Performance evaluation in $dtsPBC$

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Abstract: In [MVF01], a continuous time stochastic extension $sPBC$ of finite PBC was proposed.

In [MVCC03], iteration operator was added to $sPBC$.

Algebra $sPBC$ has interleaving semantics, but PBC has step one.

We constructed a discrete time stochastic extension $dtsPBC$ of finite PBC [Tar05].

In [Tar06], $dtsPBC$ was enriched with iteration.

Step operational semantics was defined in terms of labeled probabilistic transition systems.

Denotational semantics was defined in terms of a subclass of labeled DTSPNs (LDTSPNs) called discrete time stochastic Petri boxes (dts-boxes).

We proposed a variety of stochastic equivalences.

We demonstrated how to apply the equivalences to compare stationary behaviour.

In this talk, two case studies of performance evaluation are presented.

Keywords: stochastic Petri nets, stochastic process algebras, Petri box calculus, iteration, discrete time, stochastic equivalences, stationary behaviour, performance evaluation.

Stationary behaviour

Theoretical background

The elements $\mathcal{P}^*_{ij}$ $(1 \le i,j \le n = |DR(G)|)$ of *(one-step) transition probability matrix (TPM)* $\mathbf{P}^*$ for $DTMC^*(G)$:

$$\mathcal{P}^*_{ij} = \begin{cases} PM^*(s_i, s_j), & s_i \to s_j; \\ 0, & \text{otherwise.} \end{cases}$$

The *transient (k-step, $k \in I\!N$) probability mass function (PMF)* $\psi^*[k] = (\psi^*_1[k], \ldots, \psi^*_n[k])$ for $DTMC^*(N)$ is the solution of

$$\psi^*[k] = \psi^*[0](\mathbf{P}^*)^k,$$

where $\psi^*[0] = (\psi_1^*[0], \ldots, \psi_n^*[0])$ is the *initial PMF*:

$$\psi_i^*[0] = \begin{cases} 1, & s_i = [G]_{\simeq}; \\ 0, & \text{otherwise.} \end{cases}$$

$\psi^*[k+1] = \psi^*[k]\mathbf{P}^*,\ k \in I\!N$.

The *steady state PMF* $\psi^* = (\psi_1^*, \ldots, \psi_n^*)$ for $DTMC^*(G)$ is the solution of

$$\begin{cases} \psi^*(\mathbf{P}^* - \mathbf{E}) = \mathbf{0} \\ \psi^*\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{0}$ is a vector with n values 0, $\mathbf{1}$ is that with n values 1.

When $DTMC^*(G)$ has the steady state, $\psi^* = \lim_{k\to\infty} \psi^*[k]$.

Steady state and equivalences

For $s \in DR(G)$ with $s = s_i$ $(1 \leq i \leq n)$ we define $\psi^*[k](s) = \psi_i^*[k]$ $(k \in \mathbb{N})$ and $\psi^*(s) = \psi_i^*$.

Proposition 1 *Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$. Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \psi'^*(s').$$

Definition 1 *A* step trace *of a dynamic expression* G *is* $\Sigma = A_1 \cdots A_n \in (\mathbb{N}_f^{\mathcal{L}})^*$ *where* $\exists s \in DR(G)\ s \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \le i \le n)$.

The probability of the step trace Σ to start in the state s *is*

$$PT^*(\Sigma, s) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid s=s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\le i\le n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}).$$

Theorem 1 *Let* G, G' *be dynamic expressions with* $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ *and* Σ *be a step trace. Then* $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

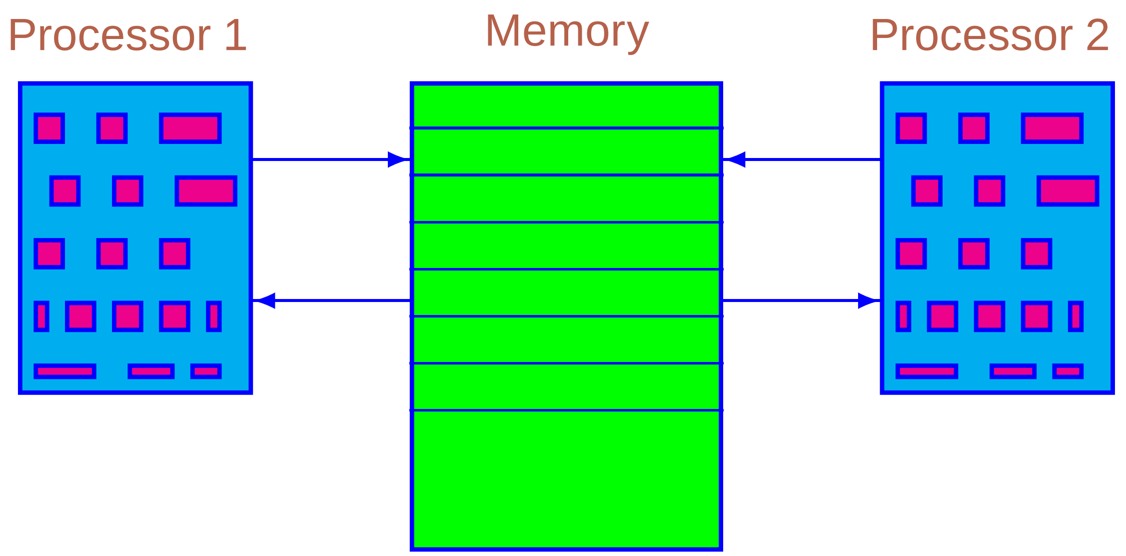
$$\sum_{s\in\mathcal{H}\cap DR(G)} \psi^*(s) PT^*(\Sigma, s) = \sum_{s'\in\mathcal{H}\cap DR(G')} {\psi'}^*(s') PT^*(\Sigma, s').$$

The result of the theorem above is valid if we replace steady state probabilities with transient ones.

Case studies

Shared memory system

A model of two processors accessing a common shared memory [MBCDF95]



The diagram of the shared memory system

After activation of the system, two processors are active, and the common memory is available. Each processor can request an access to the memory.

When a processor starts an acquisition of the memory, another processor waits until the former one ends its memory operations, and the system returns to the state with both active processors and the available common memory.

a corresponds to the system activation.

r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i.

b_i and e_i correspond to the beginning and the end of the common memory access of processor i.

The other actions are used for communication purpose only.

The static expression of the first processor is

$E_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r_1\}, \frac{1}{2}); (\{b_1, y_1\}, \frac{1}{2}); (\{e_1, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$E_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r_2\}, \frac{1}{2}); (\{b_2, y_2\}, \frac{1}{2}); (\{e_2, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

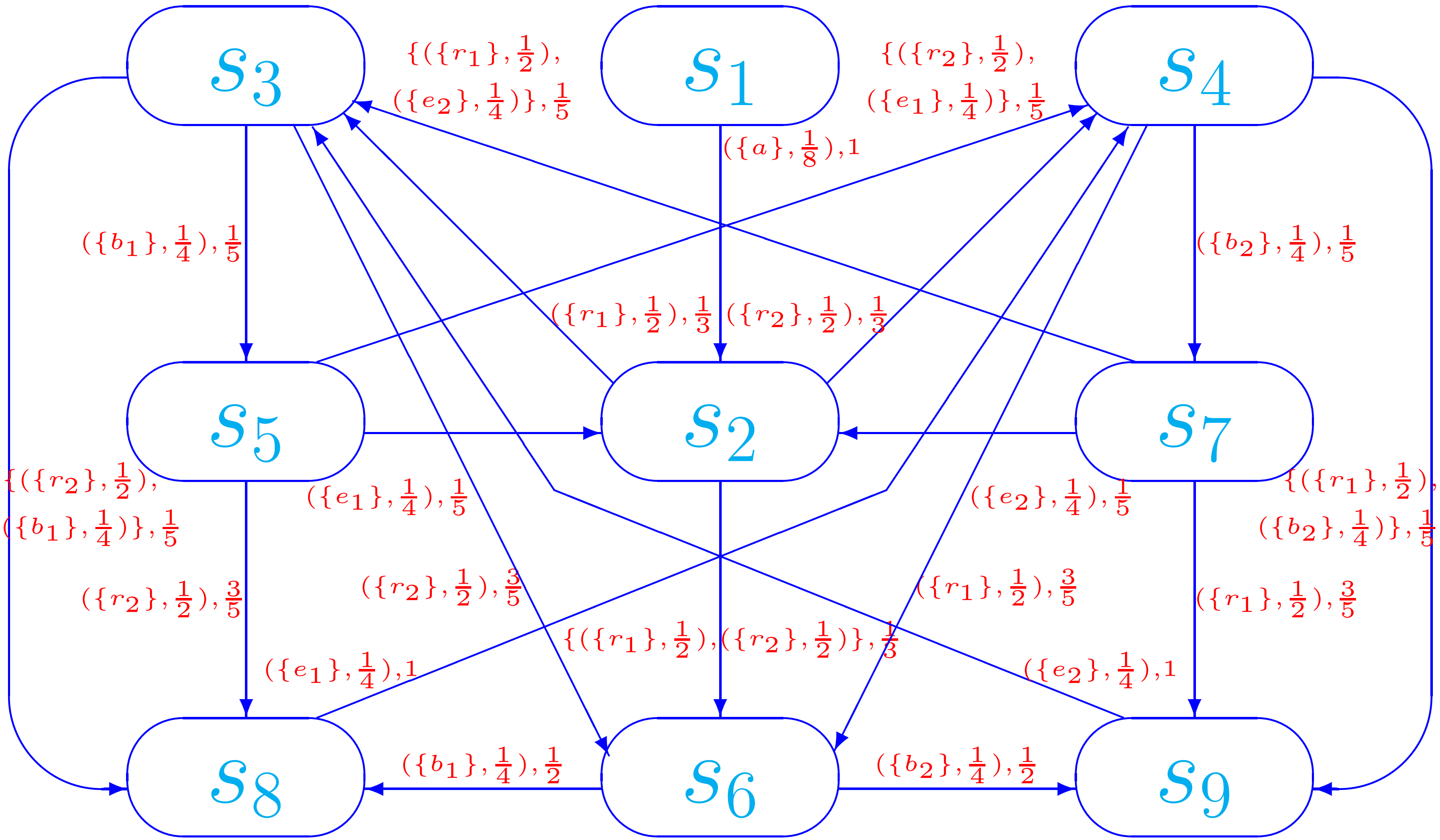
The static expression of the shared memory is

