# An Investigation of Back-Forth and Place Bisimulation Equivalences *

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**Abstract**

The paper is devoted to the investigation of behavioural equivalences of concurrent systems modelled by Petri nets. Back-forth and place bisimulation equivalences known from the literature are supplemented by new ones, and their relationship with basic behavioural equivalences is examined for the general class of nets as well as for their subclass of sequential nets (nets without concurrent transitions). In addition, the preservation of all the equivalence notions by refinements is investigated to find out which of these equivalences may be used for top-down design.

**Key words & phrases:** Petri nets, sequential nets, basic equivalences, back-forth bisimulations, place bisimulations, refinement.

## 1 Introduction

The notion of equivalence is central in any theory of systems. It allows one to compare systems taking into account particular aspects of their behaviour.

Petri nets [21] became a popular formal model for design of concurrent and distributed systems. One of the main advantages of Petri nets is their ability for structural characterization of three fundamental features of concurrent computations: causality, nondeterminism and concurrency.

In recent years, a wide range of semantic equivalences was proposed in concurrency theory. Some of them were either directly defined or transferred from other formal models to the framework of Petri nets. The following basic notions of behavioural equivalences in interleaving/true concurrency and linear time/branching time semantics are known from the literature (some of them were introduced by the author in [26, 27] to obtain the complete set of relations).

- *Trace equivalences* (they respect only protocols of behaviour of systems): interleaving ($\equiv_i$) [13], step ($\equiv_s$) [22], partial word ($\equiv_{pw}$) [11], pomset ($\equiv_{pom}$) [24] and process ($\equiv_{pr}$) [26].
- *Usual bisimulation equivalences* (they respect branching structure of behaviour of systems): interleaving ($\underline{\leftrightarrow}_i$) [20], step ($\underline{\leftrightarrow}_s$) [16], partial word ($\underline{\leftrightarrow}_{pw}$) [28], pomset ($\underline{\leftrightarrow}_{pom}$) [7] and process ($\underline{\leftrightarrow}_{pr}$) [4].
- *ST-bisimulation equivalences* (they respect the duration or maximality of events in behaviour of systems): interleaving ($\underline{\leftrightarrow}_{iST}$) [12], partial word ($\underline{\leftrightarrow}_{pwST}$) [28], pomset ($\underline{\leftrightarrow}_{pomST}$) [28] and process ($\underline{\leftrightarrow}_{prST}$) [26].
- *History preserving bisimulation equivalences* (they respect the "history" of behaviour of systems): pomset ($\underline{\leftrightarrow}_{pomh}$) [25] and process ($\underline{\leftrightarrow}_{prh}$) [26].
- *Conflict preserving equivalences* (they completely respect conflicts of events in systems): multi event structure ($\equiv_{mes}$) [26] and occurrence ($\equiv_{occ}$) [15].
- *Isomorphism* ($\simeq$) (i.e. coincidence of systems up to renaming of their components).

These basic equivalences may be represented as dots on coordinate plane in Figure 1. When moving along $X$ axis, simulation of causality grows in corresponding semantics. Moving along $Y$ axis increases modelling of nondeterminism.

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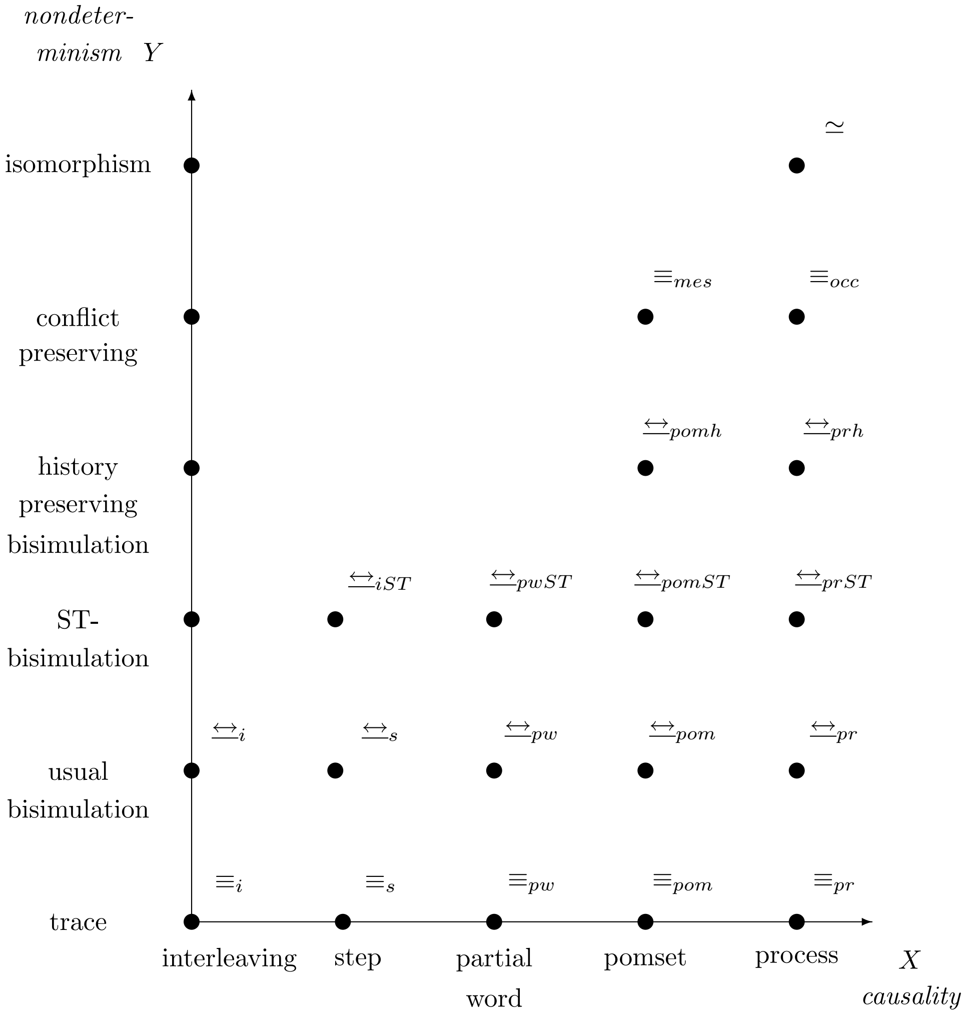


Figure 1: Classification of basic equivalences

Recently, two important groups of equivalence relations were introduced: back-forth and place bisimulation equivalences.

Back-forth bisimulation equivalences are based on the idea that bisimulation relation do not only require systems to simulate each other behaviour in the forward direction (as usually) but also when going back in history. They are closely connected with equivalences of logics with past modalities.

These equivalence notions were initially introduced in [14] in the framework of transition systems. It was shown that back-forth variant ($\underline{\leftrightarrow}_{ibif}$) of interleaving bisimulation equivalence coincide with ordinary $\underline{\leftrightarrow}_{i}$.

In [8, 9, 10] the new variants of step ($\underline{\leftrightarrow}_{sbsf}$), partial word ($\underline{\leftrightarrow}_{pwbpwf}$) and pomset ($\underline{\leftrightarrow}_{pombpomf}$) back-forth bisimulation equivalences were defined in the framework of prime event structures. The equivalence notions were compared with usual, ST- and history preserving bisimulation equivalences. It was shown that $\underline{\leftrightarrow}_{pomST}$ implies $\underline{\leftrightarrow}_{sbsf}$. The coincidence of $\underline{\leftrightarrow}_{pombpomf}$ and $\underline{\leftrightarrow}_{pomh}$ was proved, giving rise to a new, logical characterization of the latter. A stability of back-forth relations over refinement operator (allowing to consider concurrent systems on lower abstraction levels) was examined. *Only* $\underline{\leftrightarrow}_{pombpomf}$ was demonstrated to be preserved by refinements.

In [23] the new idea of differentiating the kinds of back and forth simulations appeared (following this idea, it is possible, for example, to define step back pomset forth bisimulation equivalence ($\underline{\leftrightarrow}_{sbpomf}$)). The set of all possible back-forth equivalence notions was proposed in interleaving, step, partial word and pomset semantics. Two new notions which do not coincide with known ones were obtained: step back partial word forth ($\underline{\leftrightarrow}_{sbpwf}$) and step back pomset forth ($\underline{\leftrightarrow}_{sbpomf}$) bisimulation equivalences. It was proved that the former is not preserved by refinements, and the question was addressed about the latter.

Place bisimulation equivalences were initially introduced in [1] on the basis of definition from [17, 18, 19]. Place bisimulations are relations over places instead of markings or processes. The relation on markings is obtained using the *"lifting"* of relation on places. The main application of place bisimulation equivalences is effective behaviour preserving reduction technique for Petri nets based on them.

In [1, 2] interleaving place bisimulation equivalence ($\sim_i$) was proposed. In these papers also strict interleaving place bisimulation equivalence ($\approx_i$) was defined, by imposing the additional requirement stating that corresponding transitions of nets must be related by bisimulation. The question about possibility to introduce history preserving place bisimulation equivalence was addressed.

In [4, 5] step ($\sim_s$), partial word ($\sim_{pw}$), pomset ($\sim_{pom}$), process ($\sim_{pr}$) place bisimulation equivalences and their strict analogues ($\approx_s$, $\approx_{pw}$, $\approx_{pom}$, $\approx_{pr}$) were proposed. The coincidence of $\sim_i$, $\sim_s$ and $\sim_{pw}$ was established. Also it was shown that all strict bisimulation equivalences coincide with $\sim_{pr}$. Therefore, we have only three different equivalences: $\sim_i$, $\sim_{pom}$ and $\sim_{pr}$. In addition, in these papers the polynomial algorithm of net reduction was proposed which preserves the behaviour of a net (i.e. the initial and reduced nets are bisimulation equivalent).

To choose most appropriate behavioural viewpoint on systems to be modelled, it is very important to have a complete set of equivalence notions in all semantics and understand their interrelations. This branch of research is usually called *comparative concurrency semantics.* To clarify the nature of equivalences and evaluate how they respect a concurrency, it is actual to consider also correlation of these notions on concurrency-free (sequential) nets. Treating equivalences for preservation by refinements allows one to decide which of them may be used for top-down design.

Working in the framework of Petri nets, in this paper we extend the set of back-forth equivalences from [23] by process ones and obtain as a result two new notions which cannot be reduced to the known relations: step back process forth ($\underline{\leftrightarrow}_{sbprf}$) and pomset back process forth ($\underline{\leftrightarrow}_{pombprf}$) bisimulation equivalences.

Moreother, we compare all back-forth and place equivalences with the set of basic behavioural notions from [26, 27] giving rise to the better understanding the nature of the new (and old) notions and complete the results of [10, 23, 4, 5]. In particular, we prove that $\sim_{pr}$ implies $\underline{\leftrightarrow}_{prh}$ and answer the question from [1]: $\sim_{pr}$ is strict enough to preserve the "histories" of a net functioning. Hence, it is no sense to define history preserving place bisimulation equivalence.

Since ST- and history preserving bisimulation equivalences are consequences of $\sim_{pr}$, the algorithm of net reduction from [4, 5], based on this equivalence, preserves the *timed traces* [12] of the initial net (since ST-bisimulation equivalences are *real time consistent* [12]) and "histories" of its functionings (since history preserving bisimulation equivalences respect the "past" of processes).

In [6], SM-refinement operator for Petri nets was proposed, which "replaces" their transitions by SM-nets, a special subclass of state machine nets. We treat all the considered equivalence notions for preservation by SM-refinements and establish that among back-forth relations *only* $\underline{\leftrightarrow}_{pombpomf}$ and $\underline{\leftrightarrow}_{prbprf}$ are preserved by SM-refinements (they coincide with corresponding history presrving ones for which this result holds). So, we obtained the negative answer to the question from [23]: neither $\underline{\leftrightarrow}_{sbpomf}$ nor even $\underline{\leftrightarrow}_{pombprf}$ is preserved by refinements. We prove that $\sim_{pr}$ is the only place bisimulation equivalence which is preserved by SM-refinements.

In addition, we investigate the interrelations of all the equivalence notions on sequential nets (subclass of Petri nets corresponding to transition systems where no two transitions can be fired concurrently). The merging of most of the equivalence relations in interleaving – pomset semantics is demonstrated. We prove that on sequential nets back-forth equivalences coincide with usual forth ones.

The rest of the paper is organized as follows. Basic definitions are introduced in Section 2. In Section 3 back-forth bisimulation equivalences are proposed and compared with basic equivalence relations. In Section 4 place bisimulation equivalences are defined and their interrelations with equivalence notions considered before are investigated. In Section 5 we establish which equivalence relations are preserved by SM-refinements. Section 6 is devoted to comparison of the equivalences on sequential nets. Concluding Section 7 contains a review of the main results obtained and some directions of further research.

# 2 Basic definitions

In this section we give some basic definitions used further.

## 2.1 Multisets

**Definition 2.1** *Let $X$ be some set. A finite* multiset *$M$ over $X$ is a mapping $M : X \to \mathbf{N}$ ($\mathbf{N}$ is a set of natural numbers) s.t. $|\{x \in X \mid M(x) > 0\}| < \infty$.*

$\mathcal{M}(X)$ denotes the *set of all finite multisets* over $X$. When $\forall x \in X\ M(x) \leq 1$, $M$ is a proper set. *Cardinality* of multiset $M$ is defined in such a way: $|M| = \sum_{x \in X} M(x)$. We write $x \in M$ if $M(x) > 0$ and $M \subseteq M'$, if $\forall x \in X\ M(x) \leq M'(x)$. We define $(M + M')(x) = M(x) + M'(x)$ and $(M - M')(x) = \max\{0, M(x) - M'(x)\}$.

## 2.2 Labelled nets

Let $Act = \{a, b, \ldots\}$ be a set of *action names* or *labels*.

**Definition 2.2** *A* labelled net *is a quadruple $N = \langle P_N, T_N, F_N, l_N \rangle$, where:*

- *$P_N = \{p, q, \ldots\}$ is a set of* places*;*
- *$T_N = \{t, u, \ldots\}$ is a set of* transitions*;*
- *$F_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbf{N}$ is the* flow relation *with weights ($\mathbf{N}$ denotes a set of natural numbers);*
- *$l_N : T_N \to Act$ is a* labelling *of transitions with action names.*

Given labelled nets $N = \langle P_N, T_N, F_N, l_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between $N$ and $N'$, denoted by $\beta : N \simeq N'$, if:

1. $\beta$ is a bijection s.t. $\beta(P_N) = P_{N'}$ and $\beta(T_N) = T_{N'}$;
2. $\forall p \in P_N\ \forall t \in T_N\ F_N(p, t) = F_{N'}(\beta(p), \beta(t))$ and $F_N(t, p) = F_{N'}(\beta(t), \beta(p))$;
3. $\forall t \in T_N\ l_N(t) = l_{N'}(\beta(t))$.

Labelled nets $N$ and $N'$ are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Given a labelled net $N$ and some transition $t \in T_N$, the *precondition* and *postcondition* of $t$, denoted by ${}^\bullet t$ and $t^\bullet$ respectively, are the multisets defined in such a way: $({}^\bullet t)(p) = F_N(p, t)$ and $(t^\bullet)(p) = F_N(t, p)$. Analogous definitions are introduced for places: $({}^\bullet p)(t) = F_N(t, p)$ and $(p^\bullet)(t) = F_N(p, t)$. Let ${}^\circ N = \{p \in P_N \mid {}^\bullet p = \emptyset\}$ is a set of *initial (input)* places of $N$ and $N^\circ = \{p \in P_N \mid p^\bullet = \emptyset\}$ is a set of *final (output)* places of $N$.

A labelled net $N$ is *acyclic*, if there exist no transitions $t_0, \ldots, t_n \in T_N$ s.t. $t_{i-1}^\bullet \cap {}^\bullet t_i \neq \emptyset$ $(1 \leq i \leq n)$ and $t_0 = t_n$. A labelled net $N$ is *ordinary* if $\forall p \in P_N\ {}^\bullet p$ and $p^\bullet$ are proper sets (not multisets).

Let $N = \langle P_N, T_N, F_N, l_N \rangle$ be acyclic ordinary labelled net and $x, y \in P_N \cup T_N$. Let us introduce the following notions.

