An Investigation of τ-Equivalences *

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Abstract

The paper is devoted to the investigation of behavioural equivalences of concurrent systems modelled by Petri nets with silent transitions. Basic τ-equivalences and back-forth τ-bisimulation equivalences known from the literature are supplemented by new ones, giving rise to complete set of equivalence notions in interleaving / true concurrency and linear / branching time semantics. Their interrelations are examined for the general class of nets as well as for their subclasses of nets without silent transitions and sequential nets (nets without concurrent transitions). In addition, the preservation of all the equivalence notions by refinements (allowing one to consider the systems to be modelled on a lower abstraction levels) is investigated.

Key words & phrases: Petri nets with silent transitions, sequential nets, basic τ-equivalences, back-forth τ-bisimulation equivalences, refinement.

1 Introduction

The notion of equivalence is central in any theory of systems. It allows to compare systems taking into account particular aspects of their behaviour.

Petri nets [14] became a popular formal model for design of concurrent and distributed systems. One of the main advantages of Petri nets is their ability for structural characterization of three fundamental features of concurrent computations: causality, nondeterminism and concurrency.

Silent transitions are transitions labelled by special *silent* action τ which represents an internal activity of a system to be modelled and it is invisible for external observer. It is well-known that Petri nets with silent transitions are more powerful than usual ones.

Equivalences which abstract of silent actions are called *τ-equivalences* (these are labelled by the symbol τ to distinguish them of relations not abstracting of silent actions). In recent years, a wide range of semantic equivalences was proposed in concurrency theory. Some of them were either directly defined or transferred from other formal models to Petri nets. The following basic notions of τ-equivalences are known from the literature.

- *τ-trace equivalences* (they respect only protocols of behaviour of systems): interleaving ($\equiv_i^\tau$) [15], step ($\equiv_s^\tau$) [15], partial word ($\equiv_{pw}^\tau$) [21] and pomset ($\equiv_{pom}^\tau$) [16].
- *Usual τ-bisimulation equivalences* (they respect branching structure of behaviour of systems): interleaving ($\underline{\leftrightarrow}_i^\tau$) [12], step ($\underline{\leftrightarrow}_s^\tau$) [15], partial word ($\underline{\leftrightarrow}_{pw}^\tau$) [20] and pomset ($\underline{\leftrightarrow}_{pom}^\tau$) [16].
- *ST-τ-bisimulation equivalences* (they respect the duration or maximality of events in behaviour of systems): interleaving ($\underline{\leftrightarrow}_{iST}^\tau$) [20], partial word ($\underline{\leftrightarrow}_{pwST}^\tau$) [20] and pomset ($\underline{\leftrightarrow}_{pomST}^\tau$) [20].
- *History preserving τ-bisimulation equivalences* (they respect the "past" or "history" of behaviour of systems): pomset ($\underline{\leftrightarrow}_{pomh}^\tau$) [8, 9].
- *History preserving ST-τ-bisimulation equivalences* (they respect the "history" and the duration or maximality of events in behaviour of systems): pomset ($\underline{\leftrightarrow}_{pomhST}^\tau$) [8, 9].

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- *Usual branching τ-bisimulation equivalences* (they respect branching structure of behaviour of systems taking a special care for silent actions): interleaving ($\underline{\leftrightarrow}^{\tau}_{ibr}$) [10, 11].
- *History preserving branching τ-bisimulation equivalences* (they respect "history" and branching structure of behaviour of systems taking a special care for silent actions): pomset ($\underline{\leftrightarrow}^{\tau}_{pomhbr}$) [8].
- *Isomorphism* ($\simeq$) (i.e. coincidence of systems up to renaming of their components).

Back-forth bisimulation equivalences are based on the idea that bisimulation relation do not only require systems to simulate each other behaviour in the forward direction (as usually) but also when going back in history. They are closely connected with equivalences of logics with past modalities.

These equivalence notions were initially introduced in [13]. In the framework of transition systems without silent actions interleaving back-forth bisimulation equivalence was defined. On transition systems with silent actions it was shown that back-forth variant ($\underline{\leftrightarrow}^{\tau}_{ibif}$) of interleaving τ-bisimulation equivalence coincide with $\underline{\leftrightarrow}^{\tau}_{ibr}$.

In [5, 6, 7] the new variants of step, partial word and pomset back-forth bisimulation equivalences were defined in the framework of prime event structures without silent actions.

In [17] the new idea of differentiating the kinds of back and forth simulations appeared (following this idea, it is possible, for example, to define step back pomset forth bisimulation equivalence). The set of all possible back-forth equivalence notions was proposed in interleaving, step, partial word and pomset semantics for prime event structures without silent actions. The new notion of τ-equivalence was proposed for event structures with silent actions: pomset back pomset forth ($\underline{\leftrightarrow}^{\tau}_{pombpomf}$) τ-bisimulation equivalence. Its coincidence with $\underline{\leftrightarrow}^{\tau}_{pomhbr}$ was proved.

To choose most appropriate behavioural viewpoint on systems to be modelled, it is very important to have a complete set of equivalence notions in all semantics and understand their interrelations. This branch of research is usually called *comparative concurrency semantics*. To clarify the nature of equivalences and evaluate how they respect internal activity and concurrency in systems to be modelled, it is actual to consider also correlation of these notions on nets without silent transitions and concurrency-free (sequential) ones. Treating equivalences for preservation by refinements allows one to decide which of them may be used for top-down design.

Working in the framework of Petri nets with silent transitions, in this paper we continue the research of [19] and extend the set of basic notions of τ-equivalences by *τ-conflict preserving ones* (completely respect conflicts in nets): we introduce multi event structure equivalence ($\equiv^{\tau}_{mes}$).

We complete back-forth τ-equivalences from [17] by 6 new notions: interleaving back step forth ($\underline{\leftrightarrow}^{\tau}_{ibsf}$), interleaving back partial word forth ($\underline{\leftrightarrow}^{\tau}_{ibpwf}$), interleaving back pomset forth ($\underline{\leftrightarrow}^{\tau}_{ibpomf}$), step back step forth ($\underline{\leftrightarrow}^{\tau}_{sbsf}$), step back partial word forth ($\underline{\leftrightarrow}^{\tau}_{sbpwf}$) and step back pomset forth ($\underline{\leftrightarrow}^{\tau}_{sbpomf}$) bisimulation equivalences. We compare all back-forth τ-equivalences with the set of basic behavioural relations.

We also all the considered τ-equivalences with equivalences which do not abstract of silent actions.

In addition, we investigate the interrelations of all the τ-equivalence notions on nets without silent transitions and sequential nets. We prove that on nets without silent transitions τ-equivalences coincide with equivalence notions which do not abstract of silent actions. We demonstrate that on sequential nets interleaving and pomset τ-equivalences are merged, and back-forth τ-equivalences coincide with forth τ-equivalence relations.

In [4], SM-refinement operator for Petri nets was proposed, which "replaces" their transitions by SM-nets, a special subclass of state machine nets. We treat all the considered τ-equivalence notions for preservation by SM-refinements.

The rest of the paper is organized as follows. Basic definitions are introduced in Section 2. In Section 3 we propose basic τ-equivalences and investigate their interrelations. In Section 4 back-forth τ-bisimulation equivalences are defined and compared with basic τ-equivalence notions. All the considered τ-equivalences are compared with ones which do not abstract of silent actions in Section 5. In Section 6 we establish which τ-equivalence relations are preserved by SM-refinements. Section 7 is devoted to comparison of the τ-equivalences on nets without silent transitions and sequential nets. Concluding Section 8 contains a review of the main results obtained and some directions of further research.

2 Basic definitions

In this section we give some basic definitions used further.

2.1 Multisets

Definition 2.1 *Let X be some set. A finite* multiset *M over X is a mapping $M : X \to \mathbf{N}$ ($\mathbf{N}$ is a set of natural numbers) s.t. $|\{x \in X \mid M(x) > 0\}| < \infty$.*

$\mathcal{M}(X)$ denotes the *set of all finite multisets* over X. When $\forall x \in X\ M(x) \leq 1$, M is a proper set. *Cardinality* of multiset M is defined in such a way: $|M| = \sum_{x \in X} M(x)$. We write $x \in M$ if $M(x) > 0$ and $M \subseteq M'$, if $\forall x \in X\ M(x) \leq M'(x)$. We define $(M + M')(x) = M(x) + M'(x)$ and $(M - M')(x) = \max\{0, M(x) - M'(x)\}$.

2.2 Labelled nets

Let $Act = \{a, b, \ldots\}$ be a set of *action names* or *labels*. The symbol $\tau \notin Act$ denotes a special *silent* action which represents internal activity of system to be modelled and invisible to external observer. We denote $Act_\tau = Act \cup \{\tau\}$.

Definition 2.2 *A* labelled net *is a quadruple* $N = \langle P_N, T_N, F_N, l_N \rangle$*, where:*

- $P_N = \{p, q, \ldots\}$ *is a set of* places*;*
- $T_N = \{t, u, \ldots\}$ *is a set of* transitions*;*
- $F_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbf{N}$ *is the* flow relation *with weights (*$\mathbf{N}$ *denotes a set of natural numbers);*
- $l_N : T_N \to Act_\tau$ *is a* labelling *of transitions with action names.*

Given labelled nets $N = \langle P_N, T_N, F_N, l_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', denoted by $\beta : N \simeq N'$, if:

1. β is a bijection s.t. $\beta(P_N) = P_{N'}$ and $\beta(T_N) = T_{N'}$;
2. $\forall p \in P_N\ \forall t \in T_N\ F_N(p, t) = F_{N'}(\beta(p), \beta(t))$ and $F_N(t, p) = F_{N'}(\beta(t), \beta(p))$;
3. $\forall t \in T_N\ l_N(t) = l_{N'}(\beta(t))$.

Labelled nets N and N' are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Given a labelled net N and some transition $t \in T_N$, the *precondition* and *postcondition* of t, denoted by ${}^\bullet t$ and $t^\bullet$ respectively, are the multisets defined in such a way: $({}^\bullet t)(p) = F_N(p, t)$ and $(t^\bullet)(p) = F_N(t, p)$. Analogous definitions are introduced for places: $({}^\bullet p)(t) = F_N(t, p)$ and $(p^\bullet)(t) = F_N(p, t)$. Let ${}^\circ N = \{p \in P_N \mid {}^\bullet p = \emptyset\}$ is a set of *initial (input)* places of N and $N^\circ = \{p \in P_N \mid p^\bullet = \emptyset\}$ is a set of *final (output)* places of N.

A labelled net N is *acyclic*, if there exist no transitions $t_0, \ldots, t_n \in T_N$ s.t. $t_{i-1}^\bullet \cap {}^\bullet t_i \neq \emptyset$ $(1 \leq i \leq n)$ and $t_0 = t_n$. A labelled net N is *ordinary* if $\forall p \in P_N\ {}^\bullet p$ and $p^\bullet$ are proper sets (not multisets).

Let $N = \langle P_N, T_N, F_N, l_N \rangle$ be acyclic ordinary labelled net and $x, y \in P_N \cup T_N$. Let us introduce the following notions.

- $x \prec_N y \ \Leftrightarrow\ x F_N^+ y$, where F_N^+ is a transitive closure of F_N (*strict causal dependence* relation);
- $x \preceq_N y \ \Leftrightarrow\ (x \prec_N y) \vee (x = y)$ (a relation of *causal dependence*);
- $x \#_N y \ \Leftrightarrow\ \exists t, u \in T_N\ (t \neq u,\ {}^\bullet t \cap {}^\bullet u \neq \emptyset,\ t \preceq_N x,\ u \preceq_N y)$ (a relation of *conflict*);
- $\downarrow_N x = \{y \in P_N \cup T_N \mid y \prec_N x\}$ (the set of *strict predecessors* of x).

A set $T \subseteq T_N$ is *left-closed* in N, if $\forall t \in T\ (\downarrow_N t) \cap T_N \subseteq T$.

2.3 Marked nets

A *marking* of a labelled net N is a multiset $M \in \mathcal{M}(P_N)$.

Definition 2.3 *A* marked net (net) *is a tuple* $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$*, where* $\langle P_N, T_N, F_N, l_N \rangle$ *is a labelled net and* $M_N \in \mathcal{M}(P_N)$ *is the* initial *marking.*

Given nets $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'}, M_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', denoted by $\beta : N \simeq N'$, if:

1. $\beta : \langle P_N, T_N, F_N, l_N \rangle \simeq \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$;
2. $\forall p \in P_N\ M_N(p) = M_{N'}(\beta(p))$.