$E_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the shared memory system with two processors is

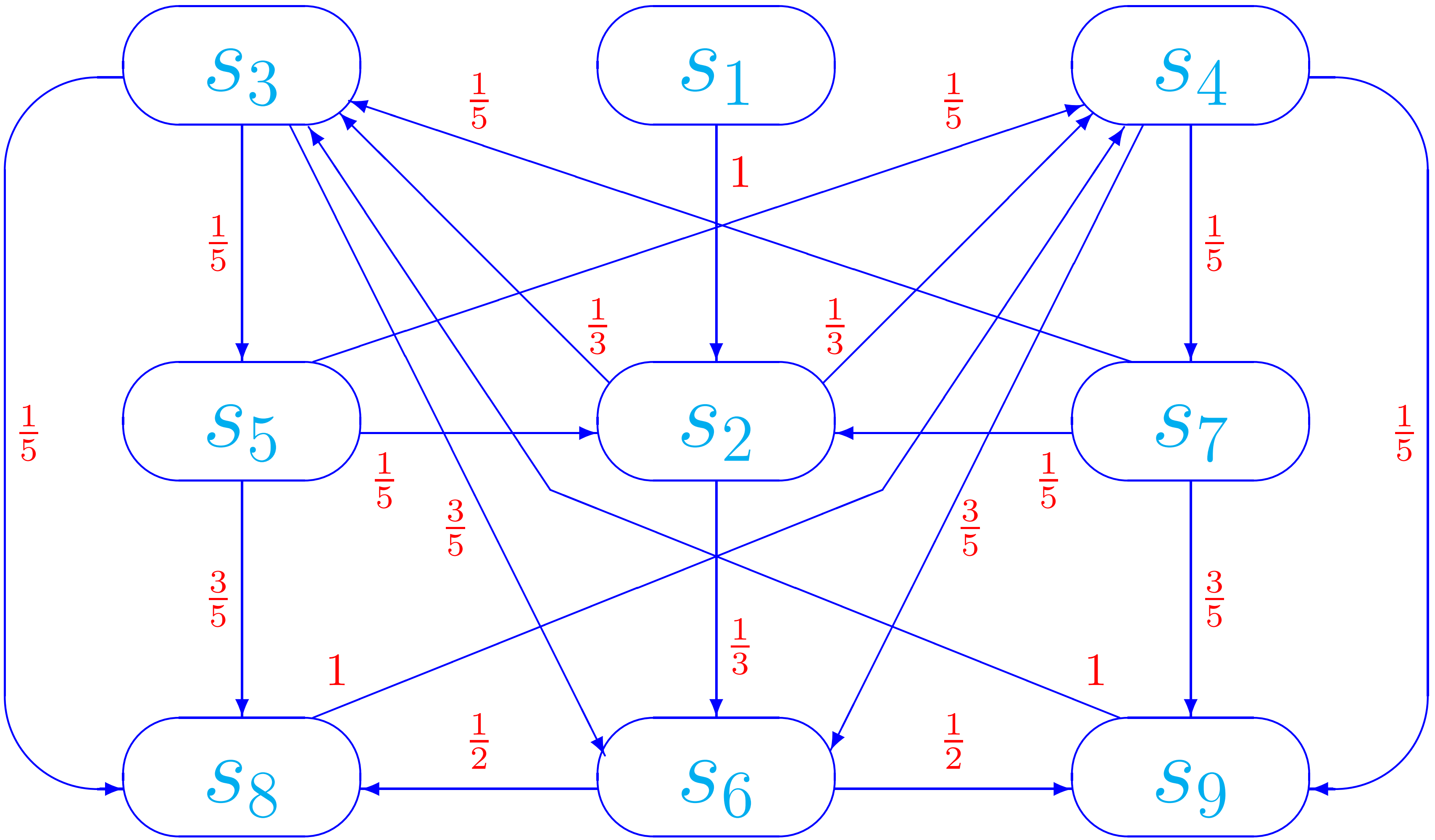
$E = (E_1 \| E_2 \| E_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$TS^*(\overline{E})$



The transition system without empty loops of the shared memory system

$DTMC^*(\overline{E})$



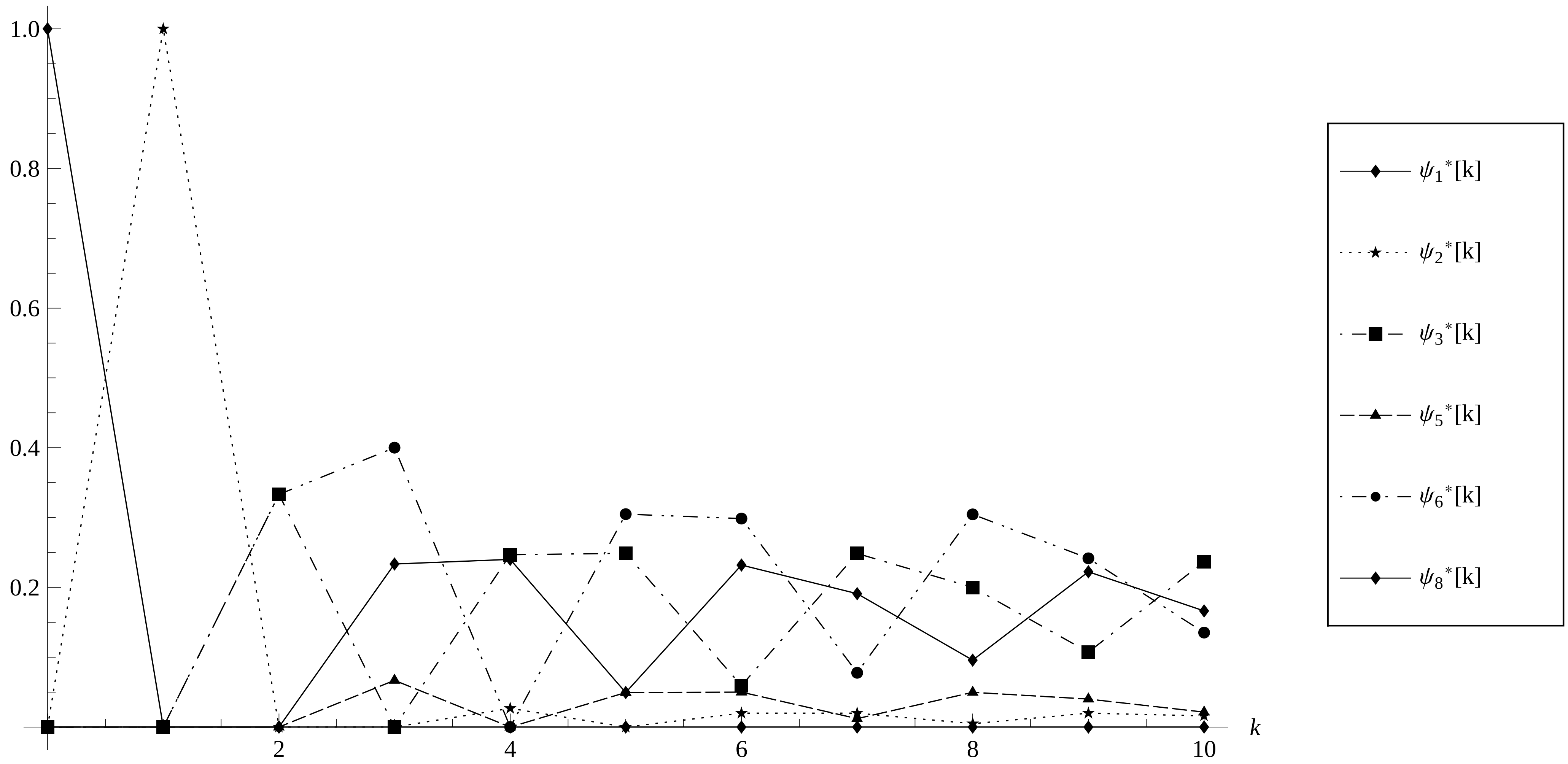
The underlying DTMC without empty loops of the shared memory system

The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The steady state PMF ψ^* for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{3}{209}, \frac{75}{418}, \frac{75}{418}, \frac{15}{418}, \frac{46}{209}, \frac{15}{418}, \frac{35}{209}, \frac{35}{209}\right).$$



Transient state probabilities of the shared memory system

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.