- $x \prec_N y \ \Leftrightarrow\ x F_N^+ y$, where $F_N^+$ is a transitive closure of $F_N$ (*strict causal dependence* relation);
- $\downarrow_N x = \{y \in P_N \cup T_N \mid y \prec_N x\}$ (the set of *strict predecessors* of $x$);

A set $T \subseteq T_N$ is *left-closed* in $N$, if $\forall t \in T\ (\downarrow_N t) \cap T_N \subseteq T$.

## 2.3 Marked nets

A *marking* of a labelled net $N$ is a multiset $M \in \mathcal{M}(P_N)$.

**Definition 2.3** *A* marked net (net) *is a tuple* $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$, *where* $\langle P_N, T_N, F_N, l_N \rangle$ *is a labelled net and* $M_N \in \mathcal{M}(P_N)$ *is the* initial *marking.*

Given nets $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'}, M_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between $N$ and $N'$, denoted by $\beta : N \simeq N'$, if:

1. $\beta : \langle P_N, T_N, F_N, l_N \rangle \simeq \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$;
2. $\forall p \in P_N \; M_N(p) = M_{N'}(\beta(p))$.

Nets $N$ and $N'$ are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Let $M \in \mathcal{M}(P_N)$ be a marking of a net $N$. A transition $t \in T_N$ is *fireable* in $M$, if ${}^\bullet t \subseteq M$. If $t$ is fireable in $M$, its firing yields a new marking $\widetilde{M} = M - {}^\bullet t + t^\bullet$, denoted by $M \xrightarrow{t} \widetilde{M}$. A marking $M$ of a net $N$ is *reachable*, if $M = M_N$ or there exists a reachable marking $\widehat{M}$ of $N$ s.t. $\widehat{M} \xrightarrow{t} M$ for some $t \in T_N$. $Mark(N)$ denotes a *set of all reachable* markings of a net $N$.

## 2.4 Partially ordered sets

**Definition 2.4** *A* labelled partially ordered set (lposet) *is a triple* $\rho = \langle X, \prec, l \rangle$, *where:*

- $X = \{x, y, \ldots\}$ *is some set;*
- $\prec \subseteq X \times X$ *is a strict partial order (irreflexive transitive relation) over* $X$*;*
- $l : X \to Act$ *is a* labelling *function.*

Let $\rho = \langle X, \prec, l \rangle$ and $\rho' = \langle X', \prec', l' \rangle$ be lposets.

A mapping $\beta : X \to X'$ is a *label-preserving bijection* between $\rho$ and $\rho'$, denoted by $\beta : \rho \asymp \rho'$, if:

1. $\beta$ is a bijection;
2. $\forall x \in X \; l(x) = l'(\beta(x))$.

We write $\rho \asymp \rho'$, if $\exists \beta : \rho \asymp \rho'$.

A mapping $\beta : X \to X'$ is a *homomorphism* between $\rho$ and $\rho'$, denoted by $\beta : \rho \sqsubseteq \rho'$, if:

1. $\beta : \rho \asymp \rho'$;
2. $\forall x, y \in X \; x \prec y \;\Rightarrow\; \beta(x) \prec' \beta(y)$.

We write $\rho \sqsubseteq \rho'$, if $\exists \beta : \rho \sqsubseteq \rho'$.

A mapping $\beta : X \to X'$ is an *isomorphism* between $\rho$ and $\rho'$, denoted by $\beta : \rho \simeq \rho'$, if $\beta : \rho \sqsubseteq \rho'$ and $\beta^{-1} : \rho' \sqsubseteq \rho$. Lposets $\rho$ and $\rho'$ are *isomorphic*, denoted by $\rho \simeq \rho'$, if $\exists \beta : \rho \simeq \rho'$.

**Definition 2.5** Partially ordered multiset (pomset) *is an isomorphism class of lposets.*

## 2.5 Event structures

**Definition 2.6** *A* labelled event structure (LES) *is a quadruple* $\xi = \langle X, \prec, \#, l \rangle$, *where:*

- $X = \{x, y, \ldots\}$ *is a set of* events*;*
- $\prec \subseteq X \times X$ *is a strict partial order, a* causal dependence *relation, which satisfies to the principle of* finite causes*:* $\forall x \in X \; |\downarrow x| < \infty$*;*
- $\# \subseteq X \times X$ *is an irreflexive symmetrical* conflict relation*, which satisfies to the principle of* conflict heredity*:* $\forall x, y, z \in X \; x \# y \prec z \;\Rightarrow\; x \# z$*;*
- $l : X \to Act$ *is a* labelling *function.*

Let $\xi = \langle X, \prec, \#, l \rangle$ and $\xi' = \langle X', \prec', \#', l' \rangle$ be LES's. A mapping $\beta : X \to X'$ is an *isomorphism* between $\xi$ and $\xi'$, denoted by $\beta : \xi \simeq \xi'$, if:

1. $\beta$ is a bijection;

2. $\forall x \in X \ l(x) = l'(\beta(x))$;

3. $\forall x, y \in X \ x \prec y \ \Leftrightarrow \ \beta(x) \prec' \beta(y)$;

4. $\forall x, y \in X \ x\#y \ \Leftrightarrow \ \beta(x)\#'\beta(y)$.

LES's $\xi$ and $\xi'$ are *isomorphic*, denoted by $\xi \simeq \xi'$, if $\exists \beta : \xi \simeq \xi'$.

**Definition 2.7** *A* multi-event structure (MES) *is an isomorphism class of LES's.*

## 2.6 C-processes

**Definition 2.8** *A* causal net *is an acyclic ordinary labelled net $C = \langle P_C, T_C, F_C, l_C \rangle$, s.t.:*

1. *$\forall r \in P_C \ |^\bullet r| \le 1$ and $|r^\bullet| \le 1$, i.e. places are unbranched;*

2. *$\forall x \in P_C \cup T_C \ | \downarrow_C x| < \infty$, i.e. a set of causes is finite.*

Let us note that on the basis of any causal net $C = \langle P_C, T_C, F_C, l_C \rangle$ one can define lposet $\rho_C = \langle T_C, \prec_N \cap (T_C \times T_C), l_C \rangle$.

The fundamental property of causal nets is [4]: if $C$ is a causal net, then there exists a sequence of transition fireings ${}^\circ C = L_0 \stackrel{v_1}{\to} \cdots \stackrel{v_n}{\to} L_n = C^\circ$ s.t. $L_i \subseteq P_C \ (0 \le i \le n)$, $P_C = \cup_{i=0}^n L_i$ and $T_C = \{v_1, \ldots, v_n\}$. Such a sequence is called a *full execution* of $C$.

**Definition 2.9** *Given a net $N$ and a causal net $C$. A mapping $\varphi : P_C \cup T_C \to P_N \cup T_N$ is an* embedding *of $C$ into $N$, denoted by $\varphi : C \to N$, if:*

1. *$\varphi(P_C) \in \mathcal{M}(P_N)$ and $\varphi(T_C) \in \mathcal{M}(T_N)$, i.e. sorts are preserved;*

2. *$\forall v \in T_C \ {}^\bullet\varphi(v) = \varphi({}^\bullet v)$ and $\varphi(v)^\bullet = \varphi(v^\bullet)$, i.e. flow relation is respected;*

3. *$\forall v \in T_C \ l_C(v) = l_N(\varphi(v))$, i.e. labelling is preserved.*

Since embeddings respect the flow relation, if ${}^\circ C \stackrel{v_1}{\to} \cdots \stackrel{v_n}{\to} C^\circ$ is a full execution of $C$, then $M = \varphi({}^\circ C) \stackrel{\varphi(v_1)}{\longrightarrow} \cdots \stackrel{\varphi(v_n)}{\longrightarrow} \varphi(C^\circ) = \widetilde{M}$ is a sequence of transition fireings in $N$.

**Definition 2.10** *A* fireable in marking $M$ C-process (process) *of a net $N$ is a pair $\pi = (C, \varphi)$, where $C$ is a causal net and $\varphi : C \to N$ is an embedding s.t. $M = \varphi({}^\circ C)$. A fireable in $M_N$ process is a* process *of $N$.*

We write $\Pi(N, M)$ for a *set of all fireable* in marking $M$ processes of a net $N$ and $\Pi(N)$ for the *set of all processes* of a net $N$. The *initial* process of a net $N$ is $\pi_N = (C_N, \varphi_N) \in \Pi(N)$, s.t. $T_{C_N} = \emptyset$. If $\pi \in \Pi(N, M)$, then firing of this process transforms a marking $M$ into $\widetilde{M} = M - \varphi({}^\circ C) + \varphi(C^\circ) = \varphi(C^\circ)$, denoted by $M \stackrel{\pi}{\to} \widetilde{M}$.

Let $\pi = (C, \varphi)$, $\tilde{\pi} = (\widetilde{C}, \tilde{\varphi}) \in \Pi(N)$, $\hat{\pi} = (\widehat{C}, \hat{\varphi}) \in \Pi(N, \varphi(C^\circ))$. A process $\pi$ is a *prefix* of a process $\tilde{\pi}$, if $T_C \subseteq T_{\widetilde{C}}$ is a left-closed set in $\widetilde{C}$. A process $\hat{\pi}$ is a *suffix* of a process $\tilde{\pi}$, if $T_{\widehat{C}} = T_{\widetilde{C}} \setminus T_C$. In such a case a process $\tilde{\pi}$ is an *extension* of $\pi$ *by process* $\hat{\pi}$, and $\hat{\pi}$ is an *extending* process for $\pi$, denoted by $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$. We write $\pi \to \tilde{\pi}$, if $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$ for some $\hat{\pi}$.

A process $\tilde{\pi}$ is an extension of a process $\pi$ *by one transition*, denoted by $\pi \stackrel{v}{\to} \tilde{\pi}$, if $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$ and $T_{\widehat{C}} = \{v\}$.

A process $\tilde{\pi}$ is an extension of a process $\pi$ *by sequence of transitions*, denoted by $\pi \stackrel{\sigma}{\to} \tilde{\pi}$, if $\exists \pi_i \in \Pi(N) \ (1 \le i \le n) \ \pi \stackrel{v_1}{\to} \pi_1 \stackrel{v_2}{\to} \ldots \stackrel{v_n}{\to} \pi_n = \tilde{\pi}$ and $\sigma = v_1 \cdots v_n$.

# 3 Back-forth bisimulation equivalences

In this section we tranfer the definitions of back-forth bisimulation equivalences from event structures [23] into the framework of Petri nets and supplement them by the new notions induced by process semantics. In addition, we investigate the interrelations of all these equivalences to understand which of them are coincide, and compare the remaining ones with basic behavioural equivalences.

## 3.1 Sequential runs

In accordance to the idea of back-forth bisimulations, it is possible to move back from a state but only "along" the path which represents the execution of a system which brought to this state. Therefore in [14] in the framework of transition systems back-forth bisimulation relations connected sequences of transitions, and in [8, 9, 10, 23] in the framework of event structures they connected sequences of events called "histories" or "runs". Such sequences contain the information about the order in which transitions (events) happen, whose execution brought to the present state. On event structures these sequences also define configurations containing the information about causal dependencies of events.

In the framework of Petri nets, obviously, it is not sufficient to consider only sequences of C-net transitions of their processes, since, depending of embedding function, different sequences may correspond to one process and, vece versa, one sequence may correspond to different processes. Therefore, to respect the information about both causal dependencies between the transitions and the order in which they have occured, we introduce a notion of sequential run as a pair consisting of process and a sequence of its C-net transitions, which extended the initial process to the present one.

**Definition 3.1** *A* sequential run *of a net $N$ is a pair $(\pi, \sigma)$, where:*

- *a process $\pi \in \Pi(N)$ contains the information about causal dependencies of transitions which brought to this state;*
- *a sequence $\sigma \in T_C^*$ s.t. $\pi_N \stackrel{\sigma}{\to} \pi$, contains the information about the order in which the transitions occur which brought to this state.*

Let us denote the *set of all sequential runs* of a net $N$ by $Runs(N)$.

The *initial* sequential run of a net $N$ is a pair $(\pi_N, \varepsilon)$, where $\varepsilon$ is an empty sequence. Let us denote by $|\sigma|$ a *length* of a sequence $\sigma$.

Let $(\pi, \sigma),\ (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$. We write $(\pi, \sigma) \stackrel{\hat{\pi}}{\to} (\tilde{\pi}, \tilde{\sigma})$, if $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$, $\exists \hat{\sigma} \in T_{\widetilde{C}}^*\ \pi \stackrel{\hat{\sigma}}{\to} \tilde{\pi}$ and $\tilde{\sigma} = \sigma\hat{\sigma}$. We write $(\pi, \sigma) \to (\tilde{\pi}, \tilde{\sigma})$, if $(\pi, \sigma) \stackrel{\hat{\pi}}{\to} (\tilde{\pi}, \tilde{\sigma})$ for some $\hat{\pi}$.

Let $(\pi, \sigma) \in Runs(N)$, $(\pi', \sigma') \in Runs(N')$ and $\sigma = v_1 \cdots v_n$, $\sigma' = v_1' \cdots v_n'$. Let us define a mapping $\beta_\sigma^{\sigma'} : T_C \to T_{C'}$ as follows: $\beta_\sigma^{\sigma'} = \{(v_i, v_i') \mid 1 \leq i \leq n\}$. Let $\beta_\varepsilon^\varepsilon = \emptyset$.

Let $(\pi, \sigma) \in Runs(N)$ and $\sigma = v_1 \cdots v_n$, $\pi_N \stackrel{v_1}{\to} \ldots \stackrel{v_i}{\to} \pi_i$ $(1 \leq i \leq n)$.

Let us introduce the following notations:

- $\pi(0) = \pi_N$,
  $\pi(i) = \pi_i$ $(1 \leq i \leq n)$;
- $\sigma(0) = \varepsilon$,
  $\sigma(i) = v_1 \cdots v_i$ $(1 \leq i \leq n)$.

Let $(\pi, \sigma) \in Runs(N)$. An *ST-process of a sequential run* $(\pi, \sigma)$ is defined as follows: $ST(\pi, \sigma) = (\pi, \pi(j))$, where $j = \min\{i \mid (\pi, \pi(i)) \in ST-\Pi(N)\}$. We denote $Past(\pi, \sigma) = \pi(j)$.

## 3.2 Definitions of back-forth bisimulation equivalences

Now we are ready to present definitions of back-forth bisimulation equivalences.

**Definition 3.2** *Let $N$ and $N'$ be some nets. A relation $\mathcal{R} \subseteq Runs(N) \times Runs(N')$ is a* $\star$-back $\star\star$-forth bisimulation *between $N$ and $N'$,* $\star, \star\star \in${interleaving, step, partial word, pomset, process}, *denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom, pr\}$, if:*

1. *$((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.*
2. *$((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$*
   - *(back)*
     *$(\tilde{\pi}, \tilde{\sigma}) \stackrel{\hat{\pi}}{\to} (\pi, \sigma)$,*
     *(a) $|T_{\widehat{C}}| = 1$, if $\star = i$;*
     *(b) $\prec_{\widehat{C}} = \emptyset$, if $\star = s$;*
     *$\Rightarrow\ \exists (\tilde{\pi}', \tilde{\sigma}') :\ (\tilde{\pi}', \tilde{\sigma}') \stackrel{\hat{\pi}'}{\to} (\pi', \sigma'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ and*

*(a)* $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$*, if* $\star = pw$*;*
*(b)* $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$*, if* $\star \in \{i, s, pom\}$*;*
*(c)* $\widehat{C} \simeq \widehat{C}'$*, if* $\star = pr$*;*

- *(forth)*

  $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$,

  *(a)* $|T_{\widehat{C}}| = 1$*, if* $\star\star = i$*;*
  *(b)* $\prec_{\widehat{C}} = \emptyset$*, if* $\star\star = s$*;*

  $\Rightarrow\ \exists(\tilde{\pi}', \tilde{\sigma}') :\ (\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*

  *(a)* $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$*, if* $\star\star = pw$*;*
  *(b)* $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$*, if* $\star\star \in \{i, s, pom\}$*;*
  *(c)* $\widehat{C} \simeq \widehat{C}'$*, if* $\star\star = pr$*.*

3. *As item 2, but the roles of* $N$ *and* $N'$ *are reversed.*

*Two nets* $N$ *and* $N'$ *are* $\star$-back $\star\star$-forth bisimulation equivalent*,* $\star, \star\star \in$ {interleaving, step, partial word, pomset, process}*, denoted by* $N \underline{\leftrightarrow}_{\star b \star\star f} N'$*, if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_{\star b \star\star f} N'$*,* $\star, \star\star \in \{i, s, pw, pom, pr\}$*.*

Let us note that back extensions of sequential runs are *deterministic*, i.e. for $(\pi, \sigma) \in Runs(N)$ there exists only one $(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ s.t. $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$ and $|\tilde{\sigma}| = i$ $(0 \le i \le |\sigma|)$. In such a case $(\tilde{\pi}, \tilde{\sigma}) = (\pi(i), \sigma(i))$.

