 26–40, 1989.

[BT00] BUCHHOLZ P., TARASYUK I.V. *A class of stochastic Petri nets with step semantics and related equivalence notions.* *Technische Berichte* **TUD-FI00-12**, 18 p., Fakultät Informatik, Technische Universität Dresden, Germany, November 2000, `ftp://ftp.inf.tu-dresden.de/pub/berichte/tud00-12.ps.gz`.

[BT01] BUCHHOLZ P., TARASYUK I.V. *Net and algebraic approaches to probablistic modeling.* *Joint Novosibirsk Computing Center and Institute of Informatics Systems Bulletin, Series Computer Science* **15**, p. 31–64, Novosibirsk, 2001, `http://www.iis.nsk.su/persons/itar/spnpancc.pdf`.

[Buc94] BUCHHOLZ P. *Markovian process algebra: composition and equivalence.* In: U. Herzog and M. Rettelbach, eds., *Proceedings of the* 2^{nd} *Workshop on Process Algebras and Performance Modelling*, *Arbeitsberichte des IMMD* **27**, p. 11–30, University of Erlangen, 1994.

[Buc95] BUCHHOLZ P. *A notion of equivalence for stochastic Petri nets.* *LNCS* **935**, p. 161–180, 1995.

[Buc98] BUCHHOLZ P. *Iterative decomposition and aggregation of labeled GSPNs.* *LNCS* **1420**, p. 226–245, 1998.

[Buc99] BUCHHOLZ P. *Exact performance equivalence — an equivalence relation for stochastic automata.* *TCS* **215(1/2)**, p. 263–287, 1999.

[Chr90] CHRISTOFF I. *Testing equivalence and fully abstract models of probabilistic processes.* *LNCS* **458**, p. 128–140, 1990.

[CMBC93] CHIOLA G., MARSAN M.A., BALBO G., CONTE G. *Generalized stochastic Petri nets: a definition at the net level and its implications.* *IEEE Transactions on Software Engineering* **19(2)**, p. 89–107, 1993.

[FM03] de Frutos D.E., Marroquín O.A. *Ambient Petri nets. Electronic Notes in Theoretical Computer Science* **85(1)**, 27 p., 2003.

[FN85] Florin G., Natkin S. *Les reseaux de Petri stochastiques. Technique et Science Informatique* **4(1)**, 1985.

[Han94] Hansson H. *Time and probability in formal design of distributed systems.* In: *Real-Time Safety Critical Systems*, Volume **1**, Elsevier, The Netherlands, 1994.

[Hil94] Hillston J. *A compositional approach for performance modelling. Ph.D. thesis*, University of Edinburgh, Department of Computer Science, 1994.

[Hil96] Hillston J. *A compositional approach to performance modelling.* Cambridge University Press, Great Britain, 1996.

[HR94] Hermanns H., Rettelbach M. *Syntax, semantics, equivalences and axioms for MTIPP.* In: Herzog U. and Rettelbach M., eds., *Proceedings of the* 2^{nd} *Workshop on Process Algebras and Performance Modelling. Arbeitsberichte des IMMD* **27**, University of Erlangen, 1994.

[KN98] Kwiatkowska M.Z., Norman G.J. *A testing equivalence for reactive probabilistic processes. Electronic Notes in Theoretical Computer Science* **16(2)**, 19 p., 1998.

[Kou00] Koutny M. *A compositional model of time Petri nets. LNCS* **1825**, p. 303–322, 2000.

[LS91] LARSEN K., SKOU A. *Bisimulation through probabilistic testing. Information and Computation* **94**, p. 1–28, 1991.

[MBCDF95] MARSAN M.A., BALBO G., CONTE G., DONATELLI S., FRANCESCHINIS G. *Modelling with generalized stochastic Petri nets. Wiley Series in Parallel Computing*, John Wiley and Sons, 316 p., 1995, `http://www.di.unito.it/~greatspn/GSPN-Wiley/`.