Performance indices

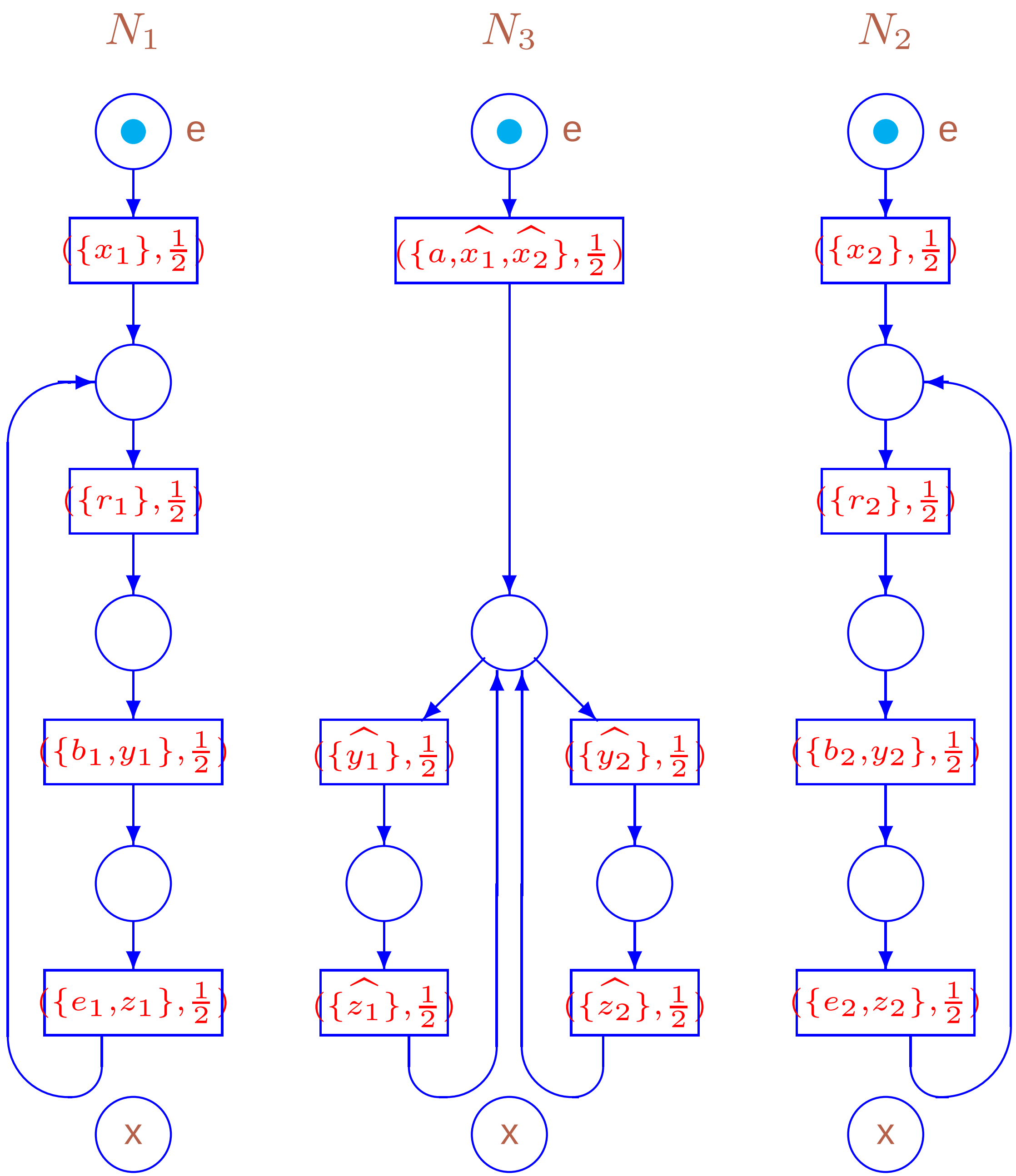
- The average recurrence time in the state s_2, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{209}{3} = 69\frac{2}{3}$.
- The common memory is available in the states s_2, s_3, s_4, s_6 only.

  The steady state probability that the memory is available is $\psi_2^* + \psi_3^* + \psi_4^* + \psi_6^* = \frac{124}{209}$.

  The steady state probability that the memory is used, the *shared memory utilization*, is $1 - \frac{124}{209} = \frac{85}{209}$.
- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is possible from the states s_2, s_4, s_7 only.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

  The *steady state probability of the shared memory request from the first processor* is
  $\psi_2^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_4^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_7^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) = \frac{3}{209} \cdot \left(\frac{1}{3} + \frac{1}{3}\right) + \frac{75}{418} \cdot \left(\frac{3}{5} + \frac{1}{5}\right) + \frac{15}{418} \cdot \left(\frac{3}{5} + \frac{1}{5}\right) = \frac{38}{209}$.



The marked dts-boxes of two processors and shared memory

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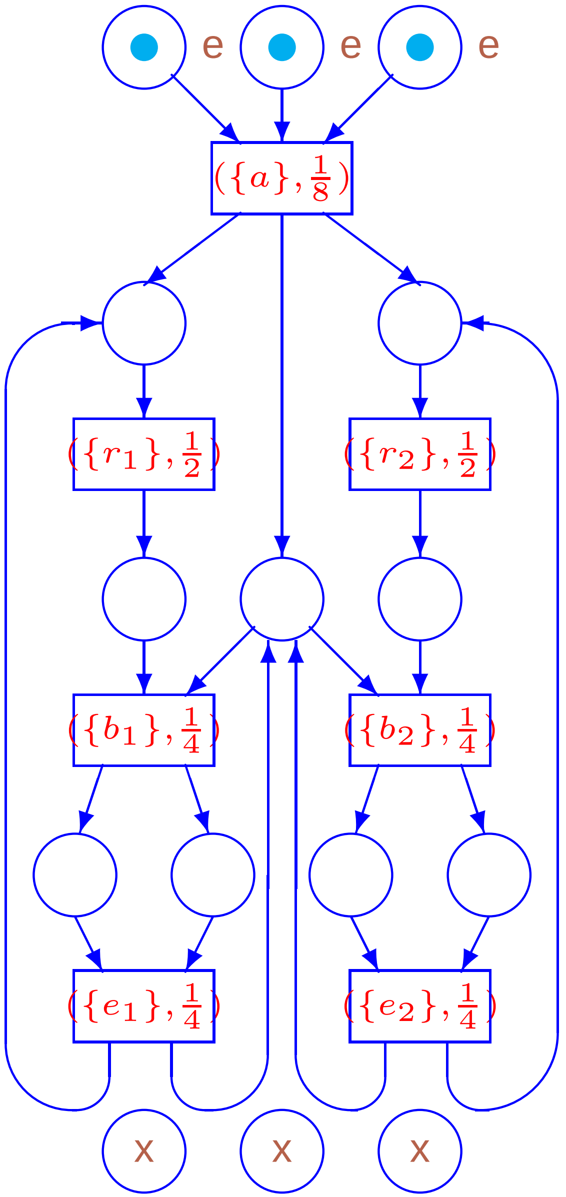
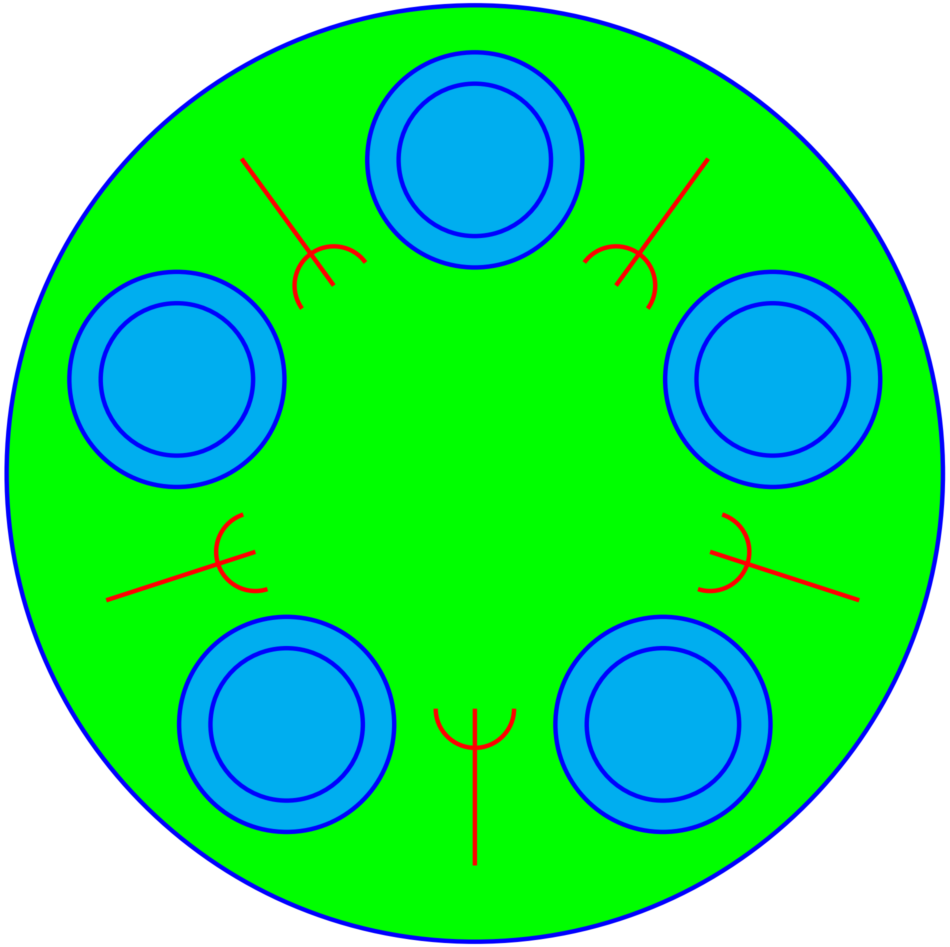


The marked dts-box of the shared memory system

Dining philosophers system

A model of five dining philosophers [P81]



The diagram of the dining philosophers system

After activation of the system, five forks appear on the table.

If the left and right forks available for a philosopher, he takes them simultaneously and begins eating.

At the end of eating, the philosopher places both his forks simultaneously back on the table.

a corresponds to the system activation.

b_i and e_i correspond to the beginning and the end of eating of philosopher i $(1 \leq i \leq 5)$.

The other actions are used for communication purpose only.

The expression of each philosopher includes two alternative subexpressions:

the second one specifies a resource (fork) sharing with the right neighbor.