## 3.3 Interrelations of back-forth bisimulation equivalences

Let us consider interrelations of back-forth bisimulation equivalences.

**Proposition 3.1** *Let* $\star \in \{i, s, pw, pom, pr\}$*. For nets* $N$ *and* $N'$ $N \underline{\leftrightarrow}_{pwb\star f} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{pomb\star f} N'$*.*

*Proof.* ($\Leftarrow$) Isomorphism of lposets is homomorphism.
($\Rightarrow$) Let $\mathcal{R} : N \underline{\leftrightarrow}_{pwb\star f} N'$. Let us prove $\mathcal{R} : N \underline{\leftrightarrow}_{pomb\star f} N'$.

1. Obviously, $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$.

   - (back)

     Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$. Then $\exists(\tilde{\pi}', \tilde{\sigma}') :\ (\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ and $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$.

     Due to the symmetry of a bisimulation, the back extension $(\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma')$ must be imitated by the extension $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$, and not by any another one, due to determinism of back extensions. Then $\rho_{\widehat{C}} \sqsubseteq \rho_{\widehat{C}'}$. Consequently, $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$.
   - (forth)

     Obviously.
3. As item 2, but the roles of $N$ and $N'$ are reversed. □

**Proposition 3.2** *Let* $\star \in \{i, s, pw, pom, pr\}$*. For nets* $N$ *and* $N'$ $N \underline{\leftrightarrow}_{\star bif} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{\star b \star f} N'$*.*

*Proof.* ($\Leftarrow$) Isomorphism of causal nets, isomorphism and homomorphism of lposets of causal nets, isomorphism of lposets of causal nets with empty precedence relation imply label preserving bijection of lposets of causal nets.
($\Rightarrow$) Let $\mathcal{R} : N \underline{\leftrightarrow}_{\star bif} N'$. Let us prove $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star f} N'$.

1. Obviously, $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$.

   - (back)

     Obviously.

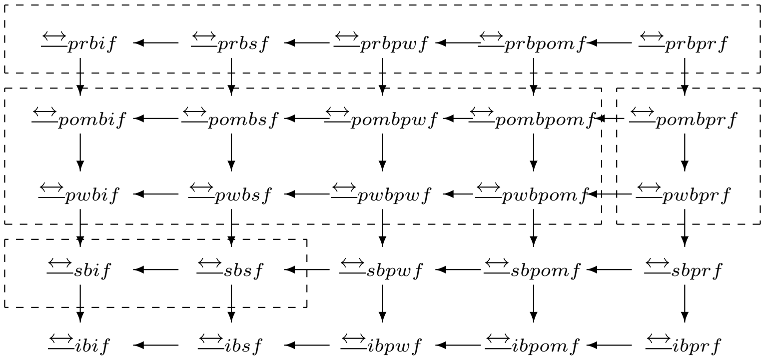


Figure 2: Merging of back-forth bisimulation equivalences

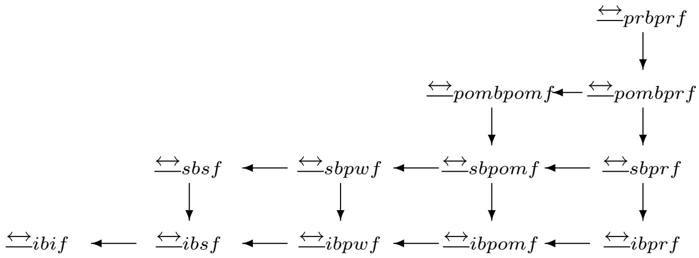


Figure 3: Interrelations of back-forth bisimulation equivalences

- (forth)

  Let $(\pi, \sigma) \stackrel{\hat{\pi}}{\to} (\tilde{\pi}, \tilde{\sigma})$. The extension by $\hat{\pi}$ corresponds to the extension by some sequence of transitions. Then $\exists (\tilde{\pi}', \tilde{\sigma}') : \ (\pi', \sigma') \stackrel{\hat{\pi}'}{\to} (\tilde{\pi}', \tilde{\sigma}'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$, where the extension by $\hat{\pi}'$ corresponds to the extension by sequence of transitions which imitates the corresponding one in the net $N$.

  Due to the symmetry of a bisimulation, the back extension $(\pi, \sigma) \stackrel{\hat{\pi}}{\to} (\tilde{\pi}, \tilde{\sigma})$ must be imitated by the extension $(\pi', \sigma') \stackrel{\hat{\pi}'}{\to} (\tilde{\pi}', \tilde{\sigma}')$, and not by any another one, due to determinism of back extensions. Then we have:

  (a) $\rho_{\widehat{C'}} \sqsubseteq \rho_{\widehat{C}}$, if $\star = pw$;

  (b) $\rho_{\widehat{C}} \simeq \rho_{\widehat{C'}}$, if $\star \in \{i, s, pom\}$;

  (c) $\widehat{C} \simeq \widehat{C}'$, if $\star = pr$.

3. As item 2, but the roles of $N$ and $N'$ are reversed. □

In Figure 2 dashed lines embrace coinciding back-forth bisimulation equivalences.
Hence, interrelations of back-forth bisimulation equivalences may be represented by graph in Figure 3.

## 3.4 Interrelations of back-forth bisimulation equivalences with basic equivalences

Let us consider interrelations of back-forth bisimulation equivalences with basic equivalences.

**Proposition 3.3** *Let $\star \in \{i, s, pw, pom, pr\}$. For nets $N$ and $N'$ $N \underline{\leftrightarrow}_{ib\star f} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{\star} N'$.*

*Proof.* ($\Leftarrow$) Let $\mathcal{R} : N \underline{\leftrightarrow}_{\star} N'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{((\pi, \sigma), (\pi', \sigma')) \mid (\pi, \sigma) \in \mathit{Runs}(N),\ (\pi', \sigma') \in \mathit{Runs}(N'),\ |\sigma| = |\sigma'|,\ l_C(\sigma) = l_{C'}(\sigma'),\ \forall i\ (0 \le i \le |\sigma|)\ (\pi(i), \pi'(i)) \in \mathcal{R}\}$. Obviously, $\mathcal{S} : N \underline{\leftrightarrow}_{ib\star f} N'$.

($\Rightarrow$) Let $\mathcal{R} : N \underline{\leftrightarrow}_{ib\star f} N'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{(\pi, \pi') \mid ((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}\}$. Obviously, $\mathcal{S} : N \underline{\leftrightarrow}_{\star} N'$. □

**Proposition 3.4** *Let $\star \in \{pom, pr\}$. For nets $N$ and $N'$ $N \underline{\leftrightarrow}_{\star b \star f} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{\star h} N'$.*

*Proof.* (⇐) Let $\mathcal{R} : N \underline{\leftrightarrow}_{\star h} N'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{((\pi, \sigma), (\pi', \sigma')) \mid (\pi, \sigma) \in Runs(N), (\pi', \sigma') \in Runs(N'),\ |\sigma| = |\sigma'|,\ \forall i\ (0 \le i \le |\sigma|)\ (\pi(i), \pi'(i), \beta_{\sigma(i)}^{\sigma'(i)}) \in \mathcal{R}\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}_{\star b \star f} N'$.

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{S}$, since $\beta_\varepsilon^\varepsilon = \emptyset$ and $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.
2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{S}$.
   - (back)
     Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$. By definition of $\mathcal{S}$, $\exists (\tilde{\pi}', \tilde{\sigma}')$ s.t. $|\tilde{\sigma}| = |\tilde{\sigma}'|$, $(\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma')$ and $((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{S}$.
     Since by definition of $\mathcal{S}$, $(\pi, \pi', \beta_\sigma^{\sigma'}) \in \mathcal{R}$ and $\mathcal{R} : N \underline{\leftrightarrow}_{\star h} N'$, we have:
     - $\beta_\sigma^{\sigma'} : \rho_C \simeq \rho_{C'}$, if $\star = pom$;
     - $C \simeq C'$, if $\star = pr$.

     Consequently,
     - $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, if $\star \in \{pom, pr\}$;
     - $\widehat{C} \simeq \widehat{C}'$, if $\star = pr$.
   - (forth)
     Let $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \sigma v_1 \cdots v_n)$. Then by definition of $\mathcal{S}$, $(\pi, \pi', \beta_\sigma^{\sigma'}) \in \mathcal{R}$ and $\exists \pi_i\ (1 \le i \le n) :\ \pi \xrightarrow{v_1} \pi_1 \xrightarrow{v_2} \ldots \xrightarrow{v_n} \pi_n = \tilde{\pi}$.
     Since $\mathcal{R} : N \underline{\leftrightarrow}_{\star h} N'$, $\exists v_i', \pi_i' :\ \pi' \xrightarrow{v_1'} \pi_1' \xrightarrow{v_2'} \ldots \xrightarrow{v_n'} \pi_n' = \tilde{\pi}'$ s.t. $(\pi_i, \pi_i', \beta_{\sigma v_1 \cdots v_i}^{\sigma' v_1' \cdots v_i'}) \in \mathcal{R}\ (1 \le i \le n)$. Consequently, for some $\hat{\pi}'$ $(\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \sigma' v_1' \cdots v_n')$ and $((\tilde{\pi}, \sigma v_1 \cdots v_n), (\tilde{\pi}', \sigma' v_1' \cdots v_n')) \in \mathcal{S}$.
     Since by definition of $\mathcal{S}$, $(\tilde{\pi}, \tilde{\pi}', \beta_{\sigma v_1 \cdots v_n}^{\sigma' v_1' \cdots v_n'}) \in \mathcal{R}$ and $\mathcal{R} : N \underline{\leftrightarrow}_{\star h} N'$, we have:
     - $\beta_{\sigma v_1 \cdots v_n}^{\sigma' v_1' \cdots v_n'} : \rho_{\widetilde{C}} \simeq \rho_{\widetilde{C}'}$, if $\star = pom$;
     - $\widetilde{C} \simeq \widetilde{C}'$, if $\star = pr$.

     Consequently,
     - $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, if $\star \in \{pom, pr\}$;
     - $\widehat{C} \simeq \widehat{C}'$, if $\star = pr$.
3. As item 2, but the roles of $N$ and $N'$ are reversed.

(⇒) Let $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star f} N'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{(\pi, \pi', \beta_\sigma^{\sigma'}) \mid ((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$ and

- $\beta_\sigma^{\sigma'} : \rho_C \simeq \rho_{C'}$, if $\star \in \{pom, pr\}$;
- $C \simeq C'$, if $\star = pr$.

Let us prove $\mathcal{S} : N \underline{\leftrightarrow}_{\star h} N'$.

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{S}$ since $\beta_\varepsilon^\varepsilon = \emptyset$ and $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. Let $(\pi, \pi', \beta_\sigma^{\sigma'}) \in \mathcal{S}$. Then by definition of $\mathcal{S}$:
   - $\beta_\sigma^{\sigma'} : \rho_C \simeq \rho_{C'}$, if $\star \in \{pom, pr\}$;
   - $C \simeq C'$, if $\star = pr$.
3. Let $(\pi, \pi', \beta_\sigma^{\sigma'}) \in \mathcal{S}$ and $\pi \xrightarrow{v} \tilde{\pi}$. Then by definition of $\mathcal{S}$, $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$ and $(\pi, \sigma) \to (\tilde{\pi}, \sigma v)$. Since $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star f} N'$, $\exists v', \tilde{\pi}' :\ (\pi', \sigma') \to (\tilde{\pi}', \sigma' v')$ and $((\tilde{\pi}, \sigma v), (\tilde{\pi}', \sigma' v')) \in \mathcal{R}$. We have $\pi' \xrightarrow{v'} \tilde{\pi}'$.
   Let us prove:
   - $\beta_{\sigma v}^{\sigma' v'} : \rho_{\widetilde{C}} \simeq \rho_{\widetilde{C}'}$, if $\star \in \{pom, pr\}$;
   - $\widetilde{C} \simeq \widetilde{C}'$, if $\star = pr$.

Let $\beta_{\sigma v}^{\sigma' v'}$ be not isomorphism. If $\sigma = v_1 \cdots v_n$, $\sigma' = v'_1 \cdots v'_n$, let us define $j = \max\{i \mid ((v_i \prec_{\widetilde{C}} v) \wedge (v'_i \not\prec_{\widetilde{C}'} v') \vee ((v_i \not\prec_{\widetilde{C}} v) \wedge (v'_i \prec_{\widetilde{C}'} v'))\}$. If $\pi(i-1) \xrightarrow{\hat{\pi}_i} \tilde{\pi}$ and $\pi'(i-1) \xrightarrow{\hat{\pi}'_i} \tilde{\pi}'$ $(1 \le i \le n)$, then $\rho_{\widehat{C}_j} \not\simeq \rho_{\widehat{C}'_j}$ and $\rho_{\widehat{C}_{j+1}} \simeq \rho_{\widehat{C}'_{j+1}}$. Consequently, the back extension $(\pi(j-1), \sigma(j-1)) \xrightarrow{\hat{\pi}_j} (\tilde{\pi}, \sigma v)$ cannot be imitated by the back extension $(\pi'(j-1), \sigma'(j-1)) \xrightarrow{\hat{\pi}'_j} (\tilde{\pi}', \sigma' v')$ such that $\rho_{\widehat{C}_j} \simeq \rho_{\widehat{C}'_j}$. We have contradiction with $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star f} N'$. Therefore $\beta_{\sigma v}^{\sigma' v'} : \rho_{\widetilde{C}} \simeq \rho_{\widetilde{C}'}$ and $(\tilde{\pi}, \tilde{\pi}', \beta_{\sigma v}^{\sigma' v'}) \in \mathcal{S}$. Hence, we proved for the case $\star = pom$.
Let us prove for the case $\star = pr$. Since $((\tilde{\pi}, \sigma v), (\tilde{\pi}', \sigma' v')) \in \mathcal{R}$ and $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star f} N'$, then the back extension $(\pi_N, \varepsilon) \xrightarrow{\tilde{\pi}} (\tilde{\pi}, \sigma v)$ is imitated by the back extension $(\pi_{N'}, \varepsilon) \xrightarrow{\tilde{\pi}'} (\tilde{\pi}', \sigma' v')$, where $\widetilde{C} \simeq \widetilde{C}'$.

4. As item 3, but the roles of $N$ and $N'$ are reversed. □

**Proposition 3.5** *Let $\star \in \{pom, pr\}$. For nets $N$ and $N'$ $N \underline{\leftrightarrow}_{\star ST} N' \Rightarrow N \underline{\leftrightarrow}_{sb\star f} N'$.*

*Proof.* Let $\mathcal{R} : N \underline{\leftrightarrow}_{\star ST} N'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{((\pi, \sigma), (\pi', \sigma')) \mid (\pi, \sigma) \in Runs(N)$, $(\pi', \sigma') \in Runs(N')$, $|\sigma| = |\sigma'|$, $\forall i\ (0 \le i \le |\sigma|)\ (ST(\pi(i), \sigma(i)), ST(\pi'(i), \sigma'(i)), \beta_{\sigma(i)}^{\sigma'(i)}) \in \mathcal{R}\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}_{sb\star f} N'$.

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{S}$, since $ST(\pi_N, \varepsilon) = (\pi_N, \pi_N)$, $ST(\pi_{N'}, \varepsilon) = (\pi_{N'}, \pi_{N'})$, $\beta_\varepsilon^\varepsilon = \emptyset$ and $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{S}$.
   - (back)

     Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$ and $\prec_{\widehat{C}} = \emptyset$. By definition of $\mathcal{S}$, $\exists (\tilde{\pi}', \tilde{\sigma}')$ such that $|\tilde{\sigma}| = |\tilde{\sigma}'|$, $(\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma')$ and $((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{S}$.