Nets N and N' are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Let $M \in \mathcal{M}(P_N)$ be a marking of a net N. A transition $t \in T_N$ is *fireable* in M, if ${}^\bullet t \subseteq M$. If t is fireable in M, its firing yields a new marking $\widetilde{M} = M - {}^\bullet t + t^\bullet$, denoted by $M \xrightarrow{t} \widetilde{M}$. A marking M of a net N is *reachable*, if $M = M_N$ or there exists a reachable marking $\widehat{M}$ of N s.t. $\widehat{M} \xrightarrow{t} M$ for some $t \in T_N$. $Mark(N)$ denotes a *set of all reachable* markings of a net N.

2.4 Partially ordered sets

Definition 2.4 *A* labelled partially ordered set (lposet) *is a triple* $\rho = \langle X, \prec, l\rangle$*, where:*

- $X = \{x, y, \ldots\}$ *is some set;*
- $\prec \subseteq X \times X$ *is a strict partial order (irreflexive transitive relation) over* X*;*
- $l : X \to Act_\tau$ *is a* labelling *function.*

Let $\rho = \langle X, \prec, l\rangle$ be lposet and $Y \subseteq X$. A *restriction* of ρ to the set Y is defined as follows: $\rho|_Y = \langle Y, \prec \cap (Y \times Y), l|_Y\rangle$.

Let $\rho = \langle X, \prec, l\rangle$ and $\rho' = \langle X', \prec', l'\rangle$ be lposets.

A mapping $\beta : X \to X'$ is a *label-preserving bijection* between ρ and ρ', denoted by $\beta : \rho \asymp \rho'$, if:

1. β is a bijection;
2. $\forall x \in X\ l(x) = l'(\beta(x))$.

We write $\rho \asymp \rho'$, if $\exists \beta : \rho \asymp \rho'$.

A mapping $\beta : X \to X'$ is a *homomorphism* between ρ and ρ', denoted by $\beta : \rho \sqsubseteq \rho'$, if:

1. $\beta : \rho \asymp \rho'$;
2. $\forall x, y \in X\ x \prec y \ \Rightarrow\ \beta(x) \prec' \beta(y)$.

We write $\rho \sqsubseteq \rho'$, if $\exists \beta : \rho \sqsubseteq \rho'$.

A mapping $\beta : X \to X'$ is an *isomorphism* between ρ and ρ', denoted by $\beta : \rho \simeq \rho'$, if $\beta : \rho \sqsubseteq \rho'$ and $\beta^{-1} : \rho' \sqsubseteq \rho$. Lposets ρ and ρ' are *isomorphic*, denoted by $\rho \simeq \rho'$, if $\exists \beta : \rho \simeq \rho'$.

Definition 2.5 Partially ordered multiset (pomset) *is an isomorphism class of lposets.*

2.5 Event structures

Definition 2.6 *A* labelled event structure (LES) *is a quadruple* $\xi = \langle X, \prec, \#, l\rangle$*, where:*

- $X = \{x, y, \ldots\}$ *is a set of* events*;*
- $\prec \subseteq X \times X$ *is a strict partial order, a* causal dependence *relation, which satisfies to the principle of* finite causes*:* $\forall x \in X\ |\downarrow x| < \infty$*;*
- $\# \subseteq X \times X$ *is an irreflexive symmetrical* conflict relation*, which satisfies to the principle of* conflict heredity*:* $\forall x, y, z \in X\ x\#y \prec z \ \Rightarrow\ x\#z$*;*
- $l : X \to Act_\tau$ *is a* labelling *function.*

Let $\xi = \langle X, \prec, \#, l\rangle$ be LES and $Y \subseteq X$. A *restriction* of ξ to the set Y is defined as follows: $\xi|_Y = \langle Y, \prec \cap (Y \times Y), \# \cap (Y \times Y), l|_Y\rangle$.

Let $\xi = \langle X, \prec, \#, l\rangle$ and $\xi' = \langle X', \prec', \#', l'\rangle$ be LES's. A mapping $\beta : X \to X'$ is an *isomorphism* between ξ and ξ', denoted by $\beta : \xi \simeq \xi'$, if:

1. β is a bijection;
2. $\forall x \in X\ l(x) = l'(\beta(x))$;
3. $\forall x, y \in X\ x \prec y \ \Leftrightarrow\ \beta(x) \prec' \beta(y)$;
4. $\forall x, y \in X\ x\#y \ \Leftrightarrow\ \beta(x)\#'\beta(y)$.

LES's ξ and ξ' are *isomorphic*, denoted by $\xi \simeq \xi'$, if $\exists \beta : \xi \simeq \xi'$.

Definition 2.7 *A* multi-event structure (MES) *is an isomorphism class of LES's.*

2.6 C-processes

Definition 2.8 *A* causal net *is an acyclic ordinary labelled net $C = \langle P_C, T_C, F_C, l_C \rangle$, s.t.:*

1. *$\forall r \in P_C$ $|{}^\bullet r| \le 1$ and $|r^\bullet| \le 1$, i.e. places are unbranched;*
2. *$\forall x \in P_C \cup T_C$ $|\downarrow_C x| < \infty$, i.e. a set of causes is finite.*

Let us note that on the basis of any causal net $C = \langle P_C, T_C, F_C, l_C \rangle$ one can define lposet $\rho_C = \langle T_C, \prec_N \cap (T_C \times T_C), l_C \rangle$.

The fundamental property of causal nets is [2]: if C is a causal net, then there exists a sequence of transition fireings ${}^\circ C = L_0 \stackrel{v_1}{\to} \cdots \stackrel{v_n}{\to} L_n = C^\circ$ s.t. $L_i \subseteq P_C$ $(0 \le i \le n)$, $P_C = \cup_{i=0}^n L_i$ and $T_C = \{v_1, \ldots, v_n\}$. Such a sequence is called a *full execution* of C.

Definition 2.9 *Given a net N and a causal net C. A mapping $\varphi : P_C \cup T_C \to P_N \cup T_N$ is an* embedding *of C into N, denoted by $\varphi : C \to N$, if:*

1. *$\varphi(P_C) \in \mathcal{M}(P_N)$ and $\varphi(T_C) \in \mathcal{M}(T_N)$, i.e. sorts are preserved;*
2. *$\forall v \in T_C$ ${}^\bullet\varphi(v) = \varphi({}^\bullet v)$ and $\varphi(v)^\bullet = \varphi(v^\bullet)$, i.e. flow relation is respected;*
3. *$\forall v \in T_C$ $l_C(v) = l_N(\varphi(v))$, i.e. labelling is preserved.*

Since embeddings respect the flow relation, if ${}^\circ C \stackrel{v_1}{\to} \cdots \stackrel{v_n}{\to} C^\circ$ is a full execution of C, then $M = \varphi({}^\circ C) \stackrel{\varphi(v_1)}{\longrightarrow} \cdots \stackrel{\varphi(v_n)}{\longrightarrow} \varphi(C^\circ) = \widetilde{M}$ is a sequence of transition fireings in N.

Definition 2.10 *A* fireable in marking M C-process (process) *of a net N is a pair $\pi = (C, \varphi)$, where C is a causal net and $\varphi : C \to N$ is an embedding s.t. $M = \varphi({}^\circ C)$. A fireable in M_N process is a* process *of N.*

We write $\Pi(N, M)$ for a *set of all fireable* in marking M processes of a net N and $\Pi(N)$ for the *set of all processes* of a net N. The *initial* process of a net N is $\pi_N = (C_N, \varphi_N) \in \Pi(N)$, s.t. $T_{C_N} = \emptyset$. If $\pi \in \Pi(N, M)$, then firing of this process transforms a marking M into $\widetilde{M} = M - \varphi({}^\circ C) + \varphi(C^\circ) = \varphi(C^\circ)$, denoted by $M \stackrel{\pi}{\to} \widetilde{M}$.

Let $\pi = (C, \varphi)$, $\tilde{\pi} = (\widetilde{C}, \tilde{\varphi}) \in \Pi(N)$, $\hat{\pi} = (\widehat{C}, \hat{\varphi}) \in \Pi(N, \varphi(C^\circ))$. A process π is a *prefix* of a process $\tilde{\pi}$, if $T_C \subseteq T_{\widetilde{C}}$ is a left-closed set in $\widetilde{C}$. A process $\hat{\pi}$ is a *suffix* of a process $\tilde{\pi}$, if $T_{\widehat{C}} = T_{\widetilde{C}} \setminus T_C$. In such a case a process $\tilde{\pi}$ is an *extension* of π *by process* $\hat{\pi}$, and $\hat{\pi}$ is an *extending* process for π, denoted by $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$. We write $\pi \to \tilde{\pi}$, if $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$ for some $\hat{\pi}$.

A process $\tilde{\pi}$ is an extension of a process π *by one transition*, denoted by $\pi \stackrel{v}{\to} \tilde{\pi}$ or $\pi \stackrel{a}{\to} \tilde{\pi}$, if $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$, $T_{\widehat{C}} = \{v\}$ and $l_{\widehat{C}}(v) = a$.

A process $\tilde{\pi}$ is an extension of a process π *by sequence of transitions*, denoted by $\pi \stackrel{\sigma}{\to} \tilde{\pi}$ or $\pi \stackrel{\omega}{\to} \tilde{\pi}$, if $\exists \pi_i \in \Pi(N)$ $(1 \le i \le n)$ $\pi \stackrel{v_1}{\to} \pi_1 \stackrel{v_2}{\to} \ldots \stackrel{v_n}{\to} \pi_n = \tilde{\pi}$, $\sigma = v_1 \cdots v_n$ and $l_{\widehat{C}}(\sigma) = \omega$.

A process $\tilde{\pi}$ is an extension of a process π *by multiset of transitions*, denoted by $\pi \stackrel{V}{\to} \tilde{\pi}$ or $\pi \stackrel{A}{\to} \tilde{\pi}$, if $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$, $\prec_{\widehat{C}} = \emptyset$, $T_{\widehat{C}} = V$ and $l_{\widehat{C}}(V) = A$.

2.7 O-processes

Definition 2.11 *An* occurrence net *is an acyclic ordinary labelled net $O = \langle P_O, T_O, F_O, l_O \rangle$, s.t.:*

1. *$\forall r \in P_O$ $|{}^\bullet r| \le 1$, i.e. there are no backwards conflicts;*
2. *$\forall x \in P_O \cup T_O$ $\neg(x \#_O x)$, i.e. conflict relation is irreflexive;*
3. *$\forall x \in P_O \cup T_O$ $|\downarrow_O x| < \infty$, i.e. set of causes is finite.*

Let $O = \langle P_O, T_O, F_O, l_O \rangle$ be occurrence net and $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ be some net. A mapping $\psi : P_O \cup T_O \to P_N \cup T_N$ is an *embedding* O into N, notation $\psi : O \to N$, if:

1. $\psi(P_O) \in \mathcal{M}(P_N)$ and $\psi(T_O) \in \mathcal{M}(T_N)$. i.e. sorts are preserved;
2. $\forall v \in T_O$ $l_O(v) = l_N(\psi(v))$, i.e. labelling is preserved;
3. $\forall v \in T_O$ ${}^\bullet\psi(v) = \psi({}^\bullet v)$ and $\psi(v)^\bullet = \psi(v^\bullet)$, i.e. flow relation is respected;
4. $\forall v, w \in T_O$ $({}^\bullet v = {}^\bullet w) \wedge (\psi(v) = \psi(w)) \Rightarrow$ $v = w$, i.e. there are no "superfluous" conflicts.

Definition 2.12 *An* O-process *of a net N is a pair $\varpi = (O, \psi)$, where O is an occurrence net and $\psi : O \to N$ is an embedding s.t. $M_N = \psi({}^\circ O)$.*

We write $\wp(N)$ for a set of *all O-processes* of a net N. The *initial* O-process of a net N coincides with its initial C-process, i.e. $\varpi_N = \pi_N$.

Let $\varpi = (O, \psi)$, $\tilde{\varpi} = (\widetilde{O}, \tilde{\psi}) \in \wp(N)$, $O = \langle P_O, T_O, F_O, l_O \rangle$, $\widetilde{O} = \langle P_{\widetilde{O}}, T_{\widetilde{O}}, F_{\widetilde{O}}, l_{\widetilde{O}} \rangle$. A process ϖ is a *prefix* of a process $\tilde{\varpi}$, if $T_O \subseteq T_{\widetilde{O}}$ is a left-closed set in $\widetilde{O}$. In such a case O-process $\tilde{\varpi}$ is an *extension* of ϖ, and $\hat{\varpi}$ is an *extending* O-process for ϖ, denoted by $\varpi \to \tilde{\varpi}$.