The static expression of the philosopher i $(1 \leq i \leq 4)$ is

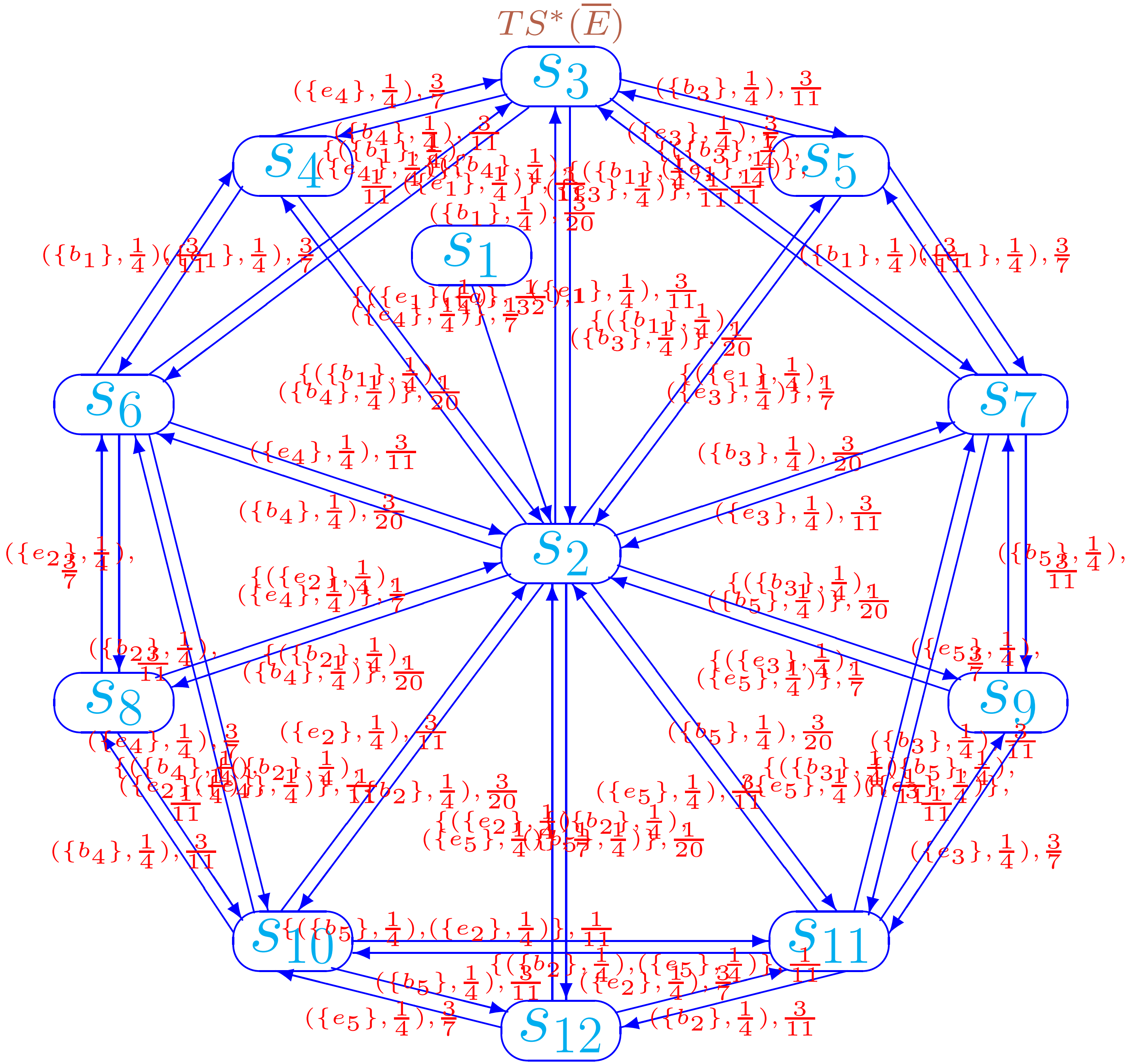
$E_i = [(\{x_i\}, \frac{1}{2}) * (((\{b_i, \widehat{y_i}\}, \frac{1}{2}); (\{e_i, \widehat{z_i}\}, \frac{1}{2}))[]((\{y_{i+1}\}, \frac{1}{2}); (\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is

$E_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \frac{1}{2}); (\{e_5, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2}); (\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}]$.

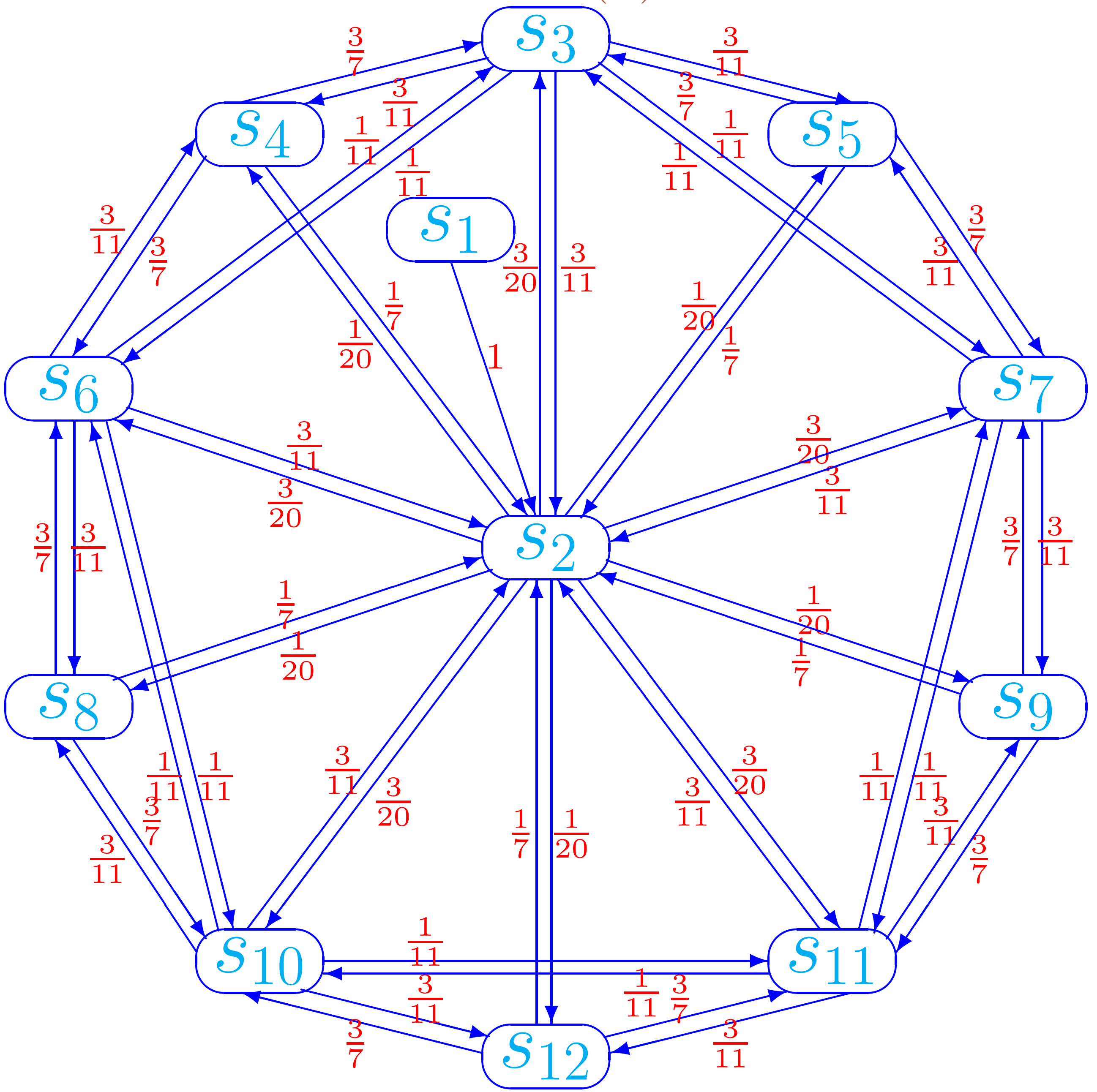
The static expression of the dining philosophers system is

$E = (E_1 \| E_2 \| E_3 \| E_4 \| E_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.