     Let $ST(\pi, \sigma) = (\pi, \bar{\pi})$, $ST(\pi', \sigma') = (\pi', \bar{\pi}')$. Then $((\pi, \bar{\pi}), (\pi', \bar{\pi}'), \beta_\sigma^{\sigma'}) \in \mathcal{R}$. If $\bar{\pi} \xrightarrow{\pi_W} \pi$ and $\bar{\pi}' \xrightarrow{\pi'_W} \pi'$, then $\prec_{C_W} = \emptyset$ and $\beta_\sigma^{\sigma'}|_{T_{C_W}} : \rho_{C_W} \simeq \rho_{C'_W}$. Since $\bar{\pi} \to \tilde{\pi} \xrightarrow{\hat{\pi}} \pi$, $\bar{\pi}' \to \tilde{\pi}' \xrightarrow{\hat{\pi}'} \pi'$, then $\prec_{\widehat{C}'} = \emptyset$ and $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$.
   - (forth)

     Let $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \sigma v_1 \cdots v_n)$. Then $\exists \pi_i\ (1 \le i \le n) : (\pi, \sigma) \to (\pi_1, \sigma v_1) \to \ldots \to (\pi_n, \sigma v_1 \cdots v_n) = (\tilde{\pi}, \sigma v_1 \cdots v_n)$. Let $ST(\pi_i, \sigma v_1 \cdots v_i) = (\pi_i, \bar{\pi}_i)$ $(1 \le i \le n)$, $ST(\pi, \sigma) = (\pi, \bar{\pi})$, $ST(\pi', \sigma') = (\pi', \bar{\pi}')$. Then $((\pi, \bar{\pi}), (\pi', \bar{\pi}'), \beta_\sigma^{\sigma'}) \in \mathcal{R}$. Since $\mathcal{R} : N \underline{\leftrightarrow}_{\star ST} N'$, $\exists v'_i, (\pi'_i, \bar{\pi}'_i) : (\pi', \bar{\pi}') \to (\pi'_1, \bar{\pi}'_1) \to \ldots \to (\pi'_n, \bar{\pi}'_n) = (\tilde{\pi}', \bar{\tilde{\pi}}'_n)$, where $(\pi', \sigma') \to (\pi'_1, \sigma' v'_1) \to \ldots \to (\pi'_n, \sigma' v'_1 \cdots v'_n) = (\tilde{\pi}', \sigma' v'_1 \cdots v'_n)$ and $((\pi_i, \bar{\pi}_i), (\pi'_i, \bar{\pi}'_i), \beta_{\sigma v_1 \cdots v_i}^{\sigma' v'_1 \cdots v'_i}) \in \mathcal{R}$ $(1 \le i \le n)$. Consequently, for some $\hat{\pi}'$ $(\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \sigma' v'_1 \cdots v'_n)$.

     Let us prove $\bar{\pi}'_i = Past(\pi'_i, \sigma' v'_1 \cdots v'_i)$ $(1 \le i \le n)$. Let $\bar{\pi}_{i-1} \xrightarrow{\check{\pi}_i} \pi_i$, $\bar{\pi}'_{i-1} \xrightarrow{\check{\pi}'_i} \pi'_i$ $(1 \le i \le n)$. Since $\mathcal{R} : N \underline{\leftrightarrow}_{\star ST} N'$, we have $\beta_{\sigma v_1 \cdots v_n}^{\sigma' v'_1 \cdots v'_n}|_{T_{\check{C}_i}} : \rho_{\check{C}_i} \simeq \rho_{\check{C}'_i}$ and $\beta_{\sigma v_1 \cdots v_n}^{\sigma' v'_1 \cdots v'_n}(T_{\overline{C}_i}) = T_{\overline{C}'_i}$ $(1 \le i \le n)$. Then the required equality follows easily.
     Let us note that we also have $((\tilde{\pi}, \sigma v_1 \cdots v_n), (\tilde{\pi}', \sigma' v'_1 \cdots v'_n)) \in \mathcal{S}$.

     Let $\bar{\pi} \xrightarrow{\check{\pi}} \tilde{\pi}$, $\bar{\pi}' \xrightarrow{\check{\pi}'} \tilde{\pi}'$. Since $\mathcal{R} : N \underline{\leftrightarrow}_{\star ST} N'$, we have:

     - $\beta_{\sigma v_1 \cdots v_n}^{\sigma' v'_1 \cdots v'_n}|_{T_{\check{C}}} : \rho_{\check{C}} \simeq \rho_{\check{C}'}$, if $\star \in \{pom, pr\}$;
     - $\check{C} \simeq \check{C}'$, if $\star = pr$.

     Since $\bar{\pi} \to \pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, $\bar{\pi}' \to \pi' \xrightarrow{\hat{\pi}'} \tilde{\pi}'$, then

     - $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, if $\star = pom$;
     - $\widehat{C} \simeq \widehat{C}'$, if $\star = pr$.
3. As item 2, but the roles of $N$ and $N'$ are reversed. □

**Theorem 3.1** *Let $\leftrightarrow, \leftrightarrow\!\!\!\!\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \simeq\}$ and $\star, \star\star \in \{i, s, pw, pom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$. For nets $N$ and $N'$ $N \leftrightarrow_\star N' \Rightarrow N \leftrightarrow\!\!\!\!\leftrightarrow_{\star\star} N'$ iff in the graph in Figure 4 there exists a directed path from $\leftrightarrow_\star$ to $\leftrightarrow\!\!\!\!\leftrightarrow_{\star\star}$.*

*Proof.* $(\Leftarrow)$ A consequence of Theorem 1 from [26, 27] and the following substantiations.

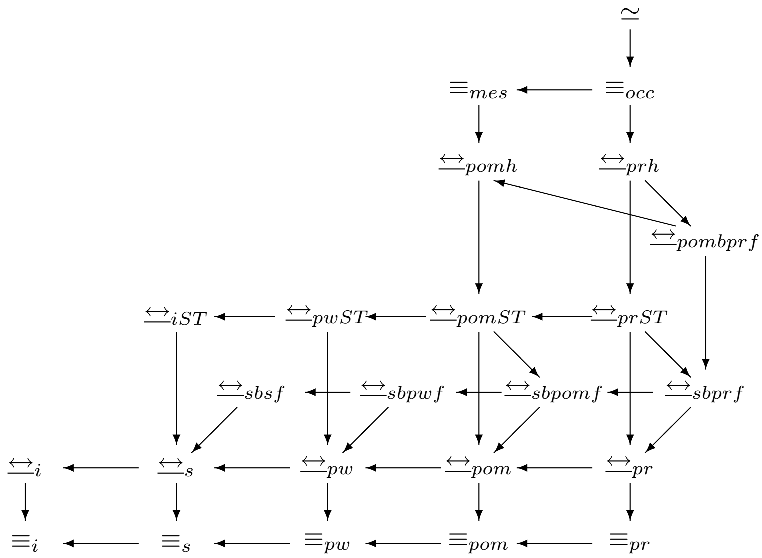


Figure 4: Interrelations of back-forth bisimulation equivalences with basic equivalences

- The implication $\underline{\leftrightarrow}_{sbpwf} \to \underline{\leftrightarrow}_{sbsf}$ is a consequence of the fact that homomorphism is isomorphism of lposets with empty precedence relation.
- The implication $\underline{\leftrightarrow}_{sbpomf} \to \underline{\leftrightarrow}_{sbpwf}$ is a consequence of the fact that isomorphism of lposets is homomorphism.
- The implication $\underline{\leftrightarrow}_{sbprf} \to \underline{\leftrightarrow}_{sbpomf}$ is a consequence of the fact that lposets of isomorphic causal nets are also isomorphic.
- The implications $\underline{\leftrightarrow}_{sb\star f} \to \underline{\leftrightarrow}_{\star}$, $\star \in \{s, pw, pom, pr\}$ is proved with constructing on the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}^{\tau}_{sb\star f}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^{\tau}_{\star}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \sigma, \sigma' \ ((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$.
- The implications $\underline{\leftrightarrow}_{\star ST} \to \underline{\leftrightarrow}_{sb\star f}$, $\star \in \{pom, pr\}$ are consequences of Proposition 3.5.
- The implication $\underline{\leftrightarrow}_{pombprf} \to \underline{\leftrightarrow}_{sbprf}$ follows from the definition of isomorphism of lposets.
- The implication $\underline{\leftrightarrow}_{prh} \to \underline{\leftrightarrow}_{pombprf}$ is a consequence of the facts that by Proposition 3.4 $\underline{\leftrightarrow}_{prh} = \underline{\leftrightarrow}_{prbprf}$ and lposets of isomorphic causal nets are also isomorphic.
- The implication $\underline{\leftrightarrow}_{pombprf} \to \underline{\leftrightarrow}_{pomh}$ is a consequence of the facts that by Proposition 3.4 $\underline{\leftrightarrow}_{pomh} = \underline{\leftrightarrow}_{pombpomf}$ and lposets of isomorphic causal nets are also isomorphic.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 4 is proved by the following examples.

- In Figure 5(a) $N\underline{\leftrightarrow}_{i}N'$, but $N \not\equiv_s N'$, since only in the net $N'$ actions $a$ and $b$ cannot happen concurrently.
- In Figure 5(c) $N\underline{\leftrightarrow}_{iST}N'$, but $N \not\equiv_{pw} N'$, since for the pomset corresponding to the net $N$ there is no even less sequential pomset in $N'$.
- In Figure 5(b) $N\underline{\leftrightarrow}_{pwh}N'$, but $N \not\equiv_{pom} N'$, since only in the net $N'$ action $b$ can depend on action $a$.
- In Figure 5(d) $N \equiv_{mes} N'$, but $N \not\equiv_{pr} N'$, since $N'$ is a causal net which is not isomorphic to the causal net $N$ (because of additional output place).
- In Figure 5(e) $N \equiv_{pr} N'$, but $N\underline{\not\leftrightarrow}_{i}N'$, since only in the net $N'$ action $a$ can happen so that action $b$ can not happen afterwards.
- In Figure 6(a) $N\underline{\leftrightarrow}_{pr}N'$, but $N\underline{\not\leftrightarrow}_{iST}N'$, since only in the net $N'$ action $a$ can start so that no action $b$ can begin working until finishing of $a$.
- In Figure 6(b) $N\underline{\leftrightarrow}_{prST}N'$, but $N\underline{\not\leftrightarrow}_{pomh}N'$, since only in the net $N'$ after action $a$ action $b$ can happen so that action $c$ must depend on $a$.

- In Figure 6(c) $N \underline{\leftrightarrow}_{prh} N'$, but $N \not\equiv_{mes} N'$, since only the MES corresponding to the net $N'$ has two conflict actions $a$.
- In Figure 6(d) $N \equiv_{occ} N'$, but $N \not\simeq N'$, since unfireable transitions of the nets $N$ and $N'$ are labelled by different actions ($a$ and $b$).
- In Figure 5(c) $N \underline{\leftrightarrow}_{sbsf} N'$, but $N \not\equiv_{pw} N'$.
- In Figure 7(a) $N \underline{\leftrightarrow}_{sbpwf} N'$, but $N \not\equiv_{pom} N'$, since only in the net $N'$ action $c$ can depend on actions $a$ and $b$.
- In Figure 7(b) $N \underline{\leftrightarrow}_{sbprf} N'$, but $N \underline{\not\leftrightarrow}_{iST} N'$, since only in the net $N'$ action $a$ can start so that:
  1. until finishing of $a$ the sequence of actions $bc$ cannot happen, and
  2. immediately after finishing of $a$ action $c$ cannot happen.
- In Figure 7(c) $N \underline{\leftrightarrow}_{pombprf} N'$, but $N \underline{\not\leftrightarrow}_{prST} N'$, since only in the net $N'$ the process with action $a$ can start so that it can be extended by process with action $b$ in the only way (i.e. so that extended process be unique).
- In Figure 5(b) $N \underline{\leftrightarrow}_{pwST} N'$, but $N \underline{\not\leftrightarrow}_{sbsf} N'$, since only in the net $N'$ the sequence of actions $ab$ can happen so that $b$ must depend on $a$.
- In Figure 6(a) $N \underline{\leftrightarrow}_{pr} N'$, but $N \underline{\not\leftrightarrow}_{sbsf} N'$, since only in the net $N'$ action $a$ can happen so that action $b$ must depend on $a$. □

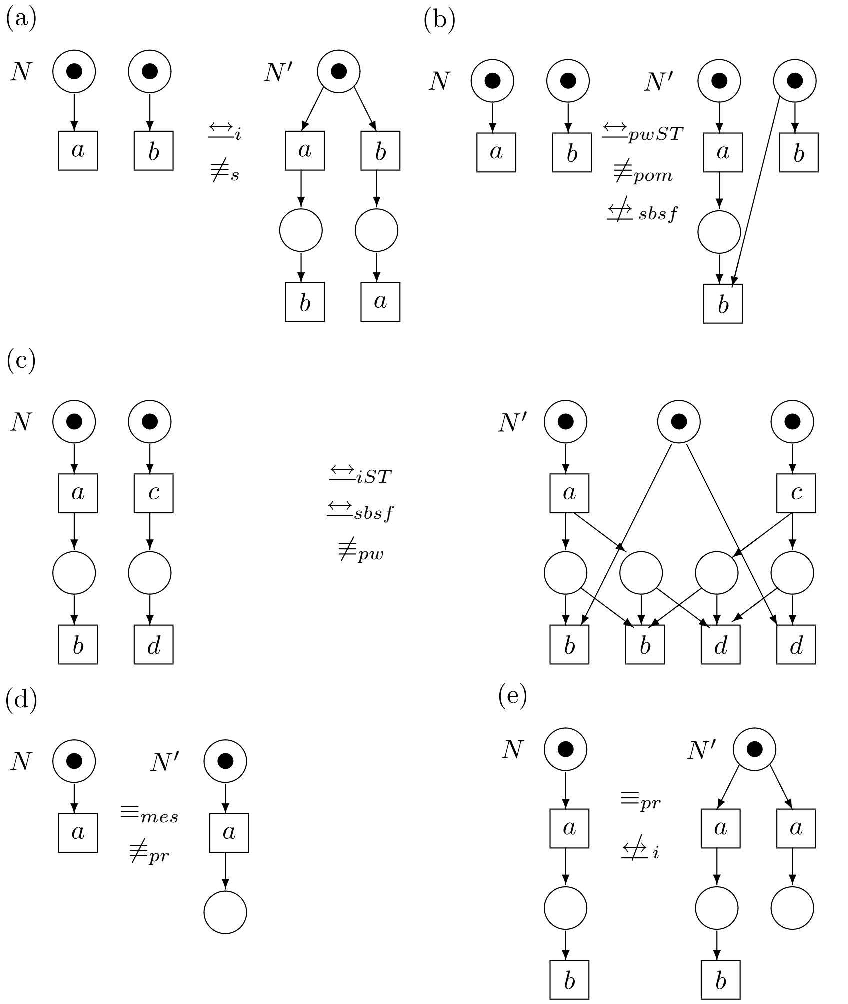


Figure 5: Examples of basic equivalences

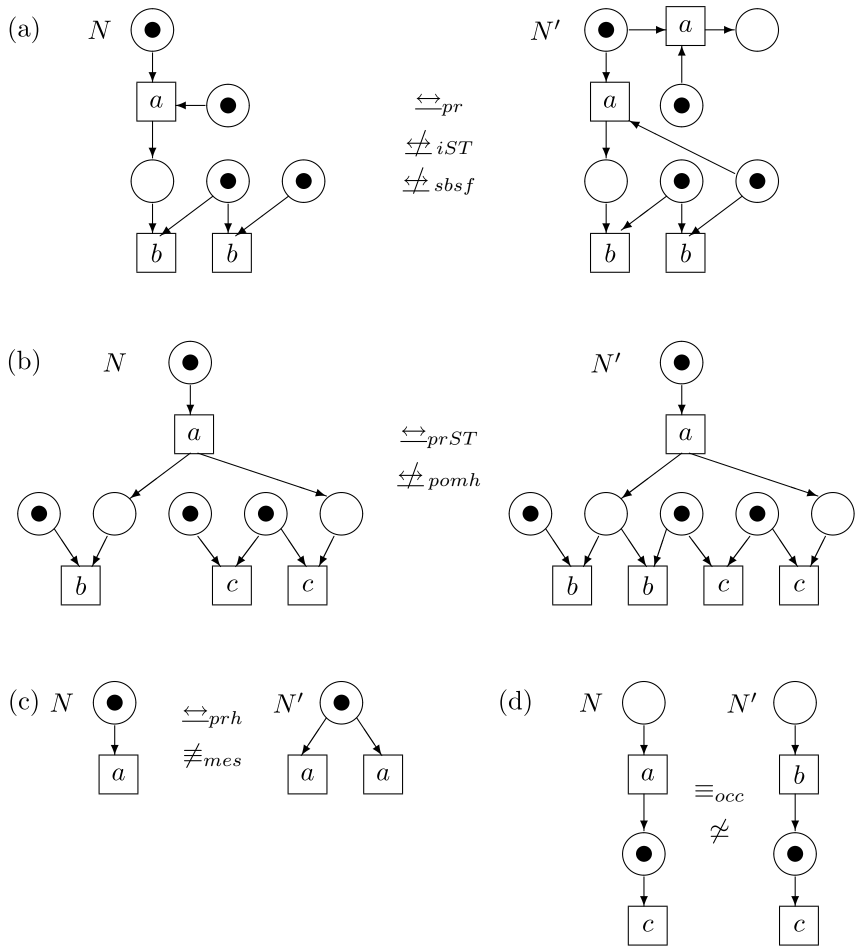


Figure 6: Examples of basic equivalences (continued)