An O-process ϖ of a net N is *maximal*, if it cannot be extended, i.e. $\forall \varpi = (O, \psi)$ s.t. $\varpi \to \tilde{\varpi} : T_{\widetilde{O}} \setminus T_O = \emptyset$. A set of all maximal O-processes of a net N consists of the unique (up to isomorphism) O-process $\varpi_{max} = (O_{max}, \psi_{max})$. In such a case an isomorphism class of occurrence net O_{max} is an *unfolding* of a net N, notation $\mathcal{U}(N)$.

Let us note that on the basis of any occurrence net O one can define LES $\xi_O = \langle T_O, \prec_O \cap (T_O \times T_O), \#_O \cap (T_O \times T_O), l_O \rangle$. Then on the basis of unfolding $\mathcal{U}(N)$ of a net N one can define MES $\mathcal{E}(N) = \xi_{\mathcal{U}(N)}$ which is an isomorphism class of LES ξ_O for $O \in \mathcal{U}(N)$.

3 Basic τ-equivalences

In this section we propose basic τ-equivalences: trace, bisimulation and conflict preserving.

3.1 τ-trace equivalences

We denote the empty string by the symbol ε.

Let $\sigma = a_1 \cdots a_n \in Act_\tau^*$. We define $vis(\sigma)$ as follows (in the following definition $a \in Act_\tau$).

1. $vis(\varepsilon) = \varepsilon$;
2. $vis(\sigma a) = \begin{cases} vis(\sigma)a, & a \neq \tau; \\ vis(\sigma), & a = \tau. \end{cases}$

Definition 3.1 *A* visible interleaving trace *of a net N is a sequence $vis(a_1 \cdots a_n) \in Act^*$ s.t. $\pi_N \stackrel{a_1}{\to} \pi_1 \stackrel{a_2}{\to} \dots \stackrel{a_n}{\to} \pi_n$, where π_N is the initial process of a net N and $\pi_i \in \Pi(N)$ $(1 \leq i \leq n)$. We denote a set of* all visible interleaving traces *of a net N by $VisIntTraces(N)$. Two nets N and N' are* interleaving τ-trace equivalent, *denoted by $N \equiv_i^\tau N'$, if $VisIntTraces(N) = VisIntTraces(N')$.*

Let $\Sigma = A_1 \cdots A_n \in (\mathcal{M}(Act_\tau))^*$. We define $vis(\Sigma)$ as follows (in the following definition $A \in \mathcal{M}(Act_\tau)$).

1. $vis(\varepsilon) = \varepsilon$;
2. $vis(\Sigma A) = \begin{cases} vis(\Sigma)(A \cap Act), & A \cap Act \neq \emptyset; \\ vis(\Sigma), & \text{otherwise}. \end{cases}$

Definition 3.2 *A* visible step trace *of a net N is a sequence $vis(A_1 \cdots A_n) \in (\mathcal{M}(Act))^*$ s.t. $\pi_N \stackrel{A_1}{\to} \pi_1 \stackrel{A_2}{\to} \dots \stackrel{A_n}{\to} \pi_n$, where π_N is the initial process of a net N and $\pi_i \in \Pi(N)$ $(1 \leq i \leq n)$. We denote a set of* all visible step traces *of a net N by $VisStepTraces(N)$. Two nets N and N' are* step τ-trace equivalent, *denoted by $N \equiv_s^\tau N'$, if $VisStepTraces(N) = VisStepTraces(N')$.*

Let $\rho = \langle X, \prec, l \rangle$ is lposet s.t. $l : X \to Act_\tau$. We denote $vis(X) = \{x \in X \mid l(x) \in Act\}$ and $vis(\rho) = \rho|_{vis(X)}$.

Definition 3.3 *A* visible pomset trace *of a net N is a pomset $vis(\rho)$, an isomorphism class of lposet $vis(\rho_C)$ for $\pi = (C, \varphi) \in \Pi(N)$. We denote a set of* all visible pomsets *of a net N by $VisPomsets(N)$. Two nets N and N' are* partial word τ-trace equivalent, *denoted by $N \equiv_{pw}^\tau N'$, if $VisPomsets(N) \sqsubseteq VisPomsets(N')$ and $VisPomsets(N') \sqsubseteq VisPomsets(N)$.*

Definition 3.4 *Two nets N and N' are* pomset τ-trace equivalent, *denoted by $N \equiv_{pom}^\tau N'$, if $VisPomsets(N) = VisPomsets(N')$.*

3.2 τ-bisimulation equivalences

Let $C = \langle P_C, T_C, F_C, l_C \rangle$ be C-net. We denote $vis(T_C) = \{v \in T_C \mid l_C(v) \in Act\}$ and $vis(\prec_C) = \prec_C \cap (vis(T_C) \times vis(T_C))$.

3.2.1 Usual τ-bisimulation equivalences

Definition 3.5 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ is a* $\star$-τ-bisimulation *between N and N',* $\star \in$ {interleaving, step, partial word, pomset}*, denoted by $\mathcal{R} : N \underline{\leftrightarrow}_\star^\tau N'$, $\star \in \{i, s, pw, pom\}$, if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.
2. $(\pi, \pi') \in \mathcal{R}$, $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$,
   (a) $|vis(T_{\widehat{C}})| = 1$, *if* $\star = i$;
   (b) $vis(\prec_{\widehat{C}}) = \emptyset$, *if* $\star = s$;

   $\Rightarrow \exists \tilde{\pi}' : \pi' \xrightarrow{\hat{\pi}'} \tilde{\pi}'$, $(\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$ *and*

   (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, *if* $\star = pw$;
   (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, *if* $\star \in \{i, s, pom\}$.
3. *As item 2, but the roles of N and N' are reversed.*

Two nets N and N' are $\star$-τ-bisimulation equivalent*,* $\star \in$ {interleaving, step, partial word, pomset}*, denoted by $N \underline{\leftrightarrow}_\star^\tau N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_\star^\tau N'$, $\star \in \{i, s, pw, pom\}$.*

3.2.2 ST-τ-bisimulation equivalences

Definition 3.6 ST-τ-process *of a net N is a pair (π_E, π_P) s.t. $\pi_E, \pi_P \in \Pi(N)$, $\pi_P \xrightarrow{\pi_W} \pi_E$ and $\forall v, w \in T_{C_E}$ $(v \prec_{C_E} w) \vee (l_{C_E}(v) = \tau) \Rightarrow v \in T_{C_P}$.*

In such a case π_E is a process which began working, π_P corresponds to the completed part of π_E, and π_W — to the still working part. Obviously, $\prec_{C_W} = \emptyset$. We denote a set of *all ST-τ-processes* of a net N by $ST^\tau - \Pi(N)$. (π_N, π_N) is the *initial ST-τ-process* of a net N. Let (π_E, π_P), $(\tilde{\pi}_E, \tilde{\pi}_P) \in ST^\tau - \Pi(N)$. We write $(\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \to \tilde{\pi}_E$ and $\pi_P \to \tilde{\pi}_P$.

Definition 3.7 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$ is a* $\star$-ST-τ-bisimulation *between N and N',* $\star \in$ {interleaving, partial word, pomset}*, denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{\star ST}^\tau N'$, $\star \in \{i, pw, pom\}$, if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_{C_E}) \asymp vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$, $(\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \Rightarrow \exists \tilde{\beta}$, $(\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P)$, $\tilde{\beta}|_{vis(T_{C_E})} = \beta$, $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$, *and if* $\pi_P \xrightarrow{\pi} \tilde{\pi}_E$, $\pi'_P \xrightarrow{\pi'} \tilde{\pi}'_E$, $\gamma = \tilde{\beta}|_{T_C}$, *then:*
   (a) $\gamma^{-1} : vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$, *if* $\star = pw$;
   (b) $\gamma : vis(\rho_C) \simeq vis(\rho_{C'})$, *if* $\star = pom$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are $\star$-ST-τ-bisimulation equivalent*,* $\star \in$ {interleaving, partial word, pomset}*, denoted by $N \underline{\leftrightarrow}_{\star ST}^\tau N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{\star ST}^\tau N'$, $\star \in \{i, pw, pom\}$.*

3.2.3 History preserving τ-bisimulation equivalences

Definition 3.8 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, is a* pomset history preserving τ-bisimulation *between N and N', denoted by $N \underline{\leftrightarrow}_{pomh}^\tau N'$, if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.
2. $(\pi, \pi', \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_C) \simeq vis(\rho_{C'})$.
3. $(\pi, \pi', \beta) \in \mathcal{R}$, $\pi \to \tilde{\pi} \Rightarrow \exists \tilde{\beta}$, $\tilde{\pi}' : \pi' \to \tilde{\pi}'$, $\tilde{\beta}|_{vis(T_C)} = \beta$, $(\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_{pomh}^\tau N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{pomh}^\tau N'$.*

3.2.4 History preserving ST-τ-bisimulation equivalences

Definition 3.9 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, is a* pomset history preserving ST-τ-bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$, if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \simeq vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \ \Rightarrow\ \exists \tilde{\beta},\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta,\ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving ST-τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{pomhST} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$.*

3.2.5 Usual branching τ-bisimulation equivalences

For some net N and $\pi, \tilde{\pi} \in \Pi(N)$ we write $\pi \Rightarrow \tilde{\pi}$ when $\exists \hat{\pi} = (\widehat{C}, \hat{\varphi})$ s.t. $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$ and $vis(T_{\widehat{C}}) = \emptyset$.

Definition 3.10 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ is an* interleaving branching τ-bisimulation *between N and N', denoted by $N \underline{\leftrightarrow}^\tau_{ibr} N'$, if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.
2. $(\pi, \pi') \in \mathcal{R},\ \pi \xrightarrow{a} \tilde{\pi} \ \Rightarrow$
   (a) $a = \tau$ *and* $(\tilde{\pi}, \pi') \in \mathcal{R}$ *or*
   (b) $a \neq \tau$ *and* $\exists \bar{\pi}',\ \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \xrightarrow{a} \tilde{\pi}',\ (\pi, \bar{\pi}') \in \mathcal{R},\ (\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$.
3. *As item 2, but the roles of N and N' are reversed.*

Two nets N and N' are interleaving branching τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{ibr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{ibr} N'$.*

3.2.6 History preserving branching τ-bisimulation equivalences

Definition 3.11 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : T_C \to T_{C'},\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, is a* pomset history preserving branching τ-bisimulation *between N and N', denoted by $N \underline{\leftrightarrow}^\tau_{pomhbr} N'$, if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.
2. $(\pi, \pi', \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_C) \simeq vis(\rho_{C'})$.
3. $(\pi, \pi', \beta) \in \mathcal{R},\ \pi \to \tilde{\pi} \ \Rightarrow$
   (a) $(\tilde{\pi}, \pi', \beta) \in \mathcal{R}$ *or*
   (b) $\exists \tilde{\beta},\ \bar{\pi}',\ \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \to \tilde{\pi}',\ \tilde{\beta}|_{vis(T_C)} = \beta,\ (\pi, \bar{\pi}', \beta) \in \mathcal{R},\ (\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving branching τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{pomhbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhbr} N'$.*

3.3 Conflict preserving τ-equivalences

Let $\xi = \langle X, \prec, \#, l \rangle$ be a LES s.t. $l : X \to Act_\tau$. We denote $vis(X) = \{x \in X \mid l(x) \in Act\}$ and $vis(\xi) = \xi|_{vis(X)}$.

Definition 3.12 A visible MES-trace *of a net N, denoted by $vis(\xi)$, is an isomorphism class of LES $vis(\xi_O)$ for $\varpi = (O, \psi) \in \wp(N)$. We denote a set of* all visible MES-traces *of a net N by $VisMEStructs(N)$. Two nets N and N' are* MES-τ-conflict preserving equivalent*, denoted by $N \equiv^\tau_{mes} N'$, if $VisMEStructs(N) = VisMEStructs(N')$. Let us note that, due to uniqueness of maximal O-process, this is the same as to require $vis(\mathcal{E}(N)) = vis(\mathcal{E}(N'))$.*

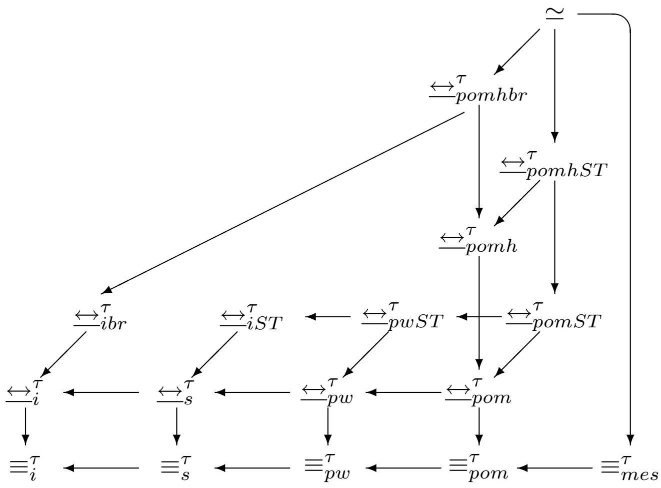

Figure 1: Interrelations of basic τ-equivalences

3.4 Interrelations of basic τ-equivalences

Let us compare basic τ-equivalences.