The transition system without empty loops of the dining philosophers system

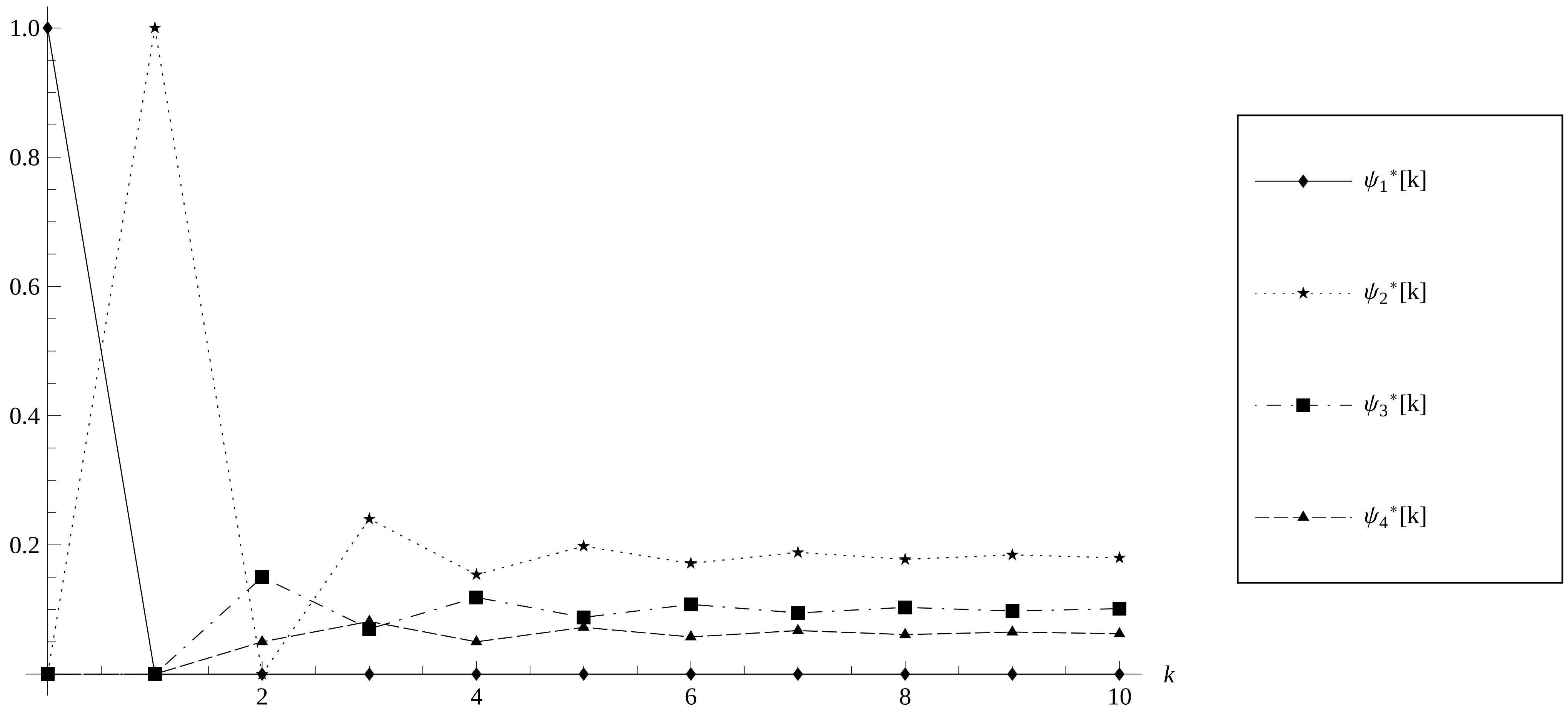
$DTMC^*(\overline{E})$



The underlying DTMC without empty loops of the dining philosophers system

The TPM for $DTMC^*(\overline{E})$ is

$$\mathbf{P}^* = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\
0 & \frac{3}{11} & 0 & \frac{3}{11} & \frac{3}{11} & \frac{1}{11} & \frac{1}{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 \\
0 & \frac{3}{11} & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & \frac{1}{11} & \frac{3}{11} \\
0 & \frac{3}{11} & 0 & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{7} & \frac{3}{7} & 0
\end{bmatrix}.$$



Transient state probabilities of the dining philosophers system

We depict the probabilities for the states $s_1, \ldots, s_4$ only, since the corresponding values coincide for the states $s_3, s_6, s_7, s_{10}, s_{11}$ as well as for $s_4, s_5, s_8, s_9, s_{12}$.

The steady state PMF ψ^* for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{2}{11}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}\right).$$

Performance indices

- The average recurrence time in the state s_2, where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{11}{2} = 5\frac{1}{2}$.

- Nobody eats at the state s_2. The *fraction of time when no philosophers dine* is $\psi_2^* = \frac{2}{11}$.

  Only one philosopher eats at the states $s_3, s_6, s_7, s_{10}, s_{11}$. The *fraction of time when only one philosopher dines* is $\psi_3^* + \psi_6^* + \psi_7^* + \psi_{10}^* + \psi_{11}^* = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}$.

  Two philosophers eat together at the states $s_4, s_5, s_8, s_9, s_{12}$. The *fraction of time when two philosophers dine* is $\psi_4^* + \psi_5^* + \psi_8^* + \psi_9^* + \psi_{12}^* = \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} = \frac{7}{22}$.

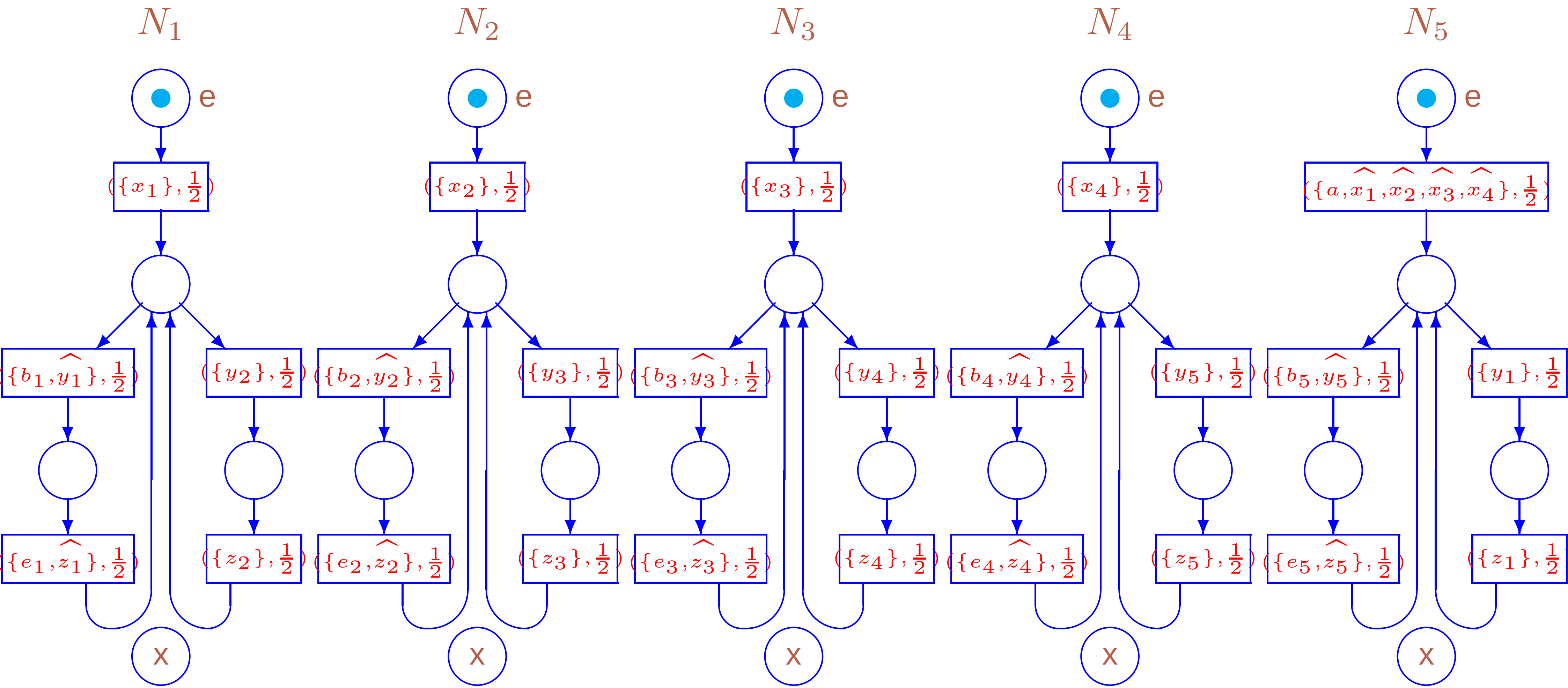
  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of first philosopher $(\{b_1\}, \frac{1}{4})$ is possible from the states s_2, s_6, s_7 only.

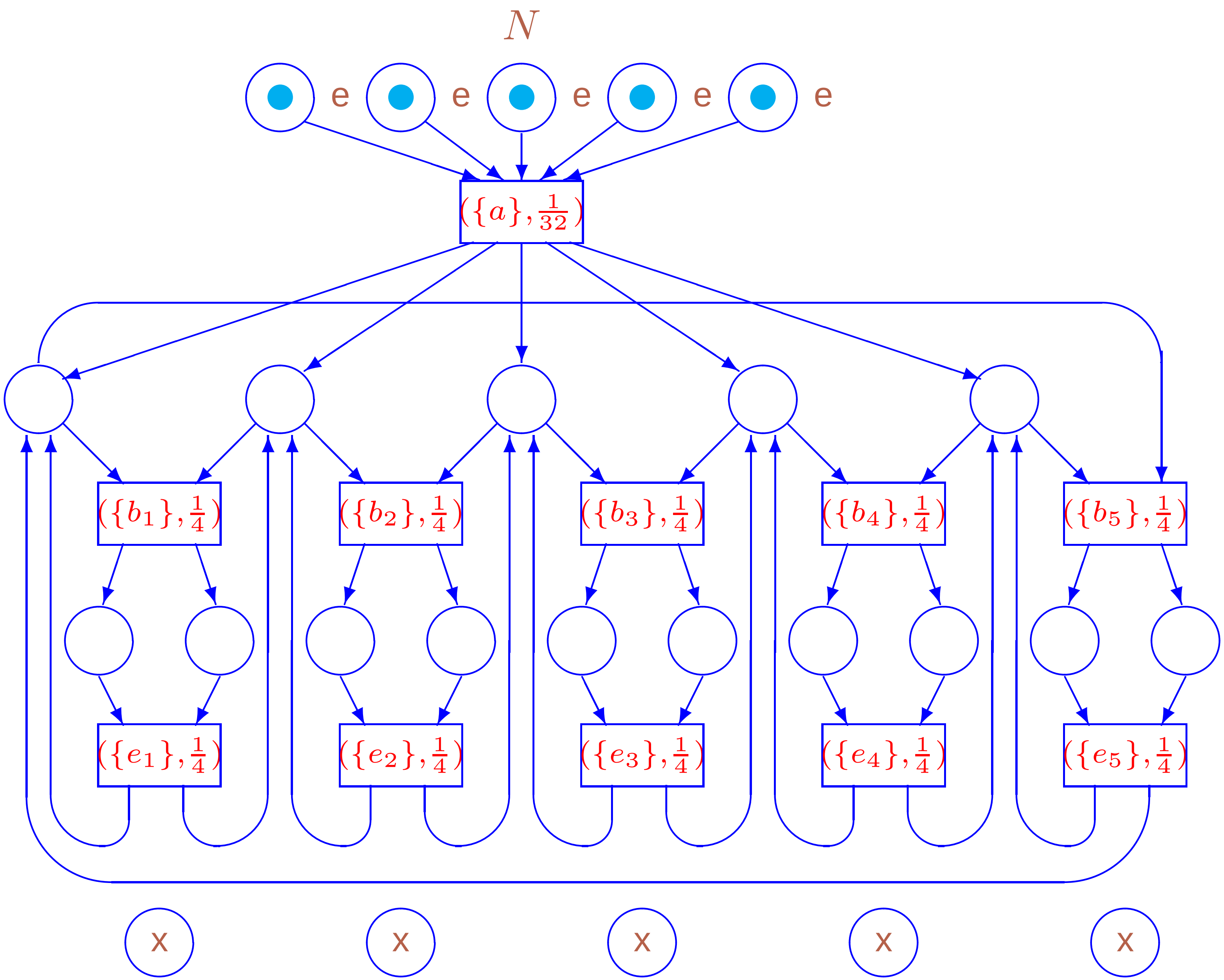
  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b_1\}, \frac{1}{4})$.