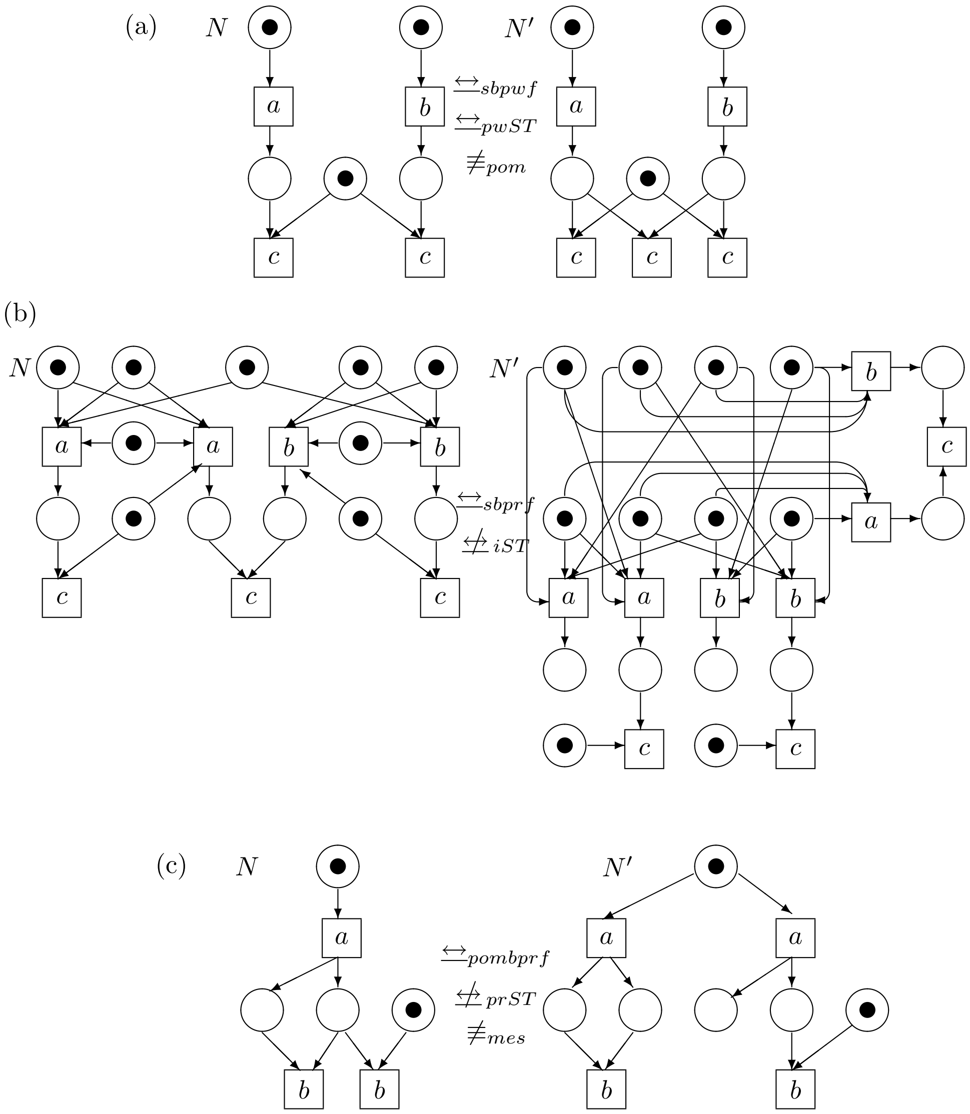


Figure 7: Examples of back-forth bisimulation equivalences

Let us note that example in Figure 7(b) is a modification of a weaker example in Figure 8, where $N\underline{\leftrightarrow}_{sbpomf}N'$, but $N\not\underline{\leftrightarrow}_{iST}N'$. The operation of a net sumation "+" is defined, for example, in [12]. It "multiplies" the input places of nets with preservation of all their outgoing arrows and unates remaining parts of the initial nets.

# 4 Place bisimulation equivalences

In this section place bisimulation equivalences from [4] are compared with back-forth bisimulation and basic equivalences.

## 4.1 Definitions of place bisimulation equivalences

Usual bisimulations may be defined on the basis of markings (instead of processes) by replacing in processes by corresponding markings in the definitions.

**Definition 4.1** *Let $N$ and $N'$ be some nets. A relation $\mathcal{R} \subseteq \mathcal{M}(N) \times \mathcal{M}(N')$ is a* $\star$-bisimulation *between $N$ and $N'$, $\star \in$* {interleaving, step, partial word, pomset, process}*, denoted by $\mathcal{R} : N\underline{\leftrightarrow}_\star N'$, $\star \in \{i, s, pw, pom, pr\}$, if:*

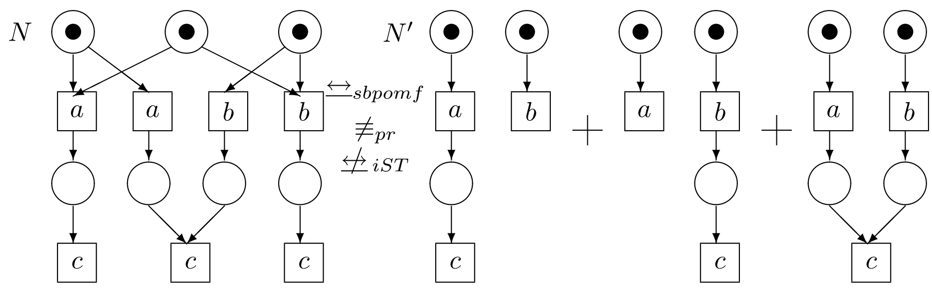


Figure 8: More clear, but weaker example of back-forth bisimulation equivalences

1. $(M_N, M_{N'}) \in \mathcal{R}$.
2. $(M, M') \in \mathcal{R},\ M \stackrel{\hat{\pi}}{\to} \widetilde{M}$,

   (a) $|T_{\widehat{C}}| = 1$, *if* $\star = i$;
   (b) $\prec_{\widehat{C}} = \emptyset$, *if* $\star = s$;

   $\Rightarrow\ \exists \widetilde{M}' :\ M' \stackrel{\hat{\pi}'}{\to} \widetilde{M}',\ (\widetilde{M}, \widetilde{M}') \in \mathcal{R}$ *and*

   (a) $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$, *if* $\star = pw$;
   (b) $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, *if* $\star \in \{i, s, pom\}$;
   (c) $\widehat{C} \simeq \widehat{C}'$, *if* $\star = pr$.
3. *As item 2, but the roles of* $N$ *and* $N'$ *are reversed.*

*Two nets* $N$ *and* $N'$ *are* $\star$-bisimulation equivalent, $\star \in$ {interleaving, step, partial word, pomset, process}, *denoted by* $N \underline{\leftrightarrow}_\star N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_\star N',\ \star \in \{i, s, pw, pom, pr\}$.

*Place bisimulations* are relations between places instead of markings. A relation on markings is obtained with use of *"lifting"* of bisimulation relation on places.

Let us note that in the definitions of bisimulations based on markings any markings may be used, not reachable only. As mentioned [4, 5], this does not change bisimulation equivalences.

**Definition 4.2** *Let for nets* $N$ *and* $N'$ $\mathcal{R} \subseteq P_N \times P_{N'}$ *be a relation between their places. A* lifting *of* $\mathcal{R}$ *is a relation* $\overline{\mathcal{R}} \subseteq \mathcal{M}(P_N) \times \mathcal{M}(P_{N'})$, *defined as follows:*

$$(M, M') \in \overline{\mathcal{R}} \ \Leftrightarrow \ \begin{cases} \exists \{(p_1, p'_1), \ldots, (p_n, p'_n)\} \in \mathcal{M}(\mathcal{R}) : \\ M = \{p_1, \ldots p_n\},\ M' = \{p'_1, \ldots p'_n\} \end{cases}$$

**Definition 4.3** *Let* $N$ *and* $N'$ *be some nets. A relation* $\mathcal{R} \subseteq P_N \times P_{N'}$ *is a* $\star$-place bisimulation *between* $N$ *and* $N'$, $\star \in$ {interleaving, step, partial word, pomset, process}, *denoted by* $\mathcal{R} : N \sim_\star N'$, *if* $\overline{\mathcal{R}} : N \underline{\leftrightarrow}_\star N',\ \star \in \{i, s, pw, pom, pr\}$.

*Two nets* $N$ *and* $N'$ *are* $\star$-place bisimulation equivalent, $\star \in$ {interleaving, step, partial word, pomset, process}, *denoted by* $N \sim_\star N'$, *if* $\exists \mathcal{R} : N \sim_\star N',\ \star \in \{i, s, pw, pom, pr\}$.

*Strict* place bisimulation equivalences are defined using the additional requirement stating that corresponding transitions of nets must be (as well as makings) related by $\overline{\mathcal{R}}$. This relation is defined on transitions as follows.

**Definition 4.4** *Let for nets* $N$ *and* $N'$ $t \in T_N,\ t' \in T_{N'}$. *Then*

$$(t, t') \in \overline{\mathcal{R}} \ \Leftrightarrow \ \begin{cases} ({}^\bullet t, {}^\bullet t') \in \overline{\mathcal{R}} \ \wedge \\ (t^\bullet, t'^\bullet) \in \overline{\mathcal{R}} \ \wedge \\ l_N(t) = l_{N'}(t') \end{cases}$$

**Definition 4.5** *Let* $N$ *and* $N'$ *be some nets. A relation* $\mathcal{R} \subseteq P_N \times P_{N'}$ *is a* strict $\star$-place bisimulation *between* $N$ *and* $N'$, $\star \in$ {interleaving, step, partial word, pomset, process}, *denoted by* $\mathcal{R} : N \approx_\star N',\ \star \in \{i, s, pw, pom, pr\}$, *if:*

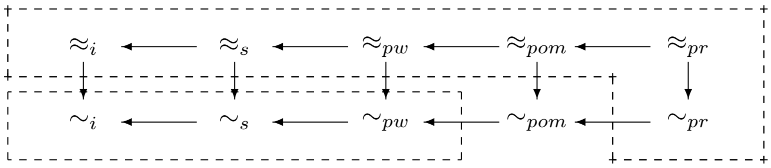


Figure 9: Merging of place bisimulation equivalences

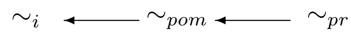


Figure 10: Interrelations of place bisimulation equivalences

1. *$\overline{\mathcal{R}} : N \underline{\leftrightarrow}_\star N'$.*

2. *In the definition of $\star$-bisimulation in item 2 (and in item 3 symmetrically) the new requirement is added: $\forall v \in T_{\widehat{C}}\ (\hat{\varphi}(v), \hat{\varphi}'(\beta(v))) \in \overline{\mathcal{R}}$, where:*

   *(a) $\beta : \rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$, if $\star = pw$;*

   *(b) $\beta : \rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, if $\star \in \{i, s, pom\}$;*

   *(c) $\beta : \widehat{C} \simeq \widehat{C}'$, if $\star = pr$.*

*Two nets $N$ and $N'$ are* strict $\star$-place bisimulation equivalent*, $\star \in${interleaving, step, partial word, pomset, process}*, denoted by $N \approx_\star N'$, if $\exists \mathcal{R} : N \approx_\star N'$, $\star \in \{i, s, pw, pom, pr\}$.*

An important property of place bisimulations is *additivity.* Let for nets $N$ and $N'$ $\mathcal{R} : N \sim_\star N'$. Then $(M_1, M_1') \in \overline{\mathcal{R}}$ and $(M_2, M_2') \in \overline{\mathcal{R}}$ implies $((M_1 + M_2), (M_1' + M_2')) \in \overline{\mathcal{R}}$. In particular, if we add $n$ tokens in each of the places $p \in P_N$ and $p' \in P_{N'}$ s.t. $(p, p') \in \mathcal{R}$, then the nets obtained as a result of such a changing of the initial markings, must be also place bisimulation equivalent.

## 4.2 Interrelations of place bisimulation equivalences

Let us consider interrelations of place bisimulation equivalences.

**Proposition 4.1** *[4, 5] For nets $N$ and $N'$:*

1. *$N \sim_i N' \Leftrightarrow N \sim_{pw} N'$;*

2. *$N \sim_{pr} N' \Leftrightarrow N \approx_i N' \Leftrightarrow N \approx_{pr} N'$.*

In Figure 9 dashed lines embrace coinciding place bisimulation equivalences.
Hence, interrelations of place bisimulation equivalences may be represented by graph in Figure 10.

## 4.3 Interrelations of place bisimulation equivalences with basic equivalences and back-forth bisimulation equivalences

Let us consider interrelations of place bisimulation equivalences with basic equivalences and back-forth bisimulation equivalences.

**Proposition 4.2** *For nets $N$ and $N'$ $N \sim_{pr} N' \Rightarrow N \underline{\leftrightarrow}_{prh} N'$.*

*Proof.* By Proposition 4.1, $\exists \mathcal{R} : N \approx_{pr} N'$. Then $\overline{\mathcal{R}} : N \underline{\leftrightarrow}_{pr} N'$ and transitions of $N$ and $N'$ are related by $\overline{\mathcal{R}}$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{(\pi, \pi', \beta) \mid \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C, \varphi') \in \Pi(N'),\ \beta = id_{T_C},\ \forall r \in P_C\ (\varphi(r), \varphi'(r)) \in \mathcal{R},\ \forall v \in T_C\ (\varphi(v), \varphi'(v)) \in \overline{\mathcal{R}}\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}_{prh} N'$.

1. Obviously, $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{S}$.

2. By definition of $\mathcal{S}$, $(\pi, \pi', \beta) \in \mathcal{S} \Rightarrow \beta : \rho_C \simeq \rho_{C'}$ and $C \simeq C'$;

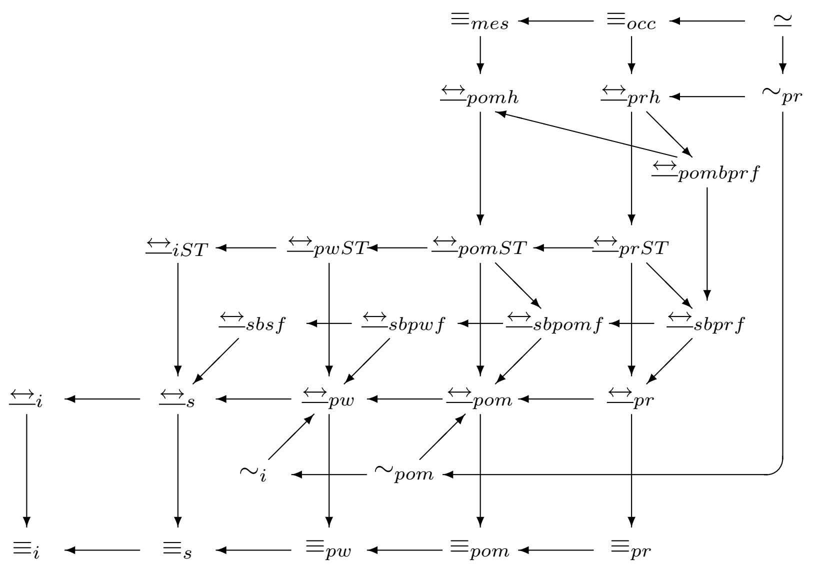


Figure 11: Interrelations of place bisimulation equivalences with basic equivalences and back-forth bisimulation equivalences

3. Let $(\pi, \pi', \beta) \in \mathcal{S},\ \pi = (C, \varphi),\ \pi' = (C, \varphi')$ and $\pi \stackrel{v}{\to} \tilde{\pi},\ \tilde{\pi} = (\widetilde{C}, \tilde{\varphi})$.