Theorem 3.1 *Let $\leftrightarrow, \twoheadleftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$, $\star, \star\star \in \{i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr,$ $pomhbr, mes\}$. For nets N and N' $N \leftrightarrow_\star N' \;\Rightarrow\; N$ doublelra$_{\star\star}N'$ iff in the graph in Figure 1 there exists a directed path from $\leftrightarrow_\star$ to $\twoheadleftrightarrow_{\star\star}$.*

Proof. ($\Leftarrow$) Let us check t he validity of the implications in the graph in Figure 1.

- The implications $\leftrightarrow^\tau_s \to \leftrightarrow^\tau_i$, $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau\}$, are valid since isomorphism of lposets with empty precedence relation is isomorphism of singleton ones.
- The implications $\leftrightarrow^\tau_{pw} \to \leftrightarrow^\tau_s$, $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau\}$, are valid since homomorphism of lposets is isomorphism of lposets with empty precedence relation.
- The implication $\underline{\leftrightarrow}^\tau_{pwST} \to \underline{\leftrightarrow}^\tau_{iST}$ is valid since homomorphism of lposets is isomorphism of singleton ones.
- The implications $\leftrightarrow^\tau_{pom} \to \leftrightarrow^\tau_{pw}$, $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau\}$, are valid since isomorphism of lposets is homomorphism.
- The implication $\equiv^\tau_{mes} \to \equiv^\tau_{pom}$ is valid since isomorphic LES's have isomorphic sets of lposets.
- The implication $\underline{\leftrightarrow}^\tau_i \to \equiv^\tau_i$ is proved as follows. Let $\mathcal{R} : N \underline{\leftrightarrow}^\tau_i N'$. If $\pi_N \stackrel{a_1}{\to} \pi_1 \stackrel{a_2}{\to} \dots \stackrel{a_n}{\to} \pi_n$, then there exists a sequence $(\pi_N, \pi_{N'}), \dots, (\pi_n, \pi'_m) \in \mathcal{R}$ s.t. $\pi_{N'} \stackrel{a'_1}{\to} \pi'_1 \stackrel{a'_2}{\to} \dots \stackrel{a'_m}{\to} \pi'_m$, $vis(a_1 \cdots a_n) = vis(a'_1 \cdots a'_m)$, and vice versa, due to the symmetry of bisimulation.
- The implication $\underline{\leftrightarrow}^\tau_s \to \equiv^\tau_s$ is proved as the previous one but with use of $A_1, \dots, A_n \in \mathcal{M}(Act_\tau)$ instead of $a_1, \dots, a_n \in Act_\tau$.
- The implication $\underline{\leftrightarrow}^\tau_{pw} \to \equiv^\tau_{pw}$ is proved as follows. Let $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pw} N'$ and $\pi = (C, \varphi) \in \Pi(N)$. Since $\pi_N \stackrel{\pi}{\to} \pi$, then $\exists (\pi, \pi') \in \mathcal{R}$ s.t. $\pi' = (C', \varphi')$ and $vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$. Hence, $VisPomsets(N') \sqsubseteq VisPomsets(N)$. The inclusion $VisPomsets(N) \sqsubseteq VisPomsets(N')$ is proved similarly, due to the symmetry of bisimulation.
- The implication $\underline{\leftrightarrow}^\tau_{pom} \to \equiv^\tau_{pom}$ is proved as the previous one but with use of isomorphism instead of homomorphism.
- The implication $\underline{\leftrightarrow}^\tau_{iST} \to \underline{\leftrightarrow}^\tau_s$ is proved as previous ones with use of the fact that a step $\pi \stackrel{A}{\to} \tilde{\pi}$, where $A = \{a_1, \dots, a_n\} \in \mathcal{M}(Act)$, corresponds to the sequence of ST-τ-processes $(\pi_0, \pi_0), \dots, (\pi_n, \pi_0), \dots, (\pi_n, \pi_n)$ s.t. $\pi = \pi_0 \stackrel{a_1}{\to} \dots \stackrel{a_n}{\to} \pi_n = \tilde{\pi}$.
- The implications $\underline{\leftrightarrow}^\tau_{\star ST} \to \underline{\leftrightarrow}^\tau_\star$, $\star \in \{pw, pom\}$ are proved with constructing on the basis of the relation $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star ST} N'$ the new relation $\mathcal{S} : N \underline{\leftrightarrow}^\tau_\star N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \beta \; ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

- The implication $\underline{\leftrightarrow}^{\tau}_{pomhST} \to \underline{\leftrightarrow}^{\tau}_{pomh}$ is proved with constructing on the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}^{\tau}_{pomhST}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^{\tau}_{pomh}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi', \beta) \mid ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

- The implication $\underline{\leftrightarrow}^{\tau}_{pomh} \to \underline{\leftrightarrow}^{\tau}_{pom}$ is proved with constructing on the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}^{\tau}_{pomh}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^{\tau}_{pom}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists\beta\ ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.

- The implication $\underline{\leftrightarrow}^{\tau}_{pomhST} \to \underline{\leftrightarrow}^{\tau}_{pomST}$ follows from the definitions.

- The implication $\underline{\leftrightarrow}^{\tau}_{ibr} \to \underline{\leftrightarrow}^{\tau}_{i}$ follows from the definitions.

- The implication $\underline{\leftrightarrow}^{\tau}_{pomhbr} \to \underline{\leftrightarrow}^{\tau}_{pomh}$ follows from the definitions.

- The implication $\underline{\leftrightarrow}^{\tau}_{pomhbr} \to \underline{\leftrightarrow}^{\tau}_{ibr}$ is proved with constructing on the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}^{\tau}_{pomhbr}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^{\tau}_{ibr}N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists\beta\ (\pi, \pi', \beta) \in \mathcal{R}\}$.

- The implication $\simeq \to \underline{\leftrightarrow}^{\tau}_{pomhbr}$ is obvious.

- The implication $\simeq \to \underline{\leftrightarrow}^{\tau}_{pomhST}$ is obvious.

- The implication $\simeq \to \equiv^{\tau}_{mes}$ is obvious.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 1 is proved by the following examples.

- In Figure 2(a) $N\underline{\leftrightarrow}^{\tau}_{ibr}N'$, but $N \not\equiv^{\tau}_{s} N'$, since only in the net N' actions a and b cannot happen concurrently.

- In Figure 2(c) $N\underline{\leftrightarrow}^{\tau}_{iST}N'$, but $N \not\equiv^{\tau}_{pw} N'$, since for the pomset corresponding to the net N there is no even less sequential pomset in N'.

- In Figure 2(b) $N\underline{\leftrightarrow}^{\tau}_{pwST}N'$, but $N \not\equiv^{\tau}_{pom} N'$, since only in the net N' action b can depend on action a.

- In Figure 4(a) $N \equiv^{\tau}_{mes} N'$, but $N\underline{\not\leftrightarrow}^{\tau}_{i}N'$, since only in the net N' action τ can happen so that in the corresponding initial state of the net N action a cannot happen.

- In Figure 3(a) $N\underline{\leftrightarrow}^{\tau}_{pom}N'$, but $N\underline{\not\leftrightarrow}^{\tau}_{iST}N'$, since only in the net N' action a can start so that no action b can begin to work until finishing a.

- In Figure 3(b) $N\underline{\leftrightarrow}^{\tau}_{pomST}N'$, but $N\underline{\not\leftrightarrow}^{\tau}_{pomh}N'$, since only in the net N' after action a action b can happen so that action c must depend on a.

- In Figure 4(b) $N\underline{\leftrightarrow}^{\tau}_{pomh}N'$, but $N\underline{\not\leftrightarrow}^{\tau}_{iST}N'$, since only in the net N' action a can start so that the action b can never occur.

- In Figure 4(c) $N\underline{\leftrightarrow}^{\tau}_{pomhST}N'$, but $N\underline{\not\leftrightarrow}^{\tau}_{ibr}N'$, since in the net N' an action a can happen so that it will be simulated by sequence of actions τa in N. Then the state of the net N reached after τ must be related with the initial state of a net N, but in such a case the occurrence of action b from the initial state of N' cannot be imitated from the corresponding state of N.

- In Figure 4(d) $N\underline{\leftrightarrow}^{\tau}_{pomhbr}N'$, but $N\underline{\not\leftrightarrow}^{\tau}_{iST}N'$, since in the net N' an action c may start so that during work of the corresponding action c in the net N an action a may happen in such a way that the action b never occur.

- In Figure 3(c) $N\underline{\leftrightarrow}^{\tau}_{pomhST}N'$, but $N \not\equiv^{\tau}_{mes} N'$, since only the MES corresponding to the net N' has two conflict actions a.

- In Figure 3(c) $N\underline{\leftrightarrow}^{\tau}_{pomhbr}N'$, but $N \not\equiv^{\tau}_{mes} N'$.

- In Figure 3(d) $N \equiv^{\tau}_{mes} N'$, but $N \not\simeq N'$, since unfireable transitions of the nets N and N' are labelled by different actions (a and b). □

4 Back-forth τ-bisimulation equivalences

In this section we propose back-forth τ-bisimulation equivalences.

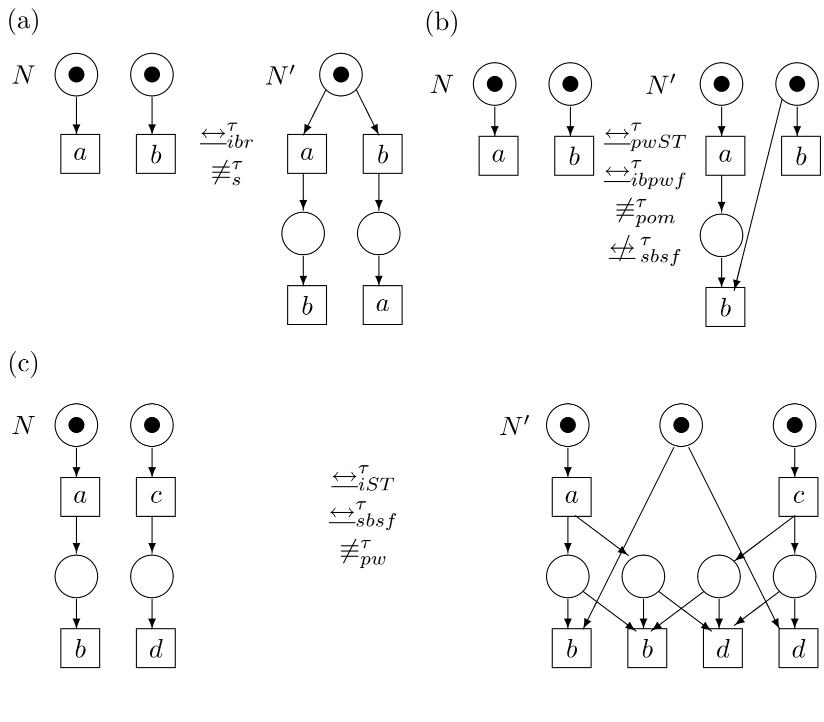

Figure 2: Examples of basic τ-equivalences

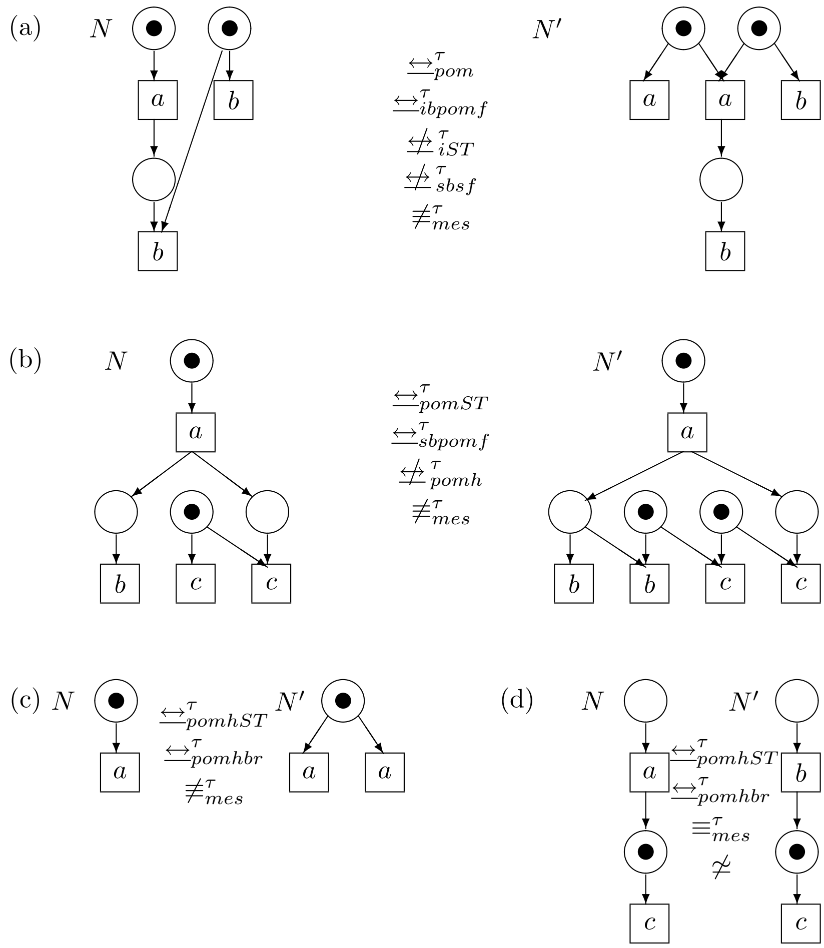

Figure 3: Examples of basic τ-equivalences (continued)