  The *steady state probability of the beginning of eating of first philosopher* is

  $\psi_2^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_6^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) +$
  $\psi_7^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) =$
  $\frac{2}{11} \cdot \left(\frac{3}{20} + \frac{1}{20} + \frac{1}{20}\right) + \frac{1}{10} \cdot \left(\frac{3}{11} + \frac{1}{11}\right) + \frac{1}{10} \cdot \left(\frac{3}{11} + \frac{1}{11}\right) = \frac{13}{110}$.



The marked dts-boxes of the dining philosophers



The marked dts-box of the dining philosophers system

Abstract dining philosophers system

The static expression of the philosopher i $(1 \leq i \leq 4)$ is

$F_i = [(\{x_i\}, \frac{1}{2}) * (((\{b, \widehat{y_i}\}, \frac{1}{2}); (\{e, \widehat{z_i}\}, \frac{1}{2}))[]((\{y_{i+1}\}, \frac{1}{2}); (\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is

$F_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * (((\{b, \widehat{y_5}\}, \frac{1}{2}); (\{e, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2}); (\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract dining philosophers system is

$F = (F_1 \| F_2 \| F_3 \| F_4 \| F_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$.

$DTMC^*(\overline{F}) = DTMC^*(\overline{E})$, thus, TPM and the steady state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

Performance indices

The first performance index and the second group of the indices are the same for the standard and abstract systems.

The following performance index: non-personalized viewpoint to the philosophers.

- The beginning of eating of a philosopher $(\{b\}, \frac{1}{4})$ is possible from the states $s_2, s_3, s_6, s_7, s_{10}, s_{11}$ only.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{1}{4})$.

  The *steady state probability of the beginning of eating of a philosopher* is
  $\psi_2^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_3) + \psi_6^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) + \psi_7^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) + \psi_{10}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{10}) + \psi_{11}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{11}) = \frac{2}{11} \cdot \left(\frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4} \cdot \left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) = \frac{6}{11}$.

Reduced abstract dining philosophers system

The static expression of the philosopher 1 is $F_1' = [(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the philosopher 2 is $F_2' = [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the reduced abstract dining philosophers system is $F' = (F_1' \| F_2')\ \mathsf{sy}\ x\ \mathsf{rs}\ x$.

$DR(\overline{F'})$ consists of isomorphism classes

$s_1' = [([\overline{(\{x\}, \frac{1}{2})} * ((\{b\}, \frac{2}{5})_1; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \|$
$[\overline{(\{a, \hat{x}\}, \frac{1}{16})} * ((\{b\}, \frac{2}{5})_2; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\simeq$,

$s_2' = [([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1}; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \|$
$[\underline{(\{a, \hat{x}\}, \frac{1}{16}) * (}\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\simeq$,

$s_3' = [([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \|$
$[(\{a, \hat{x}\}, \frac{1}{16}) * (\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\simeq$,

$s_4' = [([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1}; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \|$
$[(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\simeq$,

$s_5' = [([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \|$
$[(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}])\ \mathsf{sy}\ x\ \mathsf{rs}\ x]_\simeq$.

$\overline{F}\underline{\leftrightarrow}_{ss}\overline{F'}$ with $(DR(\overline{F}) \cup DR(\overline{F'}))/_{\underline{\leftrightarrow}_{ss}} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$, where

$\mathcal{H}_1 = \{s_1, s'_1\}$ (the initial state),

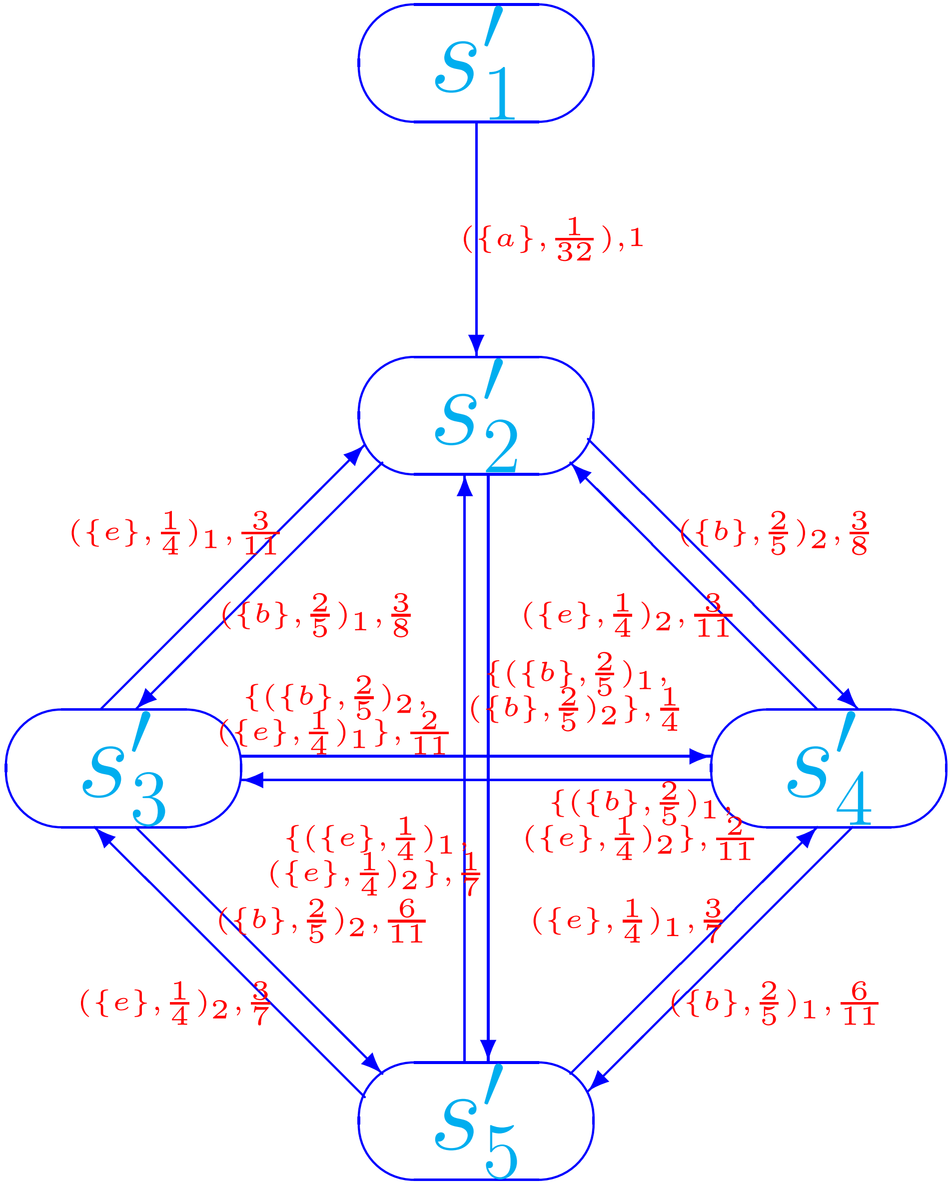
$\mathcal{H}_2 = \{s_2, s'_2\}$ (the system is activated and no philosophers dine),

$\mathcal{H}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}, s'_3, s'_4\}$ (one philosopher dines),

$\mathcal{H}_4 = \{s_4, s_5, s_8, s_9, s_{12}, s'_5\}$ (two philosophers dine).