   Let us consider the occurrence sequence $\tilde{\varphi}({}^\bullet v) \stackrel{\tilde{\varphi}(v)}{\to} \tilde{\varphi}(v^\bullet)$ in $N$. By definition of $\mathcal{S},\ (\varphi({}^\bullet v), \varphi'({}^\bullet v)) \in \overline{\mathcal{R}}$. Since $\varphi({}^\bullet v) = \tilde{\varphi}({}^\bullet v)$, we have $(\tilde{\varphi}({}^\bullet v), \varphi'({}^\bullet v)) \in \overline{\mathcal{R}}$.

   Since $\mathcal{R} : N \approx_{pr} N'$, we have $\exists u', \widetilde{M}' :\ \varphi'({}^\bullet v) \stackrel{u'}{\to} \widetilde{M}',\ (\tilde{\varphi}(v), u') \in \overline{\mathcal{R}}$ and $(\tilde{\varphi}(v^\bullet), \widetilde{M}') \in \overline{\mathcal{R}}$.

   Let $v^\bullet = \{r_1, \ldots, r_n\},\ \widetilde{M}' = \{p'_1, \ldots, p'_n\},\ \forall i\ (1 \le i \le n)\ (\tilde{\varphi}(r_i), p'_i) \in \mathcal{R}$. Let us define mapping $\tilde{\varphi}'$ as follows:

   (a) $\tilde{\varphi}'|_{(P_C \cup T_C)} = \varphi'$;
   (b) $\tilde{\varphi}'(v) = u'$;
   (c) $\forall i\ (1 \le i \le n)\ \tilde{\varphi}'(r_i) = p'_i$.

   Since by definition of $\tilde{\varphi}'$ we have $u' = \tilde{\varphi}'(v),\ \widetilde{M}' = \tilde{\varphi}'(v^\bullet),\ \varphi'({}^\bullet v) = \tilde{\varphi}'({}^\bullet v)$, then $\tilde{\varphi}'({}^\bullet v) \stackrel{\tilde{\varphi}'(v)}{\to} \tilde{\varphi}'(v^\bullet)$ is an occurrence sequence in $N'$ and $(\tilde{\varphi}(v), \tilde{\varphi}'(v)) \in \overline{\mathcal{R}},\ (\tilde{\varphi}(v^\bullet), \tilde{\varphi}'(v^\bullet)) \in \overline{\mathcal{R}}$.

   Consequently, $\tilde{\varphi}({}^\bullet v) - {}^\bullet\tilde{\varphi}(v) = \tilde{\varphi}(v^\bullet) - \tilde{\varphi}(v)^\bullet$ and $\tilde{\varphi}'({}^\bullet v) - {}^\bullet\tilde{\varphi}'(v) = \tilde{\varphi}'(v^\bullet) - \tilde{\varphi}'(v)^\bullet$. Because of additivity of place bisimulations and since $\tilde{\varphi}$ is an embedding, we have $(\emptyset, \tilde{\varphi}'({}^\bullet v) - {}^\bullet\tilde{\varphi}'(v)) \in \overline{\mathcal{R}}$ and $(\emptyset, \tilde{\varphi}'(v^\bullet) - \tilde{\varphi}'(v)^\bullet) \in \overline{\mathcal{R}}$. Consequently, $\tilde{\varphi}'({}^\bullet v) = {}^\bullet\tilde{\varphi}'(v)$ and $\tilde{\varphi}'(v^\bullet) = \tilde{\varphi}'(v)^\bullet$. Therefore $\tilde{\varphi}'$ is an embedding and $\tilde{\pi}' = (\widetilde{C}, \tilde{\varphi}') \in \Pi(N')$.

   In addition, we have $\pi' \stackrel{v}{\to} \tilde{\pi}'$. Let us define $\tilde{\beta} = id_{T_{\widetilde{C}}}$. Then $\tilde{\beta}|_{T_C} = \beta$. It is also easy to check that $(\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{S}$.

4. As item 3, but the roles of $N$ and $N'$ are reversed. □

**Theorem 4.1** *Let* $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, \sim, \simeq\}$, $\star, \star\star \in \{i, s, pw, pom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$. *For nets* $N$ *and* $N'$ $N \leftrightarrow_\star N' \ \Rightarrow\ N \twoheadleftarrow\!\!\twoheadrightarrow_{\star\star} N'$ *iff in the graph in Figure 11 there exists a directed path from* $\leftrightarrow_\star$ *to* $\twoheadleftarrow\!\!\twoheadrightarrow_{\star\star}$.

*Proof.* $(\Leftarrow)$ A consequence of Theorem 3.1 and the following substantiations.

- The implications $\sim_\star \to \underline{\leftrightarrow}_\star,\ \star \in \{i, pom, pr\}$ are valid by the definitions.
- The implication $\sim_{pr} \to \underline{\leftrightarrow}_{prh}$ is valid by Proposition 4.2.
- The implication $\sim_{pom} \to \sim_i$ is valid by the definitions.

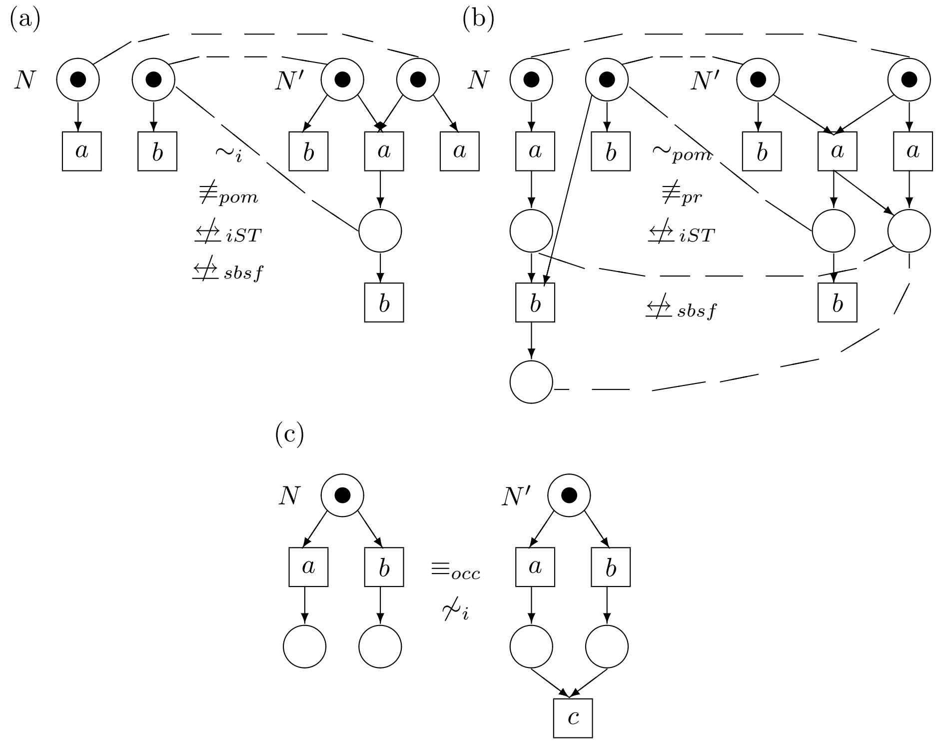


Figure 12: Examples of place bisimulation equivalences

- The implication $\sim_{pr}\to\sim_{pom}$ is valid since lposets of isomorphic nets are also isomorphic.
- The implication $\simeq\to\sim_{pr}$ is obvious.

(⇒) An absence of additional nontrivial arrows in the graph in Figure 11 is proved by Theorem 3.1 and the following examples. Let us note that dashed lines in Figure 12 connect places related by place bisimulation.

- In Figure 12(a) $N \sim_i N'$, but $N \not\equiv_{pom} N'$, since only in the net $N'$ action $b$ can depend on $a$.
- In Figure 12(b) $N \sim_{pom} N'$, but $N \not\equiv_{pr} N'$, since only in the net $N'$ the transition with label $a$ has two input (and two output) places.
- In Figure 12(c) $N \equiv_{occ} N'$, but $N \not\sim_i N'$, since any place bisimulation must relate input places of the nets $N$ and $N'$. But if we add one additional token in each of these places, then only in $N'$ the action $c$ can happen.
- In Figure 12(b) $N \sim_{pom} N'$, but $N \underline{\not\leftrightarrow}_{iST} N'$, since only in the net $N'$ action $a$ can start so that no $b$ can begin working until finishing of $a$.
- In Figure 6(c) $N \sim_{pr} N'$, but $N \not\equiv_{mes} N'$, since only the MES corresponding to the net $N'$ has two conflict actions $a$.
- In Figure 12(b) $N \sim_{pom} N'$, but $N \underline{\not\leftrightarrow}_{sbsf} N'$, since only in the net $N'$ action $a$ can happen so that $b$ must depend on $a$. □

# 5 Preservation of equivalence notions by refinements

In this section we treat the considered equivalence notions for preservation by transition refinements. We use SM-refinement, i.e. refinement by a special subclass of state-machine nets introduced in [6].

**Definition 5.1** *An* SM-net *is a net* $D = \langle P_D, T_D, F_D, l_D, M_D\rangle$ *s.t.:*

1. $\forall t \in T_D\ |{}^\bullet t| = |t^\bullet| = 1$, *i.e. each transition has exactly one input and one output place;*

2. $\exists p_{in},\ p_{out} \in P_D$ *s.t.* $p_{in} \neq p_{out}$ *and* $^\circ D = \{p_{in}\},\ D^\circ = \{p_{out}\}$, *i.e. net D has an unique input and an unique output place.*

3. $M_D = \{p_{in}\}$, *i.e. at the beginning there is an unique token in* $p_{in}$.

**Definition 5.2** *Let* $N = \langle P_N, T_N, F_N, l_N, M_N\rangle$ *be some net,* $a \in l_N(T_N)$ *and* $D = \langle P_D, T_D, F_D, l_D, M_D\rangle$ *be SM-net. An* SM-refinement, *denoted by* $ref(N, a, D)$, *is (up to isomorphism) a net* $\overline{N} = \langle P_{\overline{N}}, T_{\overline{N}}, F_{\overline{N}}, l_{\overline{N}}, M_{\overline{N}}\rangle$, *where:*

- $P_{\overline{N}} = P_N \cup \{\langle p, u\rangle \mid p \in P_D \setminus \{p_{in}, p_{out}\},\ u \in l_N^{-1}(a)\}$;
- $T_{\overline{N}} = (T_N \setminus l_N^{-1}(a)) \cup \{\langle t, u\rangle \mid t \in T_D,\ u \in l_N^{-1}(a)\}$;
- $F_{\overline{N}}(\bar{x}, \bar{y}) = \begin{cases} F_N(\bar{x}, \bar{y}), & \bar{x}, \bar{y} \in P_N \cup (T_N \setminus l_N^{-1}(a)); \\ F_D(x, y), & \bar{x} = \langle x, u\rangle,\ \bar{y} = \langle y, u\rangle,\ u \in l_N^{-1}(a); \\ F_N(\bar{x}, u), & \bar{y} = \langle y, u\rangle,\ \bar{x} \in {}^\bullet u,\ u \in l_N^{-1}(a),\ y \in p_{in}^\bullet; \\ F_N(u, \bar{y}), & \bar{x} = \langle x, u\rangle,\ \bar{y} \in {}^\bullet u,\ u \in l_N^{-1}(a),\ x \in {}^\bullet p_{out}; \\ 0, & otherwise; \end{cases}$
- $l_{\overline{N}}(\bar{u}) = \begin{cases} l_N(\bar{u}), & \bar{u} \in T_N \setminus l_N^{-1}(a); \\ l_D(t), & \bar{u} = \langle t, u\rangle,\ t \in T_D,\ u \in l_N^{-1}(a); \end{cases}$
- $M_{\overline{N}}(p) = \begin{cases} M_N(p), & p \in P_N; \\ 0, & otherwise. \end{cases}$

An equivalence is *preserved by refinements*, if equivalent nets remain equivalent after applying any refinement operator to them accordingly.

The following proposition demonstrates that some considered in the paper equivalence notions are not preserved by SM-refinements.

**Proposition 5.1** *Let* $\star \in \{i, s\}$, $\star\star \in \{i, s, pw, pom, pr, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$, $\star\star\star \in \{i, pom\}$ *Then the equivalences* $\equiv_\star$, $\underline{\leftrightarrow}_{\star\star}$, $\sim_{\star\star\star}$ *are not preserved by SM-refinements.*

*Proof.*

- In Figure 13 $N \underline{\leftrightarrow}_s N'$, but $ref(N, c, D) \not\equiv_i ref(N', c, D)$, since only in $ref(N', c, D)$ the sequence of actions $c_1abc_2$ can happen. Consequently, the equivalences between $\equiv_i$ and $\underline{\leftrightarrow}_s$ are not preserved by SM-refinements.
- In Figure 14 $N \underline{\leftrightarrow}_{sbprf} N'$, but $ref(N, a, D) \underline{\not\leftrightarrow}_i ref(N', a, D)$, since only in the net $ref(N', a, D)$ action $a_1$ can happen so that immediately after it the following holds:

  1. the sequence of actions $bc$ cannot happen, and
  2. the sequence of actions $a_2c$ cannot happen.

  Consequently, the equivalences between $\underline{\leftrightarrow}_i$ and $\underline{\leftrightarrow}_{sbprf}$ are not preserved by SM-refinements.
- In Figure 15 $N \underline{\leftrightarrow}_{pombprf} N'$, but $ref(N, a, D) \underline{\not\leftrightarrow}_{pr} ref(N', a, D)$, since only in the net $ref(N', a, D)$ action $a_1$ can happen so that after it the sequence of actions $a_2b$ can happen which has only one corresponding process (the transition labelled by $b$ connects with transition with label $a_2$ in the only way). Consequently, equivalences between $\underline{\leftrightarrow}_{pr}$ and $\underline{\leftrightarrow}_{pombprf}$ are not preserved by SM-refinements.
- In Figure 16 $N \sim_{pom} N'$, but $ref(N, a, D) \underline{\not\leftrightarrow}_i ref(N', a, D)$, since only in the net $ref(N', a, D)$ after action $a_1$ action $b$ cannot happen. Consequently, equivalences between $\underline{\leftrightarrow}_i$ and $\sim_{pom}$ are not preserved by SM-refinements.