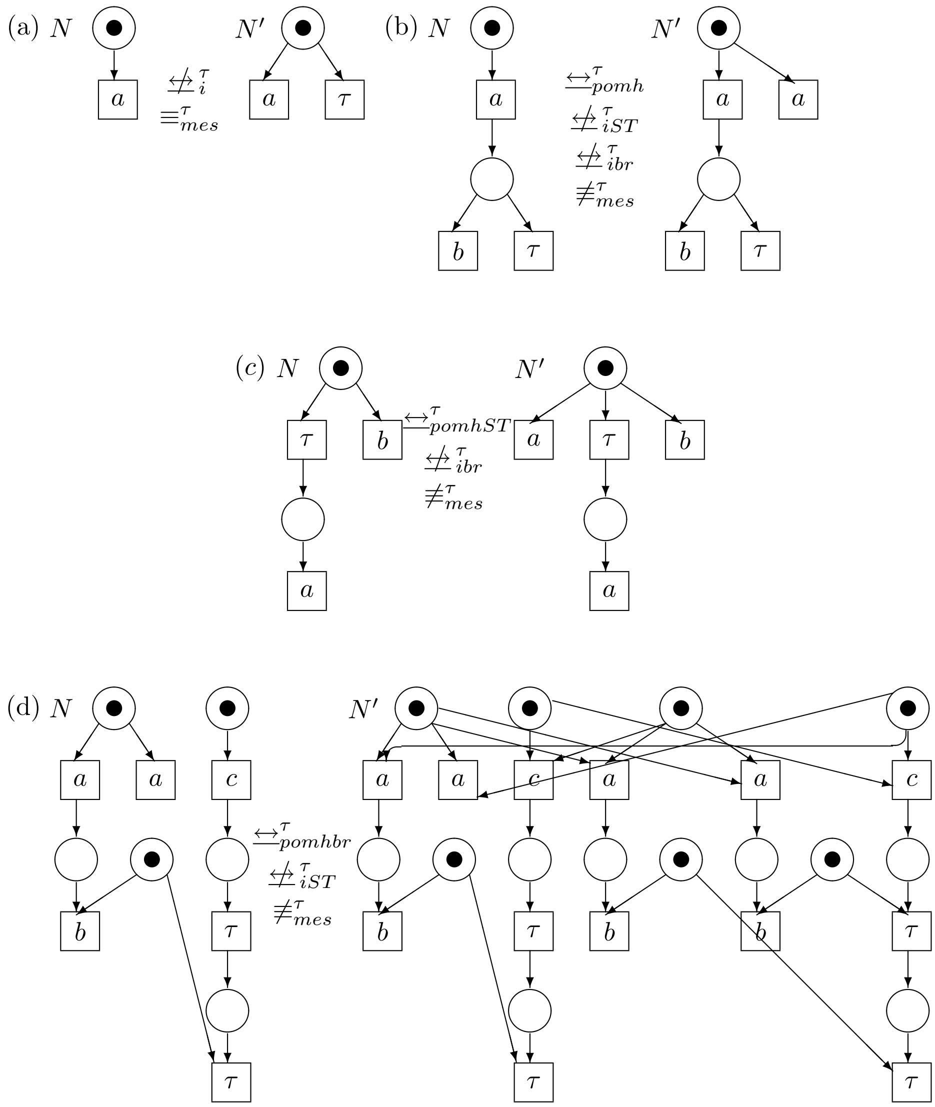

Figure 4: Examples of basic τ-equivalences (continued 2)

4.1 Sequential runs

Definition 4.1 *A* sequential run *of a net N is a pair (π, σ), where:*

- *a process $\pi \in \Pi(N)$ contains the information about causal dependencies of transitions which brought to this state;*
- *a sequence $\sigma \in T_C^*$ s.t. $\pi_N \xrightarrow{\sigma} \pi$, contains the information about the order in which the transitions occur which brought to this state.*

Let us denote the *set of all sequential runs* of a net N by $Runs(N)$.

The *initial* sequential run of a net N is a pair (π_N, ε), where ε is an empty sequence. Let us denote by $|\sigma|$ a *length* of a sequence σ.

Let $(\pi, \sigma),\ (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$. We write $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi},\ \exists \hat{\sigma} \in T_{\widetilde{C}}^*\ \pi \xrightarrow{\hat{\sigma}} \tilde{\pi}$ and $\tilde{\sigma} = \sigma\hat{\sigma}$. We write $(\pi, \sigma) \to (\tilde{\pi}, \tilde{\sigma})$, if $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$ for some $\hat{\pi}$.

Let $(\pi, \sigma) \in Runs(N),\ (\pi', \sigma') \in Runs(N')$ and $\sigma = v_1 \cdots v_n,\ \sigma' = v_1' \cdots v_n'$. Let us define a mapping $\beta_\sigma^{\sigma'} : T_C \to T_{C'}$ as follows: $\beta_\sigma^{\sigma'} = \{(v_i, v_i') \mid 1 \le i \le n\}$. Let $\beta_\varepsilon^\varepsilon = \emptyset$.

Let $(\pi, \sigma) \in Runs(N)$ and $\sigma = v_1 \cdots v_n,\ \pi_N \xrightarrow{v_1} \ldots \xrightarrow{v_i} \pi_i\ (1 \le i \le n)$.

Let us introduce the following notations:

- $\pi(0) = \pi_N$,
  $\pi(i) = \pi_i\ (1 \le i \le n)$;
- $\sigma(0) = \varepsilon$,
  $\sigma(i) = v_1 \cdots v_i\ (1 \le i \le n)$.

4.2 Definitions of back-forth τ-bisimulation equivalences

Now we are ready to present definitions of back-forth τ-bisimulation equivalences.

Definition 4.2 *Let N and N' be some nets. A relation $\mathcal{R} \subseteq Runs(N) \times Runs(N')$ is a* $\star$-back $\star\star$-forth τ-bisimulation *between N and N',*
$\star, \star\star \in${interleaving, step, partial word, pomset}*, denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom\}$, if:*

1. *$((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.*
2. *$((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$*
   - *(back)*
     $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$,
     (a) $|vis(T_{\widehat{C}})| = 1$, if $\star = i$;
     (b) $vis(\prec_{\widehat{C}}) = \emptyset$, if $\star = s$;
     $\Rightarrow\ \exists (\tilde{\pi}', \tilde{\sigma}') :\ (\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*
     (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star = pw$;
     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, if $\star \in \{i, s, pom\}$;
   - *(forth)*
     $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$,
     (a) $|vis(T_{\widehat{C}})| = 1$, if $\star\star = i$;
     (b) $vis(\prec_{\widehat{C}}) = \emptyset$, if $\star\star = s$;
     $\Rightarrow\ \exists (\tilde{\pi}', \tilde{\sigma}') :\ (\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*
     (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star\star = pw$;
     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, if $\star\star \in \{i, s, pom\}$.
3. *As item 2, but the roles of N and N' are reversed.*

Two nets N and N' are $\star$-back $\star\star$-forth τ-bisimulation equivalent, $\star, \star\star \in$ {interleaving, step, partial word, pomset}, *denoted by $N \underline{\leftrightarrow}^\tau_{\star b \star\star f} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom\}$.*

Let us note that back extensions of sequential runs are *deterministic*, i.e. for $(\pi, \sigma) \in Runs(N)$ there exists only one $(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$ s.t. $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$ and $|\tilde{\sigma}| = i\ (0 \le i \le |\sigma|)$. In such a case $(\tilde{\pi}, \tilde{\sigma}) = (\pi(i), \sigma(i))$.

4.3 Interrelations of back-forth τ-bisimulation equivalences

Let us compare back-forth τ-bisimulation equivalences.

Proposition 4.1 *Let $\star \in \{i, s, pw, pom\}$. For nets N and N' $N \underline{\leftrightarrow}^{\tau}_{pwb\star f} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{pomb\star f} N'$.*

Proof. ($\Leftarrow$) Isomorphism of lposets is homomorphism.

($\Rightarrow$) Let $\mathcal{R} : N \underline{\leftrightarrow}_{pwb\star f} N'$. Let us prove $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomb\star f} N'$.

1. Obviously, $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$.
   - (back)
     Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$. Then $\exists (\tilde{\pi}', \tilde{\sigma}') : \; (\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma'), \; ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ and $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$.

     Due to the symmetry of a bisimulation, the back extension $(\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma')$ must be imitated by some extension $(\tilde{\tilde{\pi}}, \tilde{\tilde{\sigma}}) \xrightarrow{\check{\pi}} (\pi, \sigma)$ s.t. $vis(\rho_{\check{C}}) \sqsubseteq vis(\rho_{\widehat{C}'})$. Due to determinism of back extensions, $vis(T_{\widehat{C}}) = vis(T_{\check{C}})$. Then $vis(\rho_{\widehat{C}}) = vis(\rho_{\check{C}})$. Consequently, $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$.
   - (forth)
     Obviously.
3. As item 2, but the roles of N and N' are reversed. □

Proposition 4.2 *Let $\star \in \{i, s, pw, pom\}$. For nets N and N' $N \underline{\leftrightarrow}^{\tau}_{\star bif} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{\star b \star f} N'$.*

Proof. ($\Leftarrow$) Isomorphism of causal nets, isomorphism and homomorphism of lposets of causal nets, isomorphism of lposets of causal nets with empty precedence relation imply label preserving bijection of lposets of causal nets.

($\Rightarrow$) Let $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{\star bif} N'$. Let us prove $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{\star b \star f} N'$.

1. Obviously, $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$.
   - (back)
     Obviously.
   - (forth)
     Let $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$. The extension by $\hat{\pi}$ corresponds to the extension by some sequence of transitions. Then $\exists (\tilde{\pi}', \tilde{\sigma}') : \; (\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}'), \; ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$, where the extension by $\hat{\pi}'$ corresponds to the extension by sequence of transitions which imitates the corresponding one in the net N.

     Due to the symmetry of a bisimulation, the back extension $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$ must be imitated by some extension $(\pi', \sigma') \xrightarrow{\check{\pi}'} (\tilde{\tilde{\pi}}', \tilde{\tilde{\sigma}}')$, s.t.

     (a) $vis(\rho_{\check{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, if $\star = pw$;

     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\check{C}'})$, if $\star \in \{i, s, pom\}$.

     Due to determinism of back extensions, $vis(T_{\widehat{C}'}) = vis(T_{\check{C}'})$. Then $vis(\rho_{\widehat{C}'}) = vis(\rho_{\check{C}'})$.
3. As item 2, but the roles of N and N' are reversed. □

In Figure 5 dashed lines embrace coinciding back-forth τ-bisimulation equivalences.

Hence, interrelations of back-forth τ-bisimulation equivalences may be represented by graph in Figure 6.