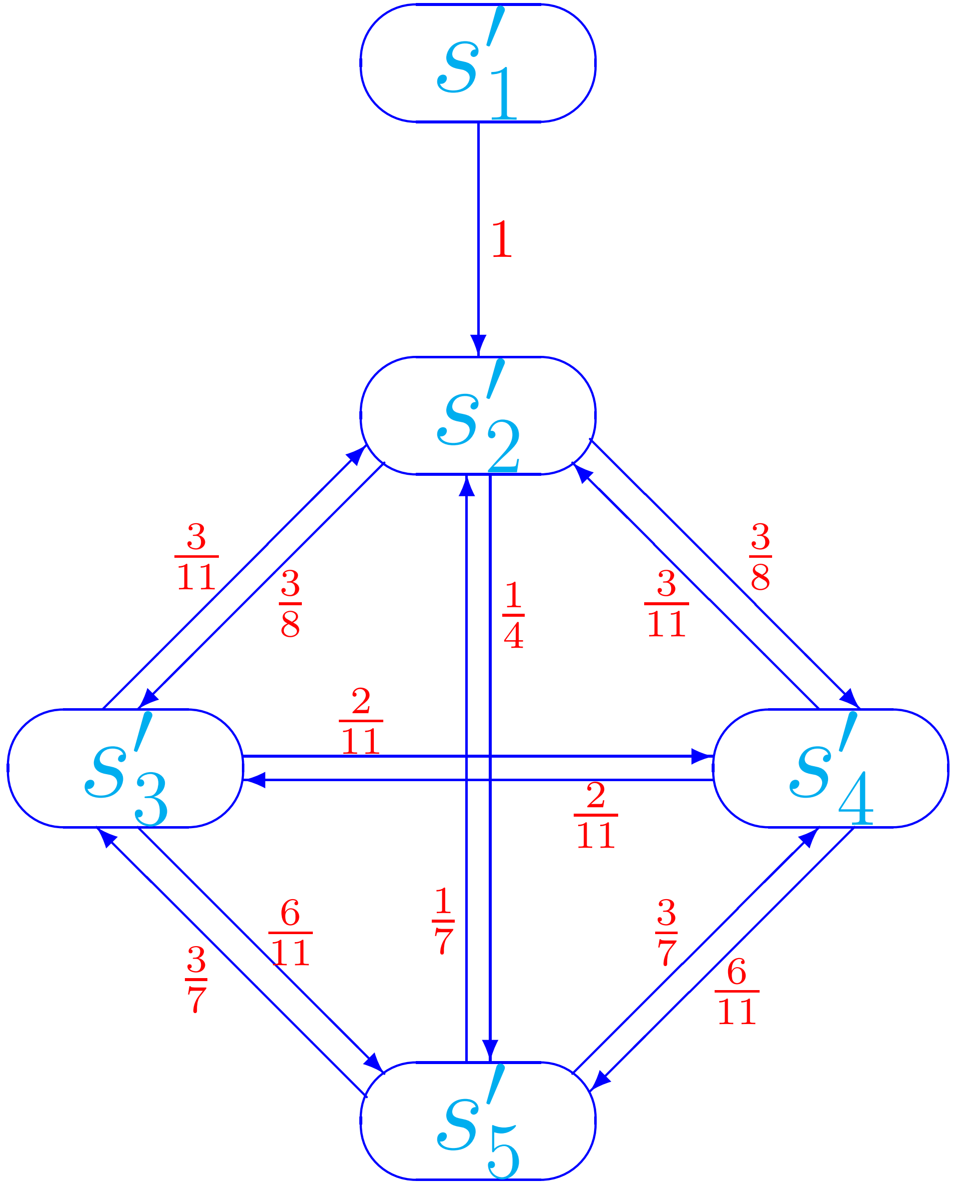
F' is a reduction of F w.r.t. $\underline{\leftrightarrow}_{ss}$.

$TS^*(\overline{F'})$



The transition system without empty loops of the reduced abstract dining philosophers system

$DTMC^*(\overline{F'})$



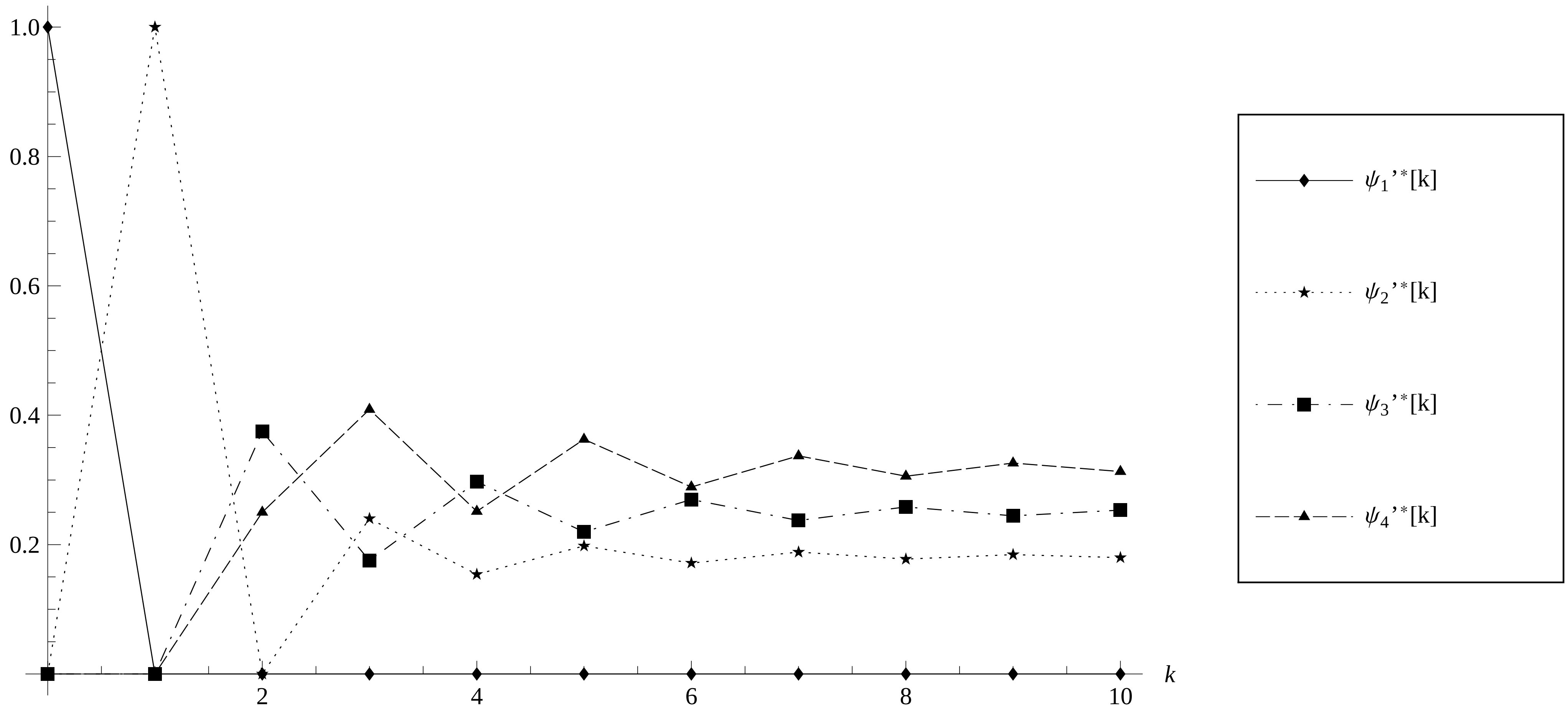
The underlying DTMC without empty loops of the reduced abstract dining philosophers system

The TPM for $DTMC^*(\overline{F'})$ is

$$\mathbf{P}'^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ 0 & \frac{3}{11} & 0 & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{3}{11} & \frac{2}{11} & 0 & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{3}{7} & \frac{3}{7} & 0 \end{bmatrix}.$$

The steady state PMF ψ'^* for $DTMC^*(\overline{F'})$ is

$$\psi'^* = \left(0, \frac{2}{11}, \frac{1}{4}, \frac{1}{4}, \frac{7}{22}\right).$$



Transient state probabilities of the reduced abstract dining philosophers system

We depict the probabilities for the states s_1, s_2, s_3, s_5 only, since the corresponding values coincide for s_3, s_4.

Performance indices

- The average recurrence time in the state s_2', where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2'^*} = \frac{11}{2} = 5\frac{1}{2}$.

- Nobody eats at the state s_2'. The *fraction of time when no philosophers dine* is $\psi_2'^* = \frac{2}{11}$.

  Only one philosopher eats at the states s_3', s_4'. The *fraction of time when only one philosopher dines* is $\psi_3'^* + \psi_4'^* = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

  Two philosophers eat together at the state s_5'. The *fraction of time when two philosophers dine* is $\psi_5'^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine with respect to when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.

- The beginning of eating of a philosopher $(\{b\}, \frac{2}{5})$ is possible from the states s_2', s_3', s_4' only.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{2}{5})$.

  The *steady state probability of the beginning of eating of a philosopher* is
  $\psi_2'^* \sum_{\{\Gamma \mid (\{b\}, \frac{2}{5}) \in \Gamma\}} PT^*(\Gamma, s_2') + \psi_3'^* \sum_{\{\Gamma \mid (\{b\}, \frac{2}{5}) \in \Gamma\}} PT^*(\Gamma, s_3') +$
  $\psi_4'^* \sum_{\{\Gamma \mid (\{b\}, \frac{2}{5}) \in \Gamma\}} PT^*(\Gamma, s_4') = \frac{2}{11} \cdot \left(\frac{3}{8} + \frac{3}{8} + \frac{1}{4}\right) + \frac{1}{4} \cdot \left(\frac{6}{11} + \frac{2}{11}\right) + \frac{1}{4} \cdot \left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}$.

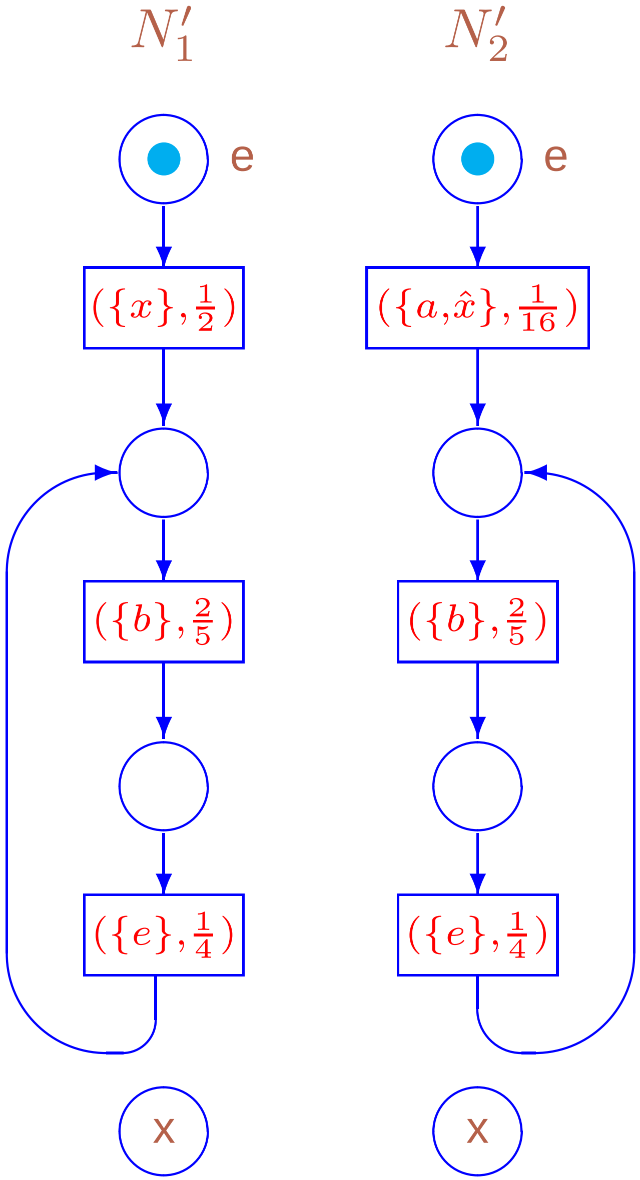
The performance indices are the same for the complete and the reduced abstract dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates the result of proposition about steady state probabilities.