In Figure 17 lines embrace equivalences which are not preserved by SM-refinements due to examples in Figures 13–16. □

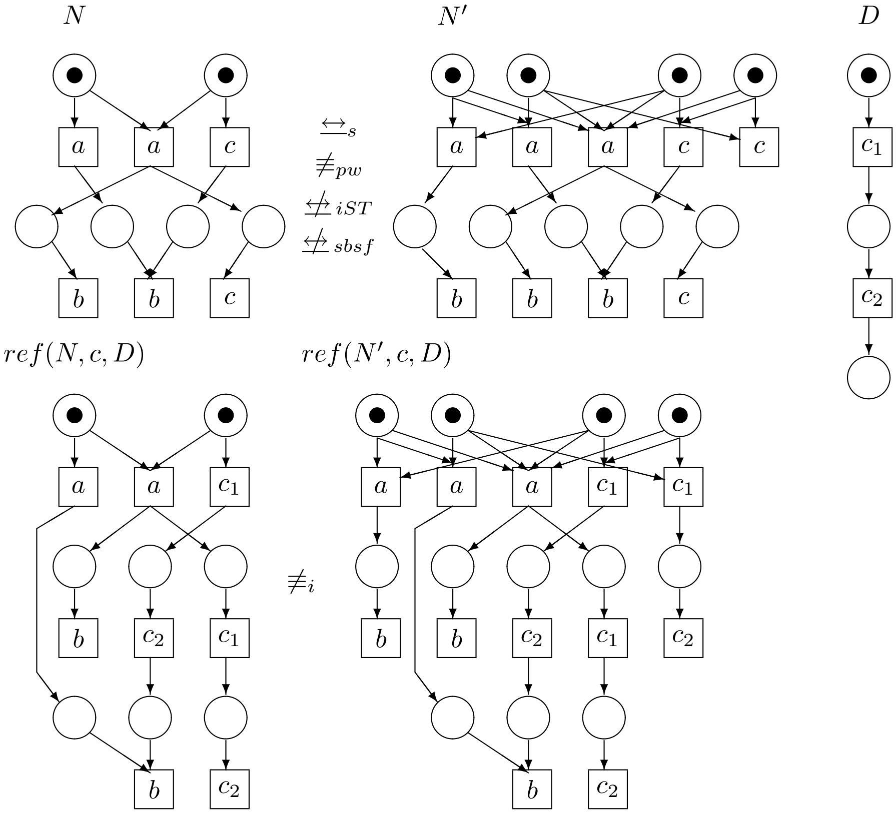


Figure 13: The equivalences between $\equiv_i$ and $\underline{\leftrightarrow}_s$ are not preserved by SM-refinements

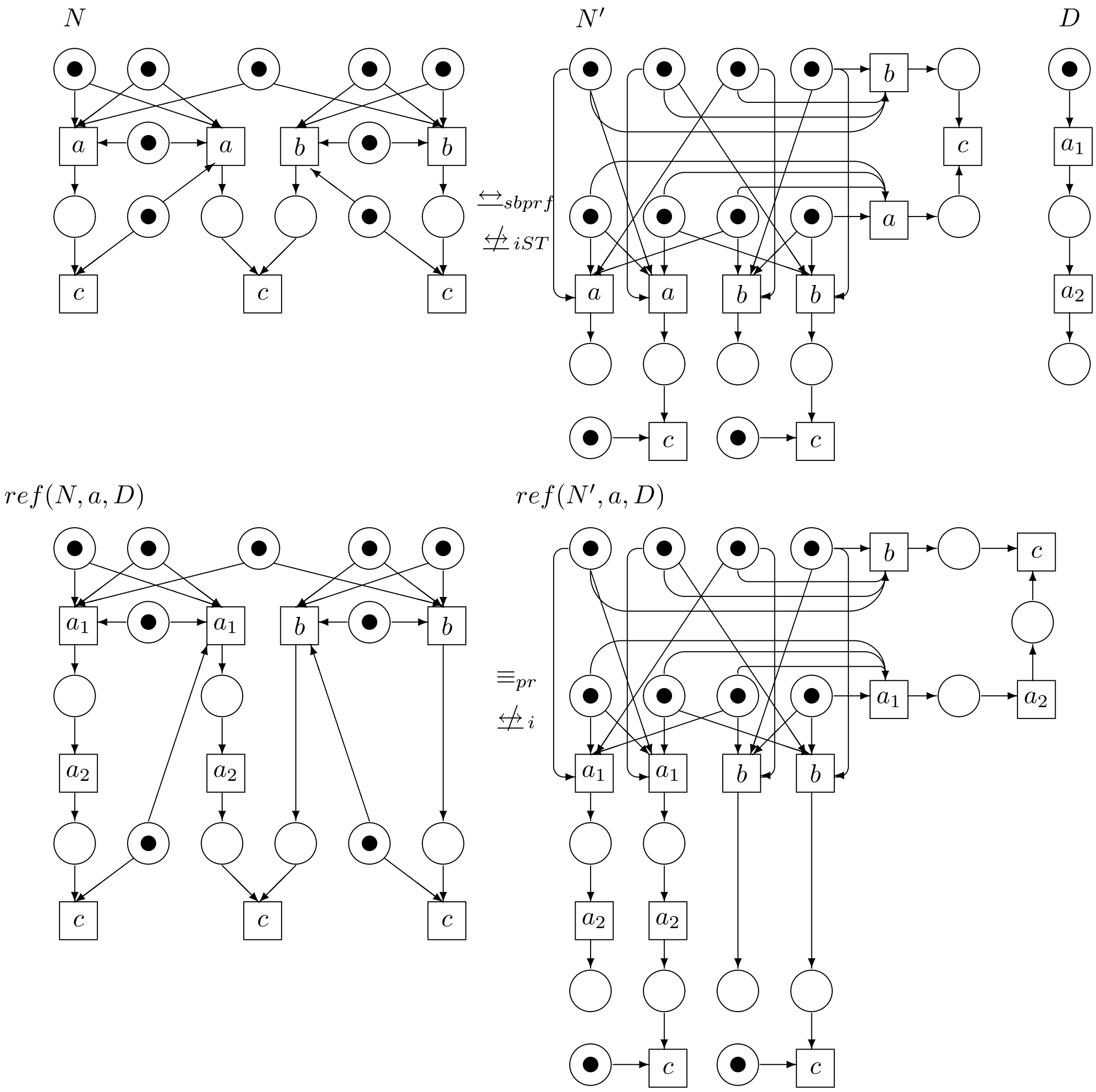


Figure 14: The equivalences between $\underline{\leftrightarrow}_i$ and $\underline{\leftrightarrow}_{sbprf}$ are not preserved by SM-refinements

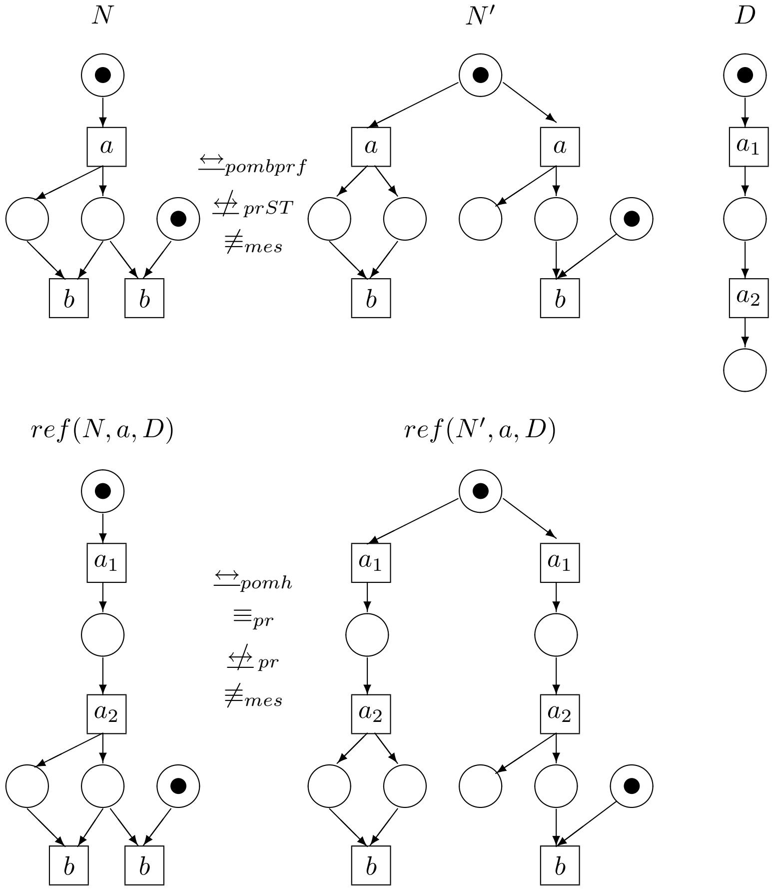


Figure 15: The equivalences between $\underline{\leftrightarrow}_{pr}$ and $\underline{\leftrightarrow}_{pombprf}$ are not preserved by SM-refinements

Let us consider which equivalences are preserved by SM-refinements.

**Proposition 5.2** *[26, 27] Let $\star \in \{pw, pom, pr, mes, occ\}$ and $\star\star \in \{iST, pwST, pomST, prST, pomh, prh\}$. For nets $N,\ N'$ s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net $D$ the following holds:*

1. $N \equiv_\star N' \;\Rightarrow\; ref(N, a, D) \equiv_\star ref(N', a, D)$;
2. $N \underline{\leftrightarrow}_{\star\star} N' \;\Rightarrow\; ref(N, a, D) \underline{\leftrightarrow}_{\star\star} ref(N', a, D)$;
3. $N \simeq N' \;\Rightarrow\; ref(N, a, D) \simeq ref(N', a, D)$.

**Proposition 5.3** *For nets $N,\ N'$ s.t. $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ and SM-net $D$ the following holds: $N \sim_{pr} N' \;\Rightarrow\; ref(N, a, D) \sim_{pr} ref(N', a, D)$.*

*Proof.* Let $\overline{N} = ref(N, a, D)$, $\overline{N}' = ref(N', a, D)$ and $\mathcal{R} : N \sim_{pr} N'$. By Proposition 4.1, $\mathcal{R} : N \approx_i N'$. It is enough to prove $\overline{N} \approx_i \overline{N}'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \mathcal{R} \cup \{(\langle p, u\rangle, \langle p, u'\rangle) \mid p \in P_D \setminus \{p_{in}, p_{out}\},\ (u, u') \in \overline{\mathcal{R}}\}$. Let us prove $\mathcal{S} : \overline{N} \approx_i \overline{N}'$.

1. $(M_{\overline{N}}, M_{\overline{N}'}) \in \mathcal{S}$, since $(M_N, M_{N'}) \in \mathcal{R}$.
2. Let $(M, M') \in \mathcal{S}$ and $M \stackrel{\bar{u}}{\to} \widetilde{M}$. Two cases are possible:
   (a) $\bar{u} = u \in T_N$;
   (b) $\bar{u} = \langle t, u\rangle,\ t \in T_D,\ u \in T_N,\ l_N(u) = a$.

   Let us consider the case (b), since the case (a) is obvious. Let ${}^\bullet t = \{p\},\ t^\bullet = \{q\}$. Then we have:

$$^\bullet\langle t, u\rangle = \begin{cases} {}^\bullet u, & t \in p_{in}^\bullet; \\ \langle p, u\rangle, & \text{otherwise.} \end{cases}$$

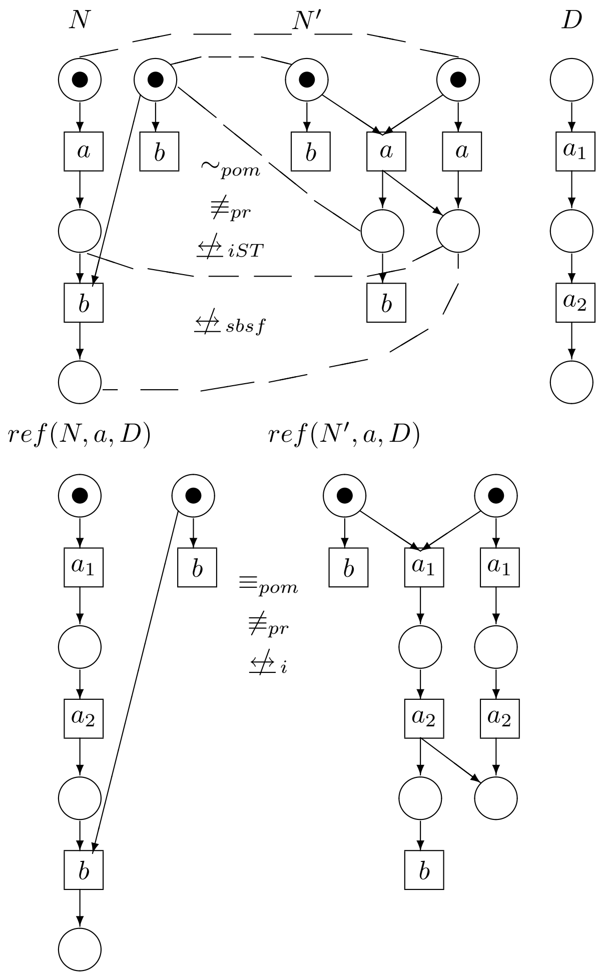


Figure 16: The equivalences between $\underline{\leftrightarrow}_i$ and $\sim_{pom}$ are not preserved by SM-refinements

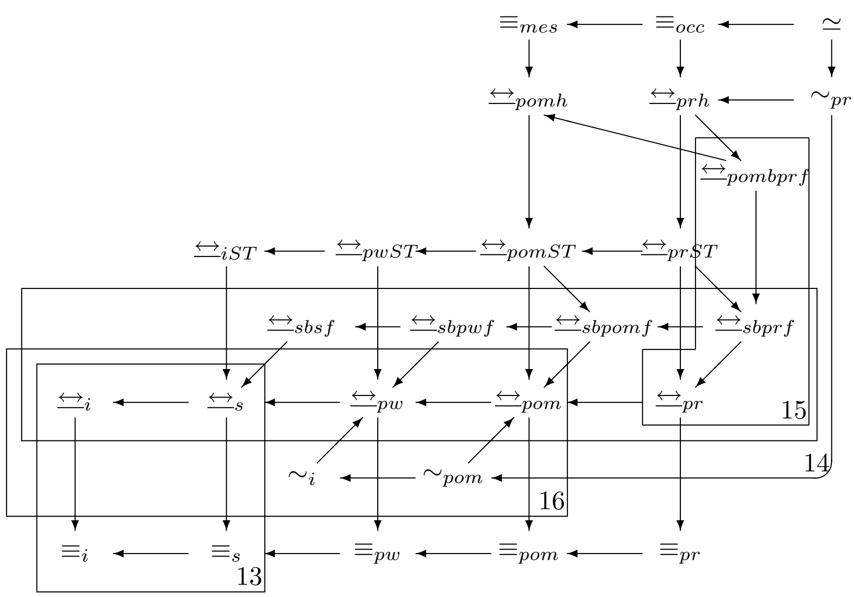


Figure 17: The equivalences which are not preserved by SM-refinements

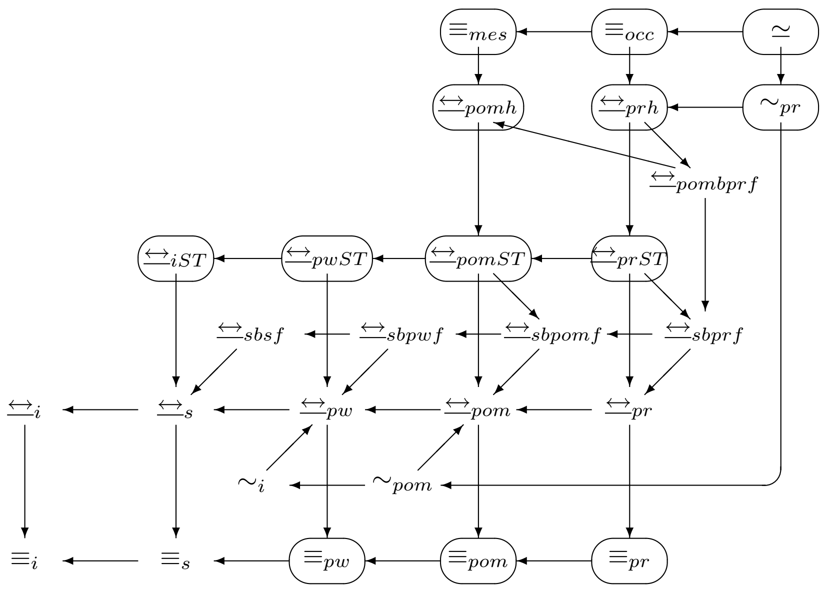


Figure 18: Preservation of the equivalences by SM-refinements

$$\langle t, u\rangle^\bullet = \begin{cases} u^\bullet, & t \in {}^\bullet p_{out}; \\ \langle q, u\rangle, & \text{otherwise.} \end{cases}$$

Four cases are possible:

(a) $t \in p_{in}^\bullet \cap {}^\bullet p_{out}$;

(b) $t \in p_{in}^\bullet \setminus {}^\bullet p_{out}$;

(c) $t \in {}^\bullet p_{out} \setminus p_{in}^\bullet$;

(d) $t \notin p_{in}^\bullet \cup {}^\bullet p_{out}$.

Let us consider the case (d), since the cases (a)–(c) are simpler. We have ${}^\bullet\langle t, u\rangle = \langle p, u\rangle \in M$. Since $(M, M') \in \overline{\mathcal{S}}$, by definition of $\mathcal{S}$ we have: $\exists u' \in T_N : \ (u, u') \in \overline{\mathcal{R}}$ and $(\langle p, u\rangle, \langle p, u'\rangle) \in \mathcal{S}, \ \langle p, u'\rangle \in M'$. Since ${}^\bullet\langle t, u'\rangle = \langle p, u'\rangle$, then $({}^\bullet\langle t, u\rangle, {}^\bullet\langle t, u'\rangle) \in \overline{\mathcal{S}}, \ {}^\bullet\langle t, u'\rangle \in M'$.