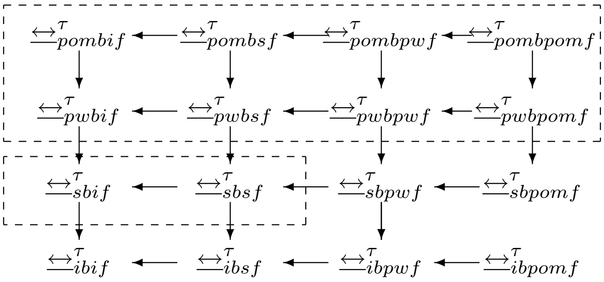

Figure 5: Merging of back-forth τ-bisimulation equivalences

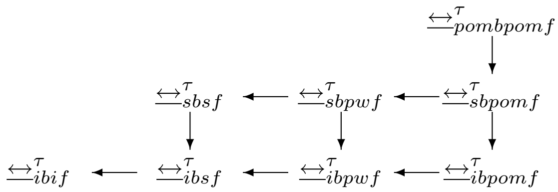

Figure 6: Interrelations of back-forth τ-bisimulation equivalences

4.4 Interrelations of back-forth τ-bisimulation equivalences with basic τ-equivalences

Let us consider compare back-forth τ-bisimulation equivalences with basic τ-equivalences.

For some net N and $(\pi,\sigma),(\tilde{\pi},\tilde{\sigma}) \in Runs(N)$ we write $(\pi,\sigma) \Rightarrow (\tilde{\pi},\tilde{\sigma})$ when $(\pi,\sigma) \to (\tilde{\pi},\tilde{\sigma})$ and $\pi \Rightarrow \tilde{\pi}$.

Let for some nets N and N' $(\pi,\sigma) \in Runs(N)$, $(\pi',\sigma') \in Runs(N')$.

We write $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{ibif}(\pi',\sigma')$ if $\exists \mathcal{R} : N\underline{\leftrightarrow}^{\tau}_{ibif}N'$ s.t. $((\pi,\sigma),(\pi',\sigma')) \in \mathcal{R}$ and analogously for $\underline{\leftrightarrow}^{\tau}_{pombpomf}$.

We write $\pi\underline{\leftrightarrow}_{ibr}\pi'$ if $\exists \mathcal{R} : N\underline{\leftrightarrow}^{\tau}_{ibr}N'$ s.t. $(\pi,\pi') \in \mathcal{R}$.

We write $\pi\underline{\leftrightarrow}^{\tau}_{pomhbr}\pi'$ if $\exists \mathcal{R} : N\underline{\leftrightarrow}^{\tau}_{pomhbr}N'$ $\exists \beta$ s.t. $(\pi,\pi',\beta) \in \mathcal{R}$.

Lemma 4.1 (X-lemma 1) *Let for nets N and N' $N\underline{\leftrightarrow}^{\tau}_{ibif}N'$ and $(\pi,\sigma),(\tilde{\pi},\tilde{\sigma}) \in Runs(N)$, $(\pi',\sigma'),(\tilde{\pi}',\tilde{\sigma}') \in Runs(N')$ s.t. $(\pi,\sigma) \Rightarrow (\tilde{\pi},\tilde{\sigma})$, $(\pi',\sigma') \Rightarrow (\tilde{\pi}',\tilde{\sigma}')$. Then $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{ibif}(\tilde{\pi}',\tilde{\sigma}')$ and $(\tilde{\pi},\tilde{\sigma})\underline{\leftrightarrow}^{\tau}_{ibif}(\pi',\sigma')$ implies $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{ibif}(\pi',\sigma')$ and $(\tilde{\pi},\tilde{\sigma})\underline{\leftrightarrow}^{\tau}_{ibif}(\tilde{\pi}',\tilde{\sigma}')$.*

Proof. As proof of the following lemma but using process extensions by one action only. □

Lemma 4.2 (X-lemma 2) *Let for nets N and N' $N\underline{\leftrightarrow}^{\tau}_{pombpomf}N'$ and $(\pi,\sigma),(\tilde{\pi},\tilde{\sigma}) \in Runs(N)$, (π',σ'), $(\tilde{\pi}',\tilde{\sigma}') \in Runs(N')$ s.t. $(\pi,\sigma) \Rightarrow (\tilde{\pi},\tilde{\sigma})$, $(\pi',\sigma') \Rightarrow (\tilde{\pi}',\tilde{\sigma}')$. Then $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{pombpomf}(\tilde{\pi}',\tilde{\sigma}')$ and $(\tilde{\pi},\tilde{\sigma})\underline{\leftrightarrow}^{\tau}_{pombpomf}(\pi',\sigma')$ implies $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{pombpomf}(\pi',\sigma')$ and $(\tilde{\pi},\tilde{\sigma})\underline{\leftrightarrow}^{\tau}_{pombpomf}(\tilde{\pi}',\tilde{\sigma}')$.*

Proof. It is enough to prove $(\tilde{\pi},\tilde{\sigma})\underline{\leftrightarrow}^{\tau}_{pombpomf}(\tilde{\pi}',\tilde{\sigma}')$, since the fact $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{pombpomf}(\pi',\sigma')$ is proved similarly.

Let $(\pi,\sigma) \Rightarrow (\tilde{\pi},\tilde{\sigma})$, $(\pi',\sigma') \Rightarrow (\tilde{\pi}',\tilde{\sigma}')$ and $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{pombpomf}(\tilde{\pi}',\tilde{\sigma}')$, $(\tilde{\pi},\tilde{\sigma})\underline{\leftrightarrow}^{\tau}_{pombpomf}(\pi',\sigma')$. We have only to check similation of the net N by N' in back and forth directions, since simulation of N' by N is proved by symmetry.

- (back)

  Let $(\bar{\pi},\bar{\sigma}) \xrightarrow{\hat{\pi}} (\tilde{\pi},\tilde{\sigma})$, $\hat{\pi} = (\widehat{C},\hat{\varphi})$. Then, since $(\tilde{\pi},\tilde{\sigma})\underline{\leftrightarrow}^{\tau}_{pombpomf}(\pi',\sigma')$, $\exists \check{\pi}' = (\check{C}',\check{\varphi}'),(\bar{\pi}',\bar{\sigma}')$ s.t. $(\bar{\pi}',\bar{\sigma}') \xrightarrow{\check{\pi}'} (\pi',\sigma')$, $(\bar{\pi},\bar{\sigma})\underline{\leftrightarrow}^{\tau}_{pombpomf}(\bar{\pi}',\bar{\sigma}')$ and $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\check{C}'})$.

  Let us note if $(\bar{\pi}',\bar{\sigma}') \xrightarrow{\hat{\pi}'} (\tilde{\pi}',\tilde{\sigma}')$, $\hat{\pi}' = (\widehat{C}',\hat{\varphi}')$ then we have $vis(\rho_{\check{C}'}) = vis(\rho_{\widehat{C}'})$. Consequently, $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$.

- (forth)

  Let $(\tilde{\pi},\tilde{\sigma}) \xrightarrow{\hat{\pi}} (\bar{\pi},\bar{\sigma})$, $\hat{\pi} = (\widehat{C},\hat{\varphi})$. Let us note if $(\pi,\sigma) \xrightarrow{\check{\pi}} (\bar{\pi},\bar{\sigma})$, $\check{\pi} = (\check{C},\check{\varphi})$ then we have $vis(\rho_{\widehat{C}}) = vis(\rho_{\check{C}})$.

  Since $(\pi,\sigma)\underline{\leftrightarrow}^{\tau}_{pombpomf}(\tilde{\pi}',\tilde{\sigma}')$, $\exists \hat{\pi}' = (\widehat{C}',\hat{\varphi}'),(\bar{\pi}',\bar{\sigma}')$ s.t. $(\tilde{\pi}',\tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\bar{\pi}',\bar{\sigma}')$, $(\bar{\pi},\bar{\sigma})\underline{\leftrightarrow}^{\tau}_{pombpomf}(\bar{\pi}',\bar{\sigma}')$ and $vis(\rho_{\check{C}}) \simeq vis(\rho_{\widehat{C}'})$. Consequently, $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$. □

Proposition 4.3 *For nets N and N' $N \underline{\leftrightarrow}^{\tau}_{ibif} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{ibr} N'$.*

Proof. As proof of the following proposition but using process extensions by one action only. □

Proposition 4.4 *For nets N and N' $N \underline{\leftrightarrow}^{\tau}_{pombpomf} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$.*

Proof. For $\pi \in \Pi(N)$ we denote $[\pi] = \{\bar{\pi} \mid \bar{\pi} \in \Pi(N),\ \pi \underline{\leftrightarrow}^{\tau}_{pomhbr} \bar{\pi}\}$. Let $(\pi, \sigma) \in Runs(N)$ and $\sigma = v_1 \cdots, v_n$. A *trace* of (π, σ) is defined by $trace(\pi, \sigma) = [\pi_N] l_C(v_1)[\pi(1)] \cdots [\pi(n-1)] l_C(v_n)[\pi(n)]$. A *trace modulo stuttering* of (π, σ), denoted by $stutt(\pi, \sigma)$, is obtained from $trace(\pi, \sigma)$ by replacing all triples of a kind $R\tau R$ by R.

($\Leftarrow$) Let $N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$, $(\pi, \sigma) \in Runs(N)$, $(\pi', \sigma') \in Runs(N')$ and $stutt(\pi, \sigma) = R_1 a_1 R_2 \cdots R_{n-1} a_n R_n$, $stutt(\pi', \sigma') = R'_1 a'_1 R'_2 \cdots R'_{m-1} a'_m R'_m$. We say that $stutt(\pi, \sigma)$ and $stutt(\pi', \sigma')$ are *isomorphic*, denoted by $stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')$, if:

1. $n = m$;
2. $\forall i\ (1 \le i \le n)\ a_i = a'_i$;
3. $\forall i\ (1 \le i \le n)$ and $\pi_i \in R_i,\ \pi'_i \in R'_i :\ \pi_i \underline{\leftrightarrow}^{\tau}_{pomhbr} \pi'_i$.

Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{((\pi, \sigma), (\pi', \sigma')) \mid (\pi, \sigma) \in Runs(N),\ (\pi', \sigma') \in Runs(N'),\ stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}^{\tau}_{pombpomf} N'$.

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{S}$, since $\pi_N \underline{\leftrightarrow}^{\tau}_{pomhbr} \pi_{N'}$.
2. Let $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{S}$.
   - (back)

     We have $\exists \beta : vis(\rho_C) \simeq vis(\rho_{C'})$. Let $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$. Then $\exists i\ (1 \le i \le n)\ (\tilde{\pi}, \tilde{\sigma}) \in R_i$ from $trace(\pi, \sigma)$. Since $stutt(\pi, \sigma) \simeq stutt(\pi', \sigma')$, then $\exists k\ (1 \le k \le n)$ s.t. R_i corresponds to R'_k from $trace(\pi', \sigma')$. Then $\tilde{\pi} \underline{\leftrightarrow}^{\tau}_{pomhbr} \pi'(k)$. Consequently, $((\tilde{\pi}, \tilde{\sigma}), (\pi'(k), \sigma'(k))) \in \mathcal{S}$ and $\exists \beta : vis(\rho_{\widetilde{C}}) \simeq vis(\rho_{C'(k)})$. Let us consider the back extension $(\pi'(k), \sigma'(k)) \xrightarrow{\hat{\pi}'} (\pi', \sigma')$. Since β and $\tilde{\beta}$ are isomorphisms, we have $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$.
   - (forth) Obviously.
3. As item 2, but the roles of N and N' are reversed.

($\Rightarrow$) Let $N \underline{\leftrightarrow}^{\tau}_{pombpomf} N'$. Let us define a relation $\mathcal{S}$ as follows: $\mathcal{S} = \{(\pi, \pi', \beta^{\sigma'}_{\sigma}) \mid (\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')\}$. Let us prove $\mathcal{S} : N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$.

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{S}$ since $\beta^{\varepsilon}_{\varepsilon} = \emptyset$ and $(\pi_N, \varepsilon) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi_{N'}, \varepsilon)$.
2. Let $(\pi, \pi', \beta^{\sigma'}_{\sigma}) \in \mathcal{S}$. Then by definition of $\mathcal{S}$, $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')$ and back extension $(\pi_N, \varepsilon) \xrightarrow{\pi} (\pi, \sigma)$ is imitated by $(\bar{\pi}', \varepsilon) \xrightarrow{\bar{\pi}'} (\pi', \sigma')$ for some $\bar{\pi}'$ s.t. $\pi_{N'} \Rightarrow \bar{\pi}'$. If $\pi = (C, \varphi)$ and $\bar{\pi}' = (\overline{C}, \bar{\varphi})$, we have $\beta^{\sigma'}_{\sigma} : vis(\rho_C) \simeq vis(\rho_{\overline{C}'})$. Since $vis(T'_C) = vis(T_{\overline{C}'})$, where $\pi' = (C', \varphi')$, we have $\beta^{\sigma'}_{\sigma} : vis(\rho_C) \simeq vis(\rho_{C'})$.
3. Let $(\pi, \pi', \beta^{\sigma'}_{\sigma}) \in \mathcal{S}$ and $\pi \xrightarrow{v} \tilde{\pi}$. Then by definition of $\mathcal{S}$, $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')$ and $(\pi, \sigma) \to (\tilde{\pi}, \sigma v)$. The following two cases are possible.