The coincidence of the third performance index is due to the theorem about step traces from steady states:

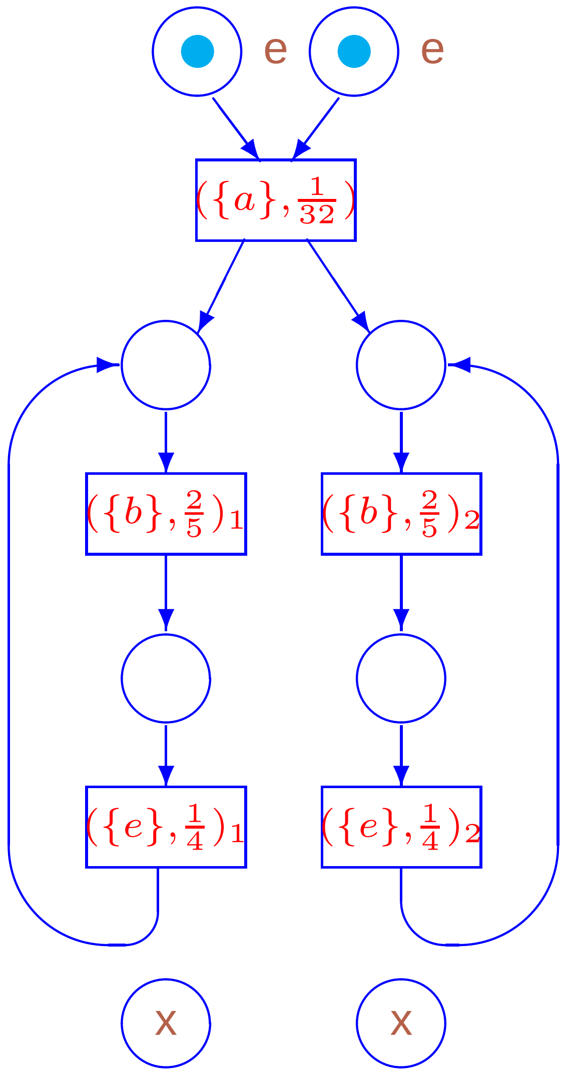
one should apply its result to the step traces $\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{F}$ and $\overline{F'}$,

and sum the left and right parts of the three resulting equalities.



The marked dts-boxes of the reduced abstract dining philosophers

N'



The marked dts-box of the reduced abstract dining philosophers system

Definition 2 *The* minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ (labeled probabilistic) transition system without empty loops *of a dynamic expression* G *is* $TS^*_{\underline{\leftrightarrow}_{ss}}(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$*:*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\underline{\leftrightarrow}_{ss}}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq I\!N_f^{\mathcal{L}} \times (0;1]$;
- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{H}, (A, \mathcal{P}), \widetilde{\mathcal{H}}) \mid \exists s \in \mathcal{H}\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{H}}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = \{[G]_{\simeq}\}$.

The transition $(\mathcal{H}, (A, \mathcal{P}), \widetilde{\mathcal{H}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{H} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{H}}$.

For $E \in RegStatExpr$ let $TS^*_{\underline{\leftrightarrow}_{ss}}(E) = TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Definition 3 *The* minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ underlying DTMC without empty loops *of a dynamic expression* G, $DTMC^*_{\underline{\leftrightarrow}_{ss}}(G)$, *has the state space* $DR(G)/_{\underline{\leftrightarrow}_{ss}}$ *and the transitions* $\mathcal{H} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{H}}$, *if* $\exists s \in \mathcal{H}\ s \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{H}}$.

For $E \in RegStatExpr$ let $DTMC^*_{\underline{\leftrightarrow}_{ss}}(E) = DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

$DR(\overline{F})/_{\underline{\leftrightarrow}_{ss}} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$, where

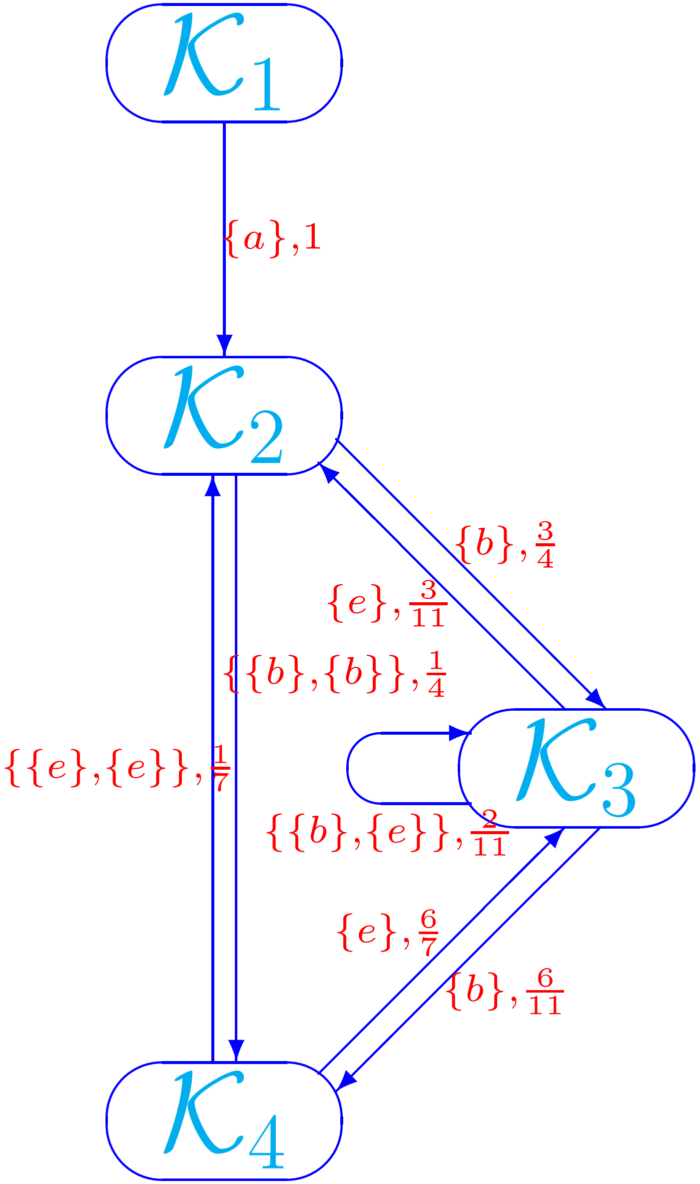
$\mathcal{K}_1 = \{s_1\}$ (the initial state),

$\mathcal{K}_2 = \{s_2\}$ (the system is activated and no philosophers dine),

$\mathcal{K}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}\}$ (one philosopher dines),

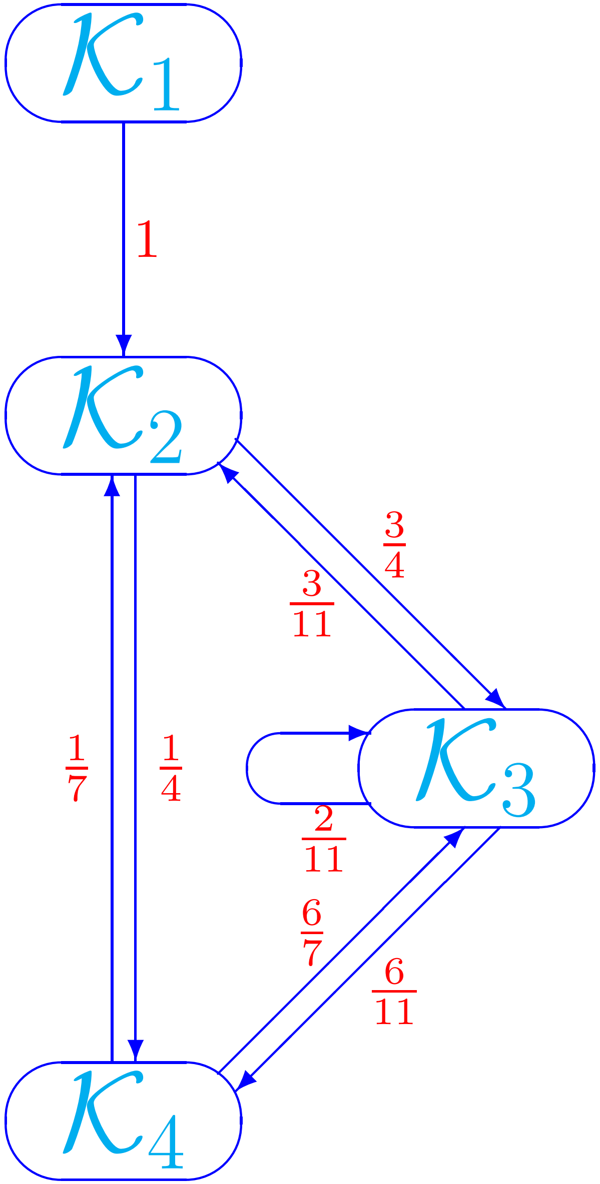
$\mathcal{K}_4 = \{s_4, s_5, s_8, s_9, s_{12}\}$ (two philosophers dine).

$TS^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



The minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ transition system without empty loops of the abstract dining philosophers system

$DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



The minimal reduced with respect to $\underline{\leftrightarrow}_{ss}$ underlying DTMC without empty loops of the abstract dining philosophers system

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