[MCB84] MARSAN M.A., CONTE G., BALBO G. *A class of generalized stochastic Petri nets for performance evaluation of multiprocessor systems. ACM Transactions on Computer Systems* **2(2)**, p. 93–122, 1984.

[MCW03] MAJSTER-CEDERBAUM M., WU J. *Adding action refinement to stochastic true concurrency models. LNCS* **2885**, p. 226–245, 2003.

[MF00] MARROQUÍN O.A., DE FRUTOS D.E. *TPBC: timed Petri box calculus. Technical Report*, Departamento de Sistemas Infofmáticos y Programación, Universidad Complutense de Madrid, Madrid, Spain, 2000 (in Spanish).

[Mol82] MOLLOY M. *Performance analysis using stochastic Petri nets. IEEE Transactions on Software Engineering* **31(9)**, p. 913–917, 1982.

[Mol85] MOLLOY M. *Discrete time stochastic Petri nets. IEEE Transactions on Software Engineering* **11(4)**, p. 417–423, 1985.

[MVC02] MACIÀ H.S., VALERO V.R., CUARTERO F.G. *A congruence relation in finite sPBC. Technical Report* **DIAB-02-01-31**, 34 p., Department of Computer Science, University of Castilla-La Mancha, Albacete, Spain, October 2002, `http://www.info-ab.uclm.es/retics/publications/2002/tr020131.ps`.

[MVCC03] MACIÀ H.S., VALERO V.R., CAZORLA D.L., CUARTERO F.G. *Introducing the iteration in sPBC. Technical Report* **DIAB-03-01-37**, 20 p., Department of Computer Science, University of Castilla-La Mancha, Albacete, Spain, September 2003, `http://www.info-ab.uclm.es/descargas/tecnicalreports/DIAB-03-01-37/diab030137.zip`.

[MVCR08] MACIÀ H.S., VALERO V.R., CUARTERO F.G., RUIZ M.C.D. *sPBC: a Markovian extension of Petri box calculus with immediate multiactions. Fundamenta Informaticae* **87(3–4)**, p. 367–406, IOS Press, Amsterdam, The Netherlands, 2008.

[MVF01] MACIÀ H.S., VALERO V.R., DE FRUTOS D.E. *sPBC: a Markovian extension of finite Petri box calculus. Proceedings of* 9^{th} *IEEE International Workshop on Petri Nets and Performance Models - 01 (PNPM'01)*, p. 207–216, Aachen, Germany, IEEE Computer Society Press, September 2001, `http://www.info-ab.uclm.es/retics/publications/2001/pnpm01.ps`.

[NFL95] NÚÑEZ M.G., DE FRUTOS D.E., LLANA L.D. *Acceptance trees for probabilistic processes. LNCS* **962**, p. 249–263, 1995.

[Nia05] NIAOURIS A. *An algebra of Petri nets with arc-based time restrictions. LNCS* **3407**, p. 447–462, 2005.

[P81] PETERSON J.L. *Petri net theory and modeling of systems.* Prentice-Hall, 1981.

[Tar05] TARASYUK I.V. *Discrete time stochastic Petri box calculus. Berichte aus dem Department für Informatik* **3/05**, 25 p., Carl von Ossietzky Universität Oldenburg, Germany, November 2005, `http://www.iis.nsk.su/persons/itar/dtspbcib_cov.pdf`.

[Tar06] TARASYUK I.V. *Iteration in discrete time stochastic Petri box calculus. Bulletin of the Novosibirsk Computing Center, Series Computer Science, IIS Special Issue* **24**, p. 129–148, NCC Publisher, Novosibirsk, 2006, `http://www.iis.nsk.su/persons/itar/dtsitncc.pdf`.

[ZG94] ZIJAL R., GERMAN R. *A new approach to discrete time stochastic Petri nets. Lecture Notes in Control and Information Science* **199**, p. 198–204, 1994.

The slides can be downloaded from Internet:

`http://www.iis.nsk.su/persons/itar/dtspbcsm.pdf`

Thank you for your attention!