   (a) $l_{\widetilde{C}}(v) \neq \tau$.

   Since $N \underline{\leftrightarrow}^{\tau}_{pombpomf} N'$, we have $\exists v'_i, w'_j\ (1 \le i \le n,\ 1 \le j \le m),\ v', \pi'_1, \pi'_2$ s.t. $(\pi', \sigma') \xrightarrow{v'_1} \cdots \xrightarrow{v'_n} (\pi'_1, \sigma' v'_1 \cdots v'_n) \xrightarrow{v'} (\pi'_2, \sigma' v'_1 \cdots v'_n v') \xrightarrow{w'_1} \cdots \xrightarrow{w'_m} (\tilde{\pi}', \sigma' v'_1 \cdots v'_n v' w'_1 \cdots w'_m)$, $(\tilde{\pi}, \sigma v) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\tilde{\pi}', \sigma' v'_1 \cdots v'_n v' w'_1 \cdots w'_m)$ and $l_{\widetilde{C}}(v) = l_{\widetilde{C}'}(v')$, $\forall i, j\ (1 \le i \le n,\ 1 \le j \le m)\ l_{\widetilde{C}'}(v'_i) = l_{\widetilde{C}'}(w'_j) = \tau$. Consequently, $\pi' \xrightarrow{v'_1} \cdots \xrightarrow{v'_n} \pi'_1 \xrightarrow{v'} \pi'_2 \xrightarrow{w'_1} \cdots \xrightarrow{w'_m} \tilde{\pi}'$.

   The back extension $(\pi'_2, \sigma' v'_1 \cdots v'_n v') \to (\tilde{\pi}', \sigma' v'_1 \cdots v'_n v' w'_1 \cdots w'_m)$ is imitated by empty back extension of $(\tilde{\pi}, \sigma v)$. Hence, $(\tilde{\pi}, \sigma v) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi'_2, \sigma' v'_1 \cdots v'_n v')$. Therefore $(\tilde{\pi}, \pi'_2, \beta^{\sigma' v'_1 \cdots v'_n v'}_{\sigma v}) \in \mathcal{S}$.

   Let us consider the back extension $(\pi'_1, \sigma' v'_1 \cdots v'_n) \to (\pi'_2, \sigma' v'_1 \cdots v'_n v')$. It is imitated by some back extension $(\bar{\pi}, \bar{\sigma}) \Rightarrow (\pi, \sigma) \to (\tilde{\pi}, \sigma v)$ s.t. $(\bar{\pi}, \bar{\sigma}) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi'_1, \sigma' v'_1 \cdots v'_n)$. Since $(\pi', \sigma') \Rightarrow (\pi'_1, \sigma' v'_1 \cdots v'_n)$ and $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')$, by X-lemma 2 we have $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi'_1, \sigma' v'_1 \cdots v'_n)$. So, we obtain $(\pi, \pi'_1, \beta^{\sigma' v'_1 \cdots v'_n}_{\sigma}) \in \mathcal{S}$.

   Hence, we have simulation, since $\pi' \Rightarrow \pi'_1 \xrightarrow{a} \tilde{\pi}'_2$ and $(\pi, \pi'_1, \beta^{\sigma' v'_1 \cdots v'_n}_{\sigma}) \in \mathcal{S}$, $(\tilde{\pi}, \pi'_2, \beta^{\sigma' v'_1 \cdots v'_n v'}_{\sigma v}) \in \mathcal{S}$.

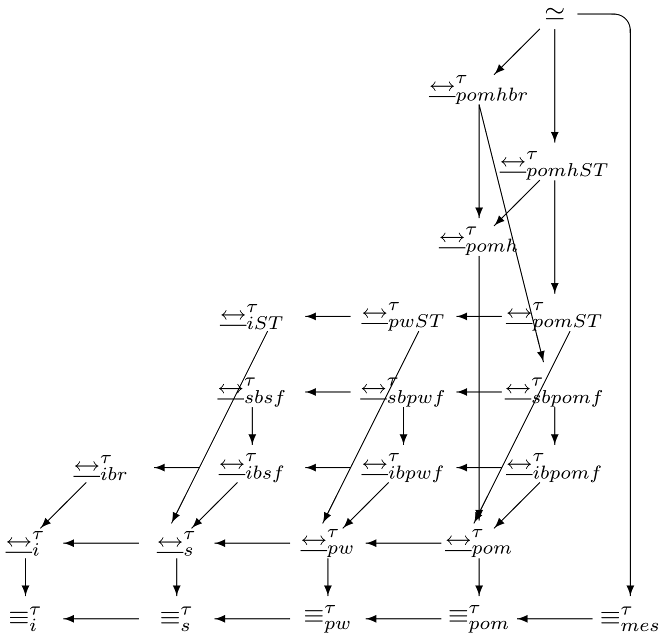

Figure 7: Interrelations of back-forth τ-bisimulation equivalences with basic τ-equivalences

(b) $l_{\widetilde{C}}(v) = \tau$.
Since $N \underline{\leftrightarrow}^{\tau}_{pombpomf} N'$, we have $\exists \pi'_i \ (1 \le i \le n)$ s.t. $(\pi', \sigma') \Rightarrow (\pi'_1, \sigma' v_1) \Rightarrow \cdots \Rightarrow (\pi'_n, \sigma' v'_1 \cdots v'_n) = (\tilde{\pi}', \sigma' v'_1 \cdots v'_n)$ and $(\tilde{\pi}, \sigma v) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\tilde{\pi}', \sigma' v'_1 \cdots v'_n)$.

i. If $n = 0$, we have proved.

ii. If $n \ge 1$, and the back extension $(\pi'_{n-1}, \sigma' v'_1 \cdots v'_{n-1}) \Rightarrow (\pi'_n, \sigma' v'_1 \cdots v'_n)$ is simulated by the empty back extension of $(\tilde{\pi}, \sigma v)$ we have proved for $n = 1$, and for $n \ge 2$ we shall continue such a reasoning. Two cases are possible.
In the first case, we shall obtain $(\tilde{\pi}, \sigma v) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi', \sigma')$ and $(\tilde{\pi}, \pi', \beta^{\sigma'}_{\sigma v}) \in \mathcal{S}$.
In the second case, we shall obtain $\exists m \ (1 \le m \le n-1)$ s.t. $(\tilde{\pi}, \sigma v) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi'_m, \sigma' v'_1 \ldots v'_m)$ and $(\tilde{\pi}, \pi'_m, \beta^{\sigma' v'_1 \ldots v'_m}_{\sigma v}) \in \mathcal{S}$.
The back extension $(\pi'_{m-1}, \sigma' v'_1 \cdots v'_{m-1}) \Rightarrow (\pi'_m, \sigma' v'_1 \cdots v'_m)$ is imitated by some back extension $(\bar{\pi}, \bar{\sigma}) \Rightarrow (\pi, \sigma)$ s.t. $(\bar{\pi}, \bar{\sigma}) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi'_{m-1}, \sigma' v'_1 \cdots v'_{m-1})$. By X-lemma 2 we have $(\pi, \sigma) \underline{\leftrightarrow}^{\tau}_{pombpomf} (\pi'_{m-1}, \sigma' v'_1 \cdots v'_{m-1})$. So, we obtain $(\pi, \pi'_{m-1}, \beta^{\sigma' v'_1 \cdots v'_{m-1}}_{\sigma}) \in \mathcal{S}$.
Hence, we have simulation, since $\pi' \Rightarrow \pi'_{m-1} \xrightarrow{\tau} \tilde{\pi}'_m$ and $(\pi, \pi'_{m-1}, \beta^{\sigma' v'_1 \cdots v'_{m-1}}_{\sigma}) \in \mathcal{S}$, $(\tilde{\pi}, \pi'_m, \beta^{\sigma' v'_1 \ldots v'_m}_{\sigma v}) \in \mathcal{S}$. □

4. As item 3, but the roles of N and N' are reversed.

Theorem 4.1 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\!\!\leftrightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$ *and* $\star, \star\star \in \{i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr, pomhbr, mes, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. *For nets* N *and* N' $N \leftrightarrow_{\star} N' \ \Rightarrow \ N \leftrightarrow\!\!\!\!\!\leftrightarrow_{\star\star} N'$ *iff in the graph in Figure 7 there exists a directed path from* $\leftrightarrow_{\star}$ *to* $\leftrightarrow\!\!\!\!\!\leftrightarrow_{\star\star}$.

Proof. ($\Leftarrow$) A consequence of Theorem 3.1 and the following substantiations.

- The implication $\underline{\leftrightarrow}^{\tau}_{ibsf} \to \underline{\leftrightarrow}^{\tau}_{ibr}$ is valid since by Proposition 4.3 $\underline{\leftrightarrow}^{\tau}_{ibr} = \underline{\leftrightarrow}^{\tau}_{ibif}$ and isomorphism of lposets with empty precedence relation is isomorphism of singleton ones.

- The implications $\underline{\leftrightarrow}^{\tau}_{\star bpwf} \to \underline{\leftrightarrow}^{\tau}_{\star bsf}$, $\star \in \{i, s\}$ is valid since homomorphism is isomorphism of lposets with empty precedence relation.

- The implications $\underline{\leftrightarrow}^{\tau}_{\star bpomf} \to \underline{\leftrightarrow}^{\tau}_{\star bpwf}$, $\star \in \{i, s\}$ is valid since isomorphism of lposets is homomorphism.

- The implications $\underline{\leftrightarrow}^{\tau}_{ib\star f} \to \underline{\leftrightarrow}^{\tau}_{\star}$, $\star \in \{s, pw, pom\}$ is proved with constructing on the basis of the relation $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{sb\star f} N'$ the new relation $\mathcal{S} : N \underline{\leftrightarrow}^{\tau}_{\star} N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \sigma, \sigma' \ ((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}\}$.

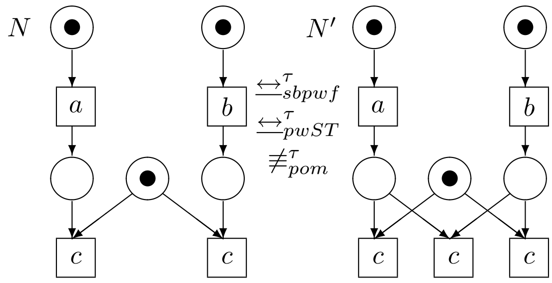

Figure 8: Example of back-forth τ-bisimulation equivalences

- The implications $\underline{\leftrightarrow}^\tau_{sb\star f} \to \underline{\leftrightarrow}^\tau_{ib\star f},\ \star \in \{s, pw, pom\}$ are valid since isomorphism of lposets with empty precedence relation is isomorphism of singleton ones.
- The implication $\underline{\leftrightarrow}^\tau_{pomhbr} \to \underline{\leftrightarrow}^\tau_{sbpomf}$ is valid since by Proposition 4.4 $\underline{\leftrightarrow}^\tau_{pomhbr} = \underline{\leftrightarrow}^\tau_{pombpomf}$ and homomorphism is isomorphism of lposets with empty precedence relation.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 7 is proved by the following examples.

- In Figure 2(c) $N\underline{\leftrightarrow}^\tau_{sbsf}N'$, but $N \not\equiv^\tau_{pw} N'$.
- In Figure 8 $N\underline{\leftrightarrow}^\tau_{sbpwf}N'$, but $N \not\equiv^\tau_{pom} N'$.
- In Figure 3(a) $N\underline{\leftrightarrow}^\tau_{ibpomf}N'$, but $N \underline{\not\leftrightarrow}^\tau_{sbsf} N'$. □

5 Interrelations of equivalences with τ-equivalences

In this section we compare equivalences which do not abstract of silent actions with all the considered τ-equivalencees.

Proposition 5.1 *Let* $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$, $\star \in \{i, s, pw, pom, iST, pwST, pomST, mes, sbsf, sbpwf, sbpomf\}$, $\star\star \in \{s, pw, pom\}$. *For nets N and N':*

1. $N \leftrightarrow_\star N' \ \Rightarrow\ N \leftrightarrow^\tau_\star N'$;
2. $N\underline{\leftrightarrow}_{pomh}N' \ \Rightarrow\ N\underline{\leftrightarrow}^\tau_{pomhST}N'$;
3. $N\underline{\leftrightarrow}_{i}N' \ \Rightarrow\ N\underline{\leftrightarrow}^\tau_{ibr}N'$;
4. $N\underline{\leftrightarrow}_{pomh}N' \ \Rightarrow\ N\underline{\leftrightarrow}^\tau_{pomhbr}N'$;
5. $N\underline{\leftrightarrow}_{\star\star}N' \ \Rightarrow\ N\underline{\leftrightarrow}^\tau_{ib\star\star f}N'$.

and all the implications are strict.

Proof.

1. By definitions.
2. We prove with construction one the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}_{pomh}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^\tau_{pomhST}N$, defined as follows: $\mathcal{S} = \{((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \mid (\pi_E, \pi'_E, \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \in ST^\tau - \Pi(N),\ (\pi'_E, \pi'_P) \in ST^\tau - \Pi(N'),\ \beta(T_{C_P}) = T_{C'_P}\}$.
3. By definitions.
4. By definitions.
5. We prove with construction one the basis of the relation $\mathcal{R} : N\underline{\leftrightarrow}_{\star\star}N'$ the new relation $\mathcal{S} : N\underline{\leftrightarrow}^\tau_{ib\star\star f}N'$, defined as follows: $\mathcal{S} = \{((\pi, \sigma), (\pi', \sigma')) \mid (\pi, \sigma) \in \mathit{Runs}(N),\ (\pi', \sigma') \in \mathit{Runs}(N'),\ |\sigma| = |\sigma'|,\ l_C(\sigma) = l_{C'}(\sigma'),\ \forall i\ (0 \le i \le |\sigma|)\ (\pi(i), \pi'(i)) \in \mathcal{R}\}$.

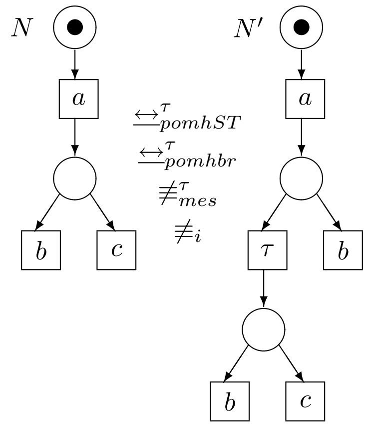

Figure 9: Example of interrelations of equivalences and τ-equivalences

The strictness of the implications is proved by the following examples.

1. In Figure 4(c) $N\underline{\leftrightarrow}^{\tau}_{pomhST}N'$, but $N \not\equiv_i N'$, since only in the net N' an action a can happen in the initial state.
2. In Figure 4(a) $N \equiv^{\tau}_{mes} N'$, but $N \not\equiv_i N'$, since only in the net N' an action τ can happen in the initial state.
3. In Figure 9 $N\underline{\leftrightarrow}^{\tau}_{pomhbr}N'$, but $N \not\equiv_i N'$, since only in the net N' a sequence of actions $a\tau$ can happen from the initial state. □

6 Preservation of the τ-equivalences by refinements

In this section we treat the considered τ-equivalences for preservation by transition refinements. We use SM-refinement, i.e. refinement by a special subclass of state-machine nets introduced in [4].

Definition 6.1 *An* SM-net *is a net* $D = \langle P_D, T_D, F_D, l_D, M_D\rangle$ *s.t.:*

1. $\forall t \in T_D\ |^{\bullet}t| = |t^{\bullet}| = 1$, *i.e. each transition has exactly one input and one output place;*
2. $\exists p_{in},\ p_{out} \in P_D$ *s.t.* $p_{in} \neq p_{out}$ *and* $^{\circ}D = \{p_{in}\},\ D^{\circ} = \{p_{out}\}$, *i.e. net D has unique input and unique output place.*
3. $M_D = \{p_{in}\}$, *i.e. at the beginning there is unique token in* p_{in}.

Definition 6.2 *Let* $N = \langle P_N, T_N, F_N, l_N, M_N\rangle$ *be some net,* $a \in l_N(T_N)$ *and* $D = \langle P_D, T_D, F_D, l_D, M_D\rangle$ *be SM-net. An* SM-refinement*, denoted by* $ref(N, a, D)$*, is (up to isomorphism) a net* $\overline{N} = \langle P_{\overline{N}}, T_{\overline{N}}, F_{\overline{N}}, l_{\overline{N}}, M_{\overline{N}}\rangle$*, where:*

- $P_{\overline{N}} = P_N \cup \{\langle p, u\rangle \mid p \in P_D \setminus \{p_{in}, p_{out}\},\ u \in l_N^{-1}(a)\}$;
- $T_{\overline{N}} = (T_N \setminus l_N^{-1}(a)) \cup \{\langle t, u\rangle \mid t \in T_D,\ u \in l_N^{-1}(a)\}$;
- $F_{\overline{N}}(\bar{x}, \bar{y}) = \begin{cases} F_N(\bar{x}, \bar{y}), & \bar{x}, \bar{y} \in P_N \cup (T_N \setminus l_N^{-1}(a)); \\ F_D(x, y), & \bar{x} = \langle x, u\rangle,\ \bar{y} = \langle y, u\rangle,\ u \in l_N^{-1}(a); \\ F_N(\bar{x}, u), & \bar{y} = \langle y, u\rangle,\ \bar{x} \in {}^{\bullet}u,\ u \in l_N^{-1}(a),\ y \in p_{in}^{\bullet}; \\ F_N(u, \bar{y}), & \bar{x} = \langle x, u\rangle,\ \bar{y} \in {}^{\bullet}u,\ u \in l_N^{-1}(a),\ x \in {}^{\bullet}p_{out}; \\ 0, & otherwise; \end{cases}$
- $l_{\overline{N}}(\bar{u}) = \begin{cases} l_N(\bar{u}), & \bar{u} \in T_N \setminus l_N^{-1}(a); \\ l_D(t), & \bar{u} = \langle t, u\rangle,\ t \in T_D,\ u \in l_N^{-1}(a); \end{cases}$

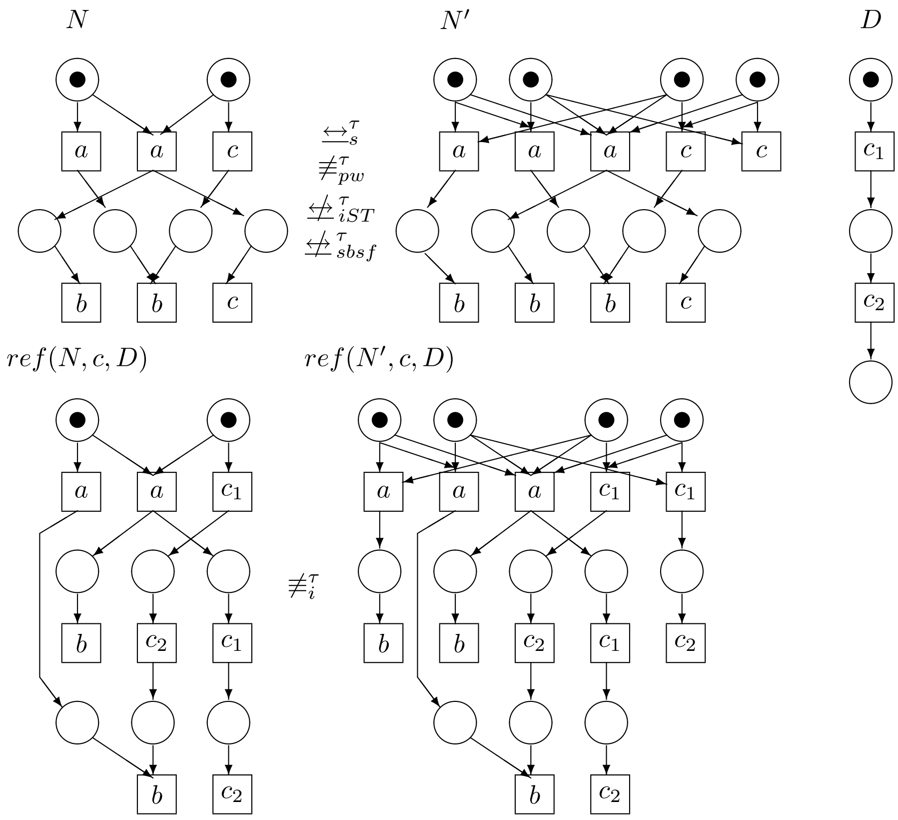

Figure 10: The τ-equivalences between $\equiv_i^\tau$ and $\underline{\leftrightarrow}_s^\tau$ are not preserved by SM-refinements

- $M_{\overline{N}}(p) = \begin{cases} M_N(p), & p \in P_N; \\ 0, & \textit{otherwise}. \end{cases}$

An equivalence is *preserved by refinements*, if equivalent nets remain equivalent after applying any refinement operator to them accordingly.

The following proposition demonstrates that some considered in the paper equivalence notions are not preserved by SM-refinements.

Proposition 6.1 *Let* $\star \in \{i, s\}$, $\star\star \in \{i, s, pw, pom, pomh, ibr, pomhbr, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. *Then the* τ*-equivalences* $\equiv_\star^\tau$, $\underline{\leftrightarrow}_{\star\star}^\tau$ *are not preserved by SM-refinements.*

Proof.

- In Figure 10 $N\underline{\leftrightarrow}_s^\tau N'$, but $ref(N, c, D) \not\equiv_i^\tau ref(N', c, D)$, since only in $ref(N', c, D)$ the sequence of actions c_1abc_2 can happen. Consequently, the τ-equivalences between $\equiv_i^\tau$ and $\underline{\leftrightarrow}_s^\tau$ are not preserved by SM-refinements.

- In Figure 11 $N\underline{\leftrightarrow}_{pom}^\tau N'$, but $ref(N, a, D)\not\underline{\leftrightarrow}_i^\tau ref(N', a, D)$, since only in $ref(N', a, D)$ after occurrence of action a_1 action b can not happen. Consequently, no equivalence between $\underline{\leftrightarrow}_i^\tau$ and $\underline{\leftrightarrow}_{pom}^\tau$ is preserved by SM-refinements.

- In Figure 12 $N\underline{\leftrightarrow}_{pomhbr}^\tau N'$, but $ref(N, a, D)\not\underline{\leftrightarrow}_i^\tau ref(N', a, D)$, since only in $ref(N', a, D)$ an action c_1 may happen so that after the corresponding action c_1 in the net N an action a may happen in such a way that the action b never occur. Consequently, no equivalence between $\underline{\leftrightarrow}_i^\tau$ and $\underline{\leftrightarrow}_{pomhbr}^\tau$ is preserved by SM-refinements.

In Figure 13 lines embrace τ-equivalences which are not preserved by SM-refinements due to examples in Figures 10–12. $\square$

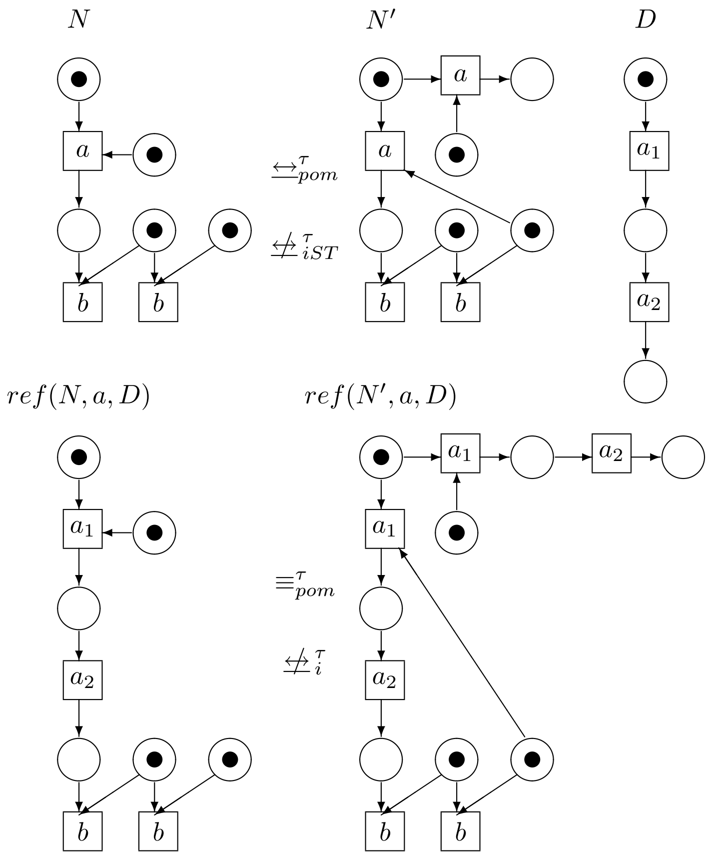

Figure 11: The τ-equivalences between $\underline{\leftrightarrow}^{\tau}_{i}$ and $\underline{\leftrightarrow}^{\tau}_{pom}$ are not preserved by SM-refinements