Then $\exists \widetilde{M}' : \ M' \stackrel{\langle t,u'\rangle}{\to} \widetilde{M}'$. We have: $l_{\overline{N}}(\langle t, u\rangle) = l_D(t) = l_{\overline{N}'}(\langle t, u'\rangle)$. Since $\langle t, u\rangle^\bullet = \langle q, u\rangle$, by definition of $\mathcal{S}$ we have $(\langle q, u\rangle, \langle q, u'\rangle) \in \mathcal{S}$. Since $\langle t, u'\rangle^\bullet = \langle q, u'\rangle$, then $(\langle t, u\rangle^\bullet, \langle t, u'\rangle^\bullet) \in \overline{\mathcal{S}}$.

Hence, $(\langle t, u\rangle, \langle t, u'\rangle) \in \overline{\mathcal{S}}$ and $(\widetilde{M}, \widetilde{M}') \in \overline{\mathcal{S}}$.

3. As item 2, but the roles of $\overline{N}$ and $\overline{N}'$ are reversed. □

**Theorem 5.1** *Let* $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \sim, \simeq\}$ *and* $\star \in \{i, s, pw, pom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$. *For nets* $N$, $N'$ *s.t.* $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ *and SM-net* $D$ *the following holds:* $N \leftrightarrow_\star N' \ \Rightarrow \ ref(N, a, D) \leftrightarrow_\star ref(N', a, D)$ *iff the equivalence* $\leftrightarrow_\star$ *is in oval in Figure 18.*

*Proof.* By Propositions 5.1–5.3. □

# 6 The equivalences on sequential nets

Let us consider the equivalences on sequential nets, where no two transitions can be fired concurrently.

**Definition 6.1** *A net* $N = \langle P_N, T_N, F_N, l_N, M_N\rangle$ *is* sequential*, if* $\forall M \in Mark(N) \ \neg\exists t, u \in T_N : \ {}^\bullet t + {}^\bullet u \subseteq M$.

**Proposition 6.1** *[26, 27] For sequential nets* $N$ *and* $N'$*:*

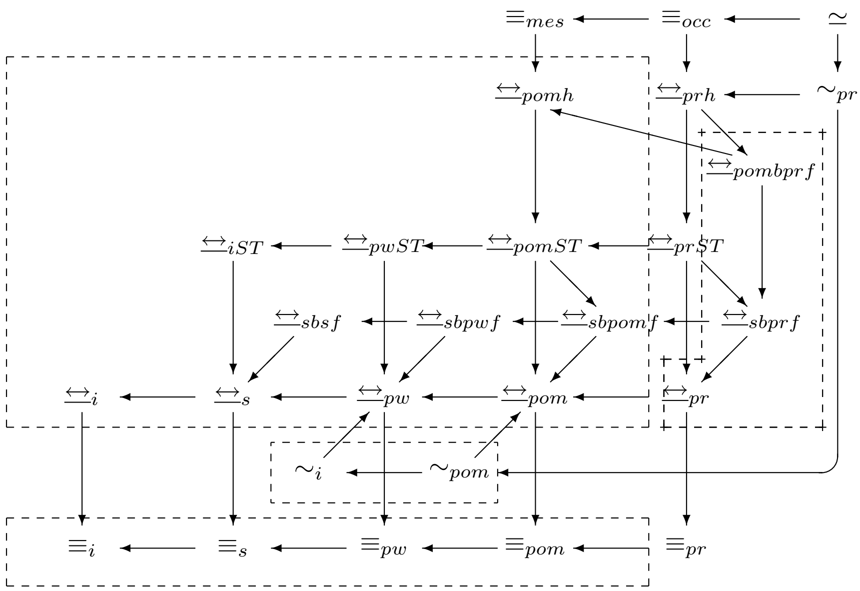


Figure 19: Merging of the equivalences on sequential nets

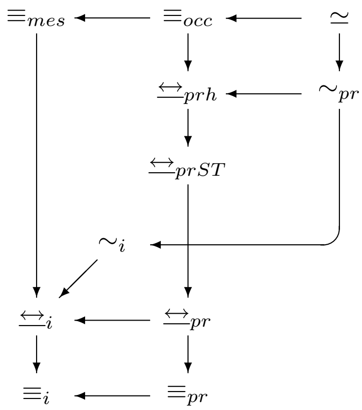


Figure 20: Interrelations of the equivalences on sequential nets

1. $N \equiv_i N' \;\Leftrightarrow\; N \equiv_{pom} N'$;
2. $N \underline{\leftrightarrow}_i N' \;\Leftrightarrow\; N \underline{\leftrightarrow}_{pomh} N'$.

**Proposition 6.2** *For sequential nets $N$ and $N'$ $N \underline{\leftrightarrow}_{pr} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}_{pombprf} N'$.*

*Proof.* ($\Leftarrow$) By Theorem 3.1.

($\Rightarrow$) We have $N \underline{\leftrightarrow}_{pr} N'$. By Proposition 3.3, $N \underline{\leftrightarrow}_{ibprf} N'$. Since for sequential nets lposets of causal nets of their processes are totally ordered, these are isomrphic to the sequences of actions corresponding to the order in which the actions occur. Hence, $N \underline{\leftrightarrow}_{pombprf} N'$. ☐

**Proposition 6.3** *For sequential nets $N$ and $N'$ $N \sim_i N' \;\Leftrightarrow\; N \sim_{pom} N'$.*

*Proof.* ($\Leftarrow$) By Theorem 4.1.

($\Rightarrow$) We have: $\exists \mathcal{R} : N \sim_i N'$. By definition of place bisimulations, $\overline{\mathcal{R}} : N \underline{\leftrightarrow}_i N'$. Since by Proposition 6.1 all the equivalences between $\underline{\leftrightarrow}_i$ and $\underline{\leftrightarrow}_{pomh}$ coincide on sequential nets, we have $\overline{\mathcal{R}} : N \underline{\leftrightarrow}_{pom} N'$. Again by definition of place bisimulations, $\mathcal{R} : N \sim_{pom} N'$. ☐

In Figure 19 dashed lines embrace the equivalences coinciding on sequential nets.

**Theorem 6.1** *Let $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, \sim, \simeq\}$, $\star, \star\star \in \{i, pr, prST, prh, mes, occ\}$. For sequential nets $N$ and $N'$ $N \leftrightarrow_\star N' \;\Rightarrow\; N \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} N'$ iff in the graph in Figure 20 there exists a directed path from $\leftrightarrow_\star$ to $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.*

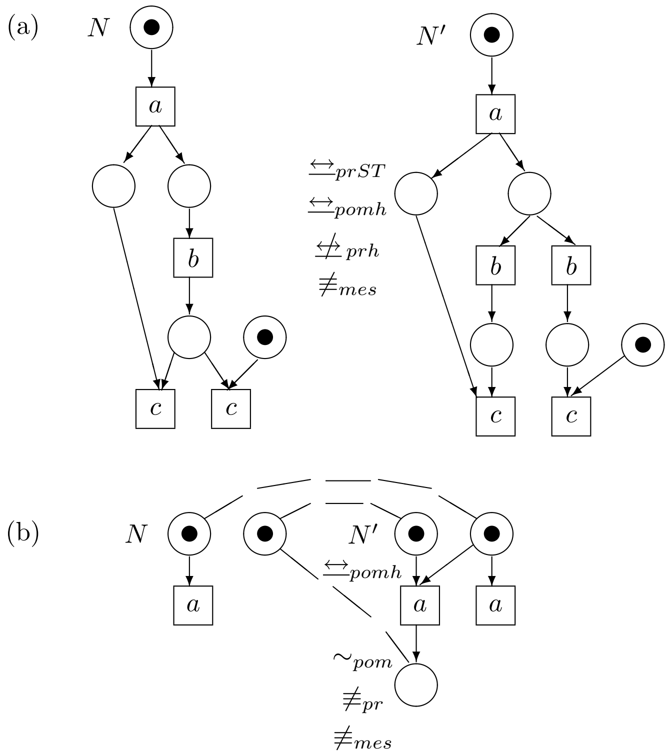


Figure 21: Examples of the equivalences on sequential nets

*Proof.* (⇐) By Theorem 4.1.

(⇒) An absence of additional nontrivial arrows in the graph in Figure 20 is proved by the following examples on sequential nets.

- In Figure 5(d) $N \equiv_{mes} N'$, but $N \not\equiv_{pr} N'$.
- In Figure 5(e) $N \equiv_{pr} N'$, but $N \not\underline{\leftrightarrow}_i N'$.
- In Figure 7(c) $N \underline{\leftrightarrow}_{pr} N'$, but $N \not\underline{\leftrightarrow}_{prST} N'$, since only in the net $N'$ the process with action $a$ can start so that it can be extended by action $b$ in the only way (i.e. so that extended process to be unique).
- In Figure 21(a) $N \underline{\leftrightarrow}_{prST} N'$, but $N \not\underline{\leftrightarrow}_{prh} N'$, since only in the net $N'$ there is process with actions $a$ and $b$ s.t. it can be extended by process with action $c$ in the only way. (i.e. so that connection of causal net with action $c$ and $a$-containing subnet of causal net with actions $a$ and $b$ be unique).
- In Figure 6(c) $N \underline{\leftrightarrow}_{prh} N'$, but $N \not\equiv_{mes} N'$.
- In Figure 6(d) $N \equiv_{occ} N'$, but $N \not\simeq N'$.
- In Figure 21(b) $N \sim_i N'$, but $N \not\equiv_{pr} N'$, since only in the net $N'$ the transition with label $a$ has two input places.
- In Figure 12(c) $N \equiv_{occ} N'$, but $N \not\sim_i N'$.
- In Figure 6(c) $N \sim_{pr} N'$, but $N \not\equiv_{mes} N'$. □

# 7 Conclusion

In this paper, we supplemented by new ones and examined a group of back-forth and place bisimulation equivalences. We compared them with basic ones on the whole class of Petri nets as well as on their subclass of sequential nets. All the considered equivalences were treated for preservation by SM-refinements to establish which of them may be used for top-down design of concurrent systems.

Further research may consist in the investigation of analogues of the considered equivalences on Petri nets with $\tau$-actions ($\tau$-equivalences). $\tau$-actions are used to abstract of internal, invisible to external observer behaviour of systems to be modelled. In the framework of Petri nets with $\tau$-actions interrelations of equivalences are drastically changed.

For example, let us try to define $\tau$-equivalences in process semantics. We abstract of $\tau$-labelled transitions of C-nets by removing these transitions and multiplication of their input and output places. Then all causal dependencies of transitions with visible labels are preserved, and process $\tau$-equivalences will imply corresponding pomset ones. But while such an abstraction the quantity of input and output places of some transitions with visible labels may be changed. The consequence is, in particular, that history preserving $\tau$-bisimulation equivalences do not imply usual $\tau$-bisimulation ones.

Therefore, it is no sense to introduce process $\tau$-equivalences. By similar reasons, it is no sense to define strict place $\tau$-bisimulation equivalences. In addition, multi event structure $\tau$-equivalence does not imply even usual $\tau$-bisimulation relations, but only $\tau$-trace ones.

In the literature, a number of $\tau$-equivalences were defined.

Some basic $\tau$-equivalences were considered on Petri nets and event structures in [6, 22, 28]. It was shown the independence of ST- and history preserving $\tau$-bisimulation equivalences.

In [14] interleaving back interleaving forth $\tau$-bisimulation equivalence was defined on transition systems. Its coincidence with interleaving branching $\tau$-bisimulation equivalence was proved. Similar result was obtained in [23], where pomset back pomset forth history preserving $\tau$-bisimulation equivalence was introduced, and its merging with new notion of branching pomset history preserving $\tau$-bisimulation equivalence was established.

In [5, 3] interleaving place $\tau$-bisimulation and interleaving place $\tau p$-bisimulation equivalences were introduced.

In future, we plan to define $\tau$-analogues of all the equivalence relations considered in this paper and exam them following the same pattern.

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# References

[1] Autant C., Belmesk Z., Schnoebelen Ph. *Strong bisimularity on nets revisited. Research Report* **847-I**, LIFIA-IMAG, Grenoble, France, 28 p., March 1991.

[2] Autant C., Belmesk Z., Schnoebelen Ph. *Strong bisimularity on nets revisited. Extended abstract. LNCS* **506**, p. 295–312, June 1991.

[3] Autant C., Pfister W., Schnoebelen Ph. *Place bisimulations for the reduction of labelled Petri nets with silent moves. Proceedings of International Conference on Computing and Information*, 1994.

[4] Autant C., Schnoebelen Ph. *Place bisimulations in Petri nets. LNCS* **616**, p. 45–61, June 1992.

[5] Autant C. *Petri nets for the semantics and the implementation of parallel processes. Ph.D. thesis*, Institut National Polytechnique de Grenoble, May 1993 (in French).

[6] Best E., Devillers R., Kiehn A., Pomello L. *Concurrent bisimulations in Petri nets. Acta Informatica* **28**, p. 231–264, 1991.

[7] Boudol G., Castellani I. *On the semantics of concurrency: partial orders and transition systems. LNCS* **249**, p. 123–137, 1987.

[8] Cherief F. *Back and forth bisimulations on prime event structures. LNCS* **605**, p. 843–858, June 1992.

[9] Cherief F. *Contributions à la sémantique du parallélisme: bisimulations pour le raffinement et le vrai parallélisme. Ph.D. thesis*, Institut National Politechnique de Grenoble, France, October 1992 (in French).

[10] Cherief F. *Investigations of back and forth bisimulations on prime event structures. Computers and Artificial Intelligence* **11**(5), p. 481–496, 1992.

[11] Grabowski J. *On partial languages. Fundamenta Informaticae* **IV(2)**, p. 428–498, 1981.

[12] van Glabbeek R.J., Vaandrager F.W. *Petri net models for algebraic theories of concurrency. LNCS* **259**, p. 224–242, 1987.

[13] Hoare C.A.R. *Communicating sequential processes, on the construction of programs.* (McKeag R.M., Macnaghten A.M., eds.) Cambridge University Press, p. 229–254, 1980.

[14] De Nicola R., Montanari U., Vaandrager F.W. *Back and forth bisimulations. LNCS* **458**, p. 152–165, 1990.

[15] Nielsen M., Plotkin G., Winskel G. *Petri nets, event structures and domains. TCS* **13**, p. 85–108, 1981.

[16] Nielsen M., Thiagarajan P.S. *Degrees of non-determinizm and concurrency: A Petri net view. LNCS* **181**, p. 89–117, December 1984.

[17] Olderog E.-R. *Nets, terms and formulas, three views of concurrent processes and their relationship. Habilitationsschrift*, Christian-Albrechts Univ., Kiel, July 1989.

[18] Olderog E.-R. *Strong bisimularity on nets: a new concept for comparsing net semantics. LNCS* **354**, p. 549–573, 1989.

[19] Olderog E.-R. *Nets, terms and formulas. Cambridge Tracts in Theoretical Computer Science* **23**, Cambridge University Press, 1991.

[20] Park D.M.R. *Concurrency and automata on infinite sequences. LNCS* **104**, p. 167–183, March 1981.

[21] Petri C.A. *Kommunikation mit Automaten. Ph.D. thesis*, Universität Bonn, Schriften des Instituts für Instrumentelle Mathematik, 1962 (in German).

[22] Pomello L. *Some equivalence notions for concurrent systems. An overview. LNCS* **222**, p. 381–400, 1986.

[23] Pinchinat S. *Bisimulations for the semantics of reactive systems. Ph.D. thesis*, Institut National Politechnique de Grenoble, January 1993 (in French).

[24] Pratt V.R. *On the composition of processes. Proceedings of* $9^{th}$ *POPL*, p. 213–223, 1982.

[25] Rabinovitch A., Trakhtenbrot B.A. *Behaviour structures and nets. Fundamenta Informaticae* **XI**, p. 357–404, 1988.

[26] Tarasyuk I.V. *Equivalence notions for design of concurrent systems using Petri nets. Hildesheimer Informatik-Bericht* **4/96**, part 1, 19 p., Institut für Informatik, Universität Hildesheim, Hildesheim, Germany, January 1996.

[27] Tarasyuk I.V. *Petri net equivalences for design of concurrent systems.* Proceedings of Workshop "Concurrency, Specification and Programming - 96", *Informatik-Bericht* **69**, p. 190–204, Institut für Informatik, Humboldt-Universität zu Berlin, Berlin, Germany, 1996.

[28] Vogler W. *Bisimulation and action refinement. LNCS* **480**, p. 309–321, 1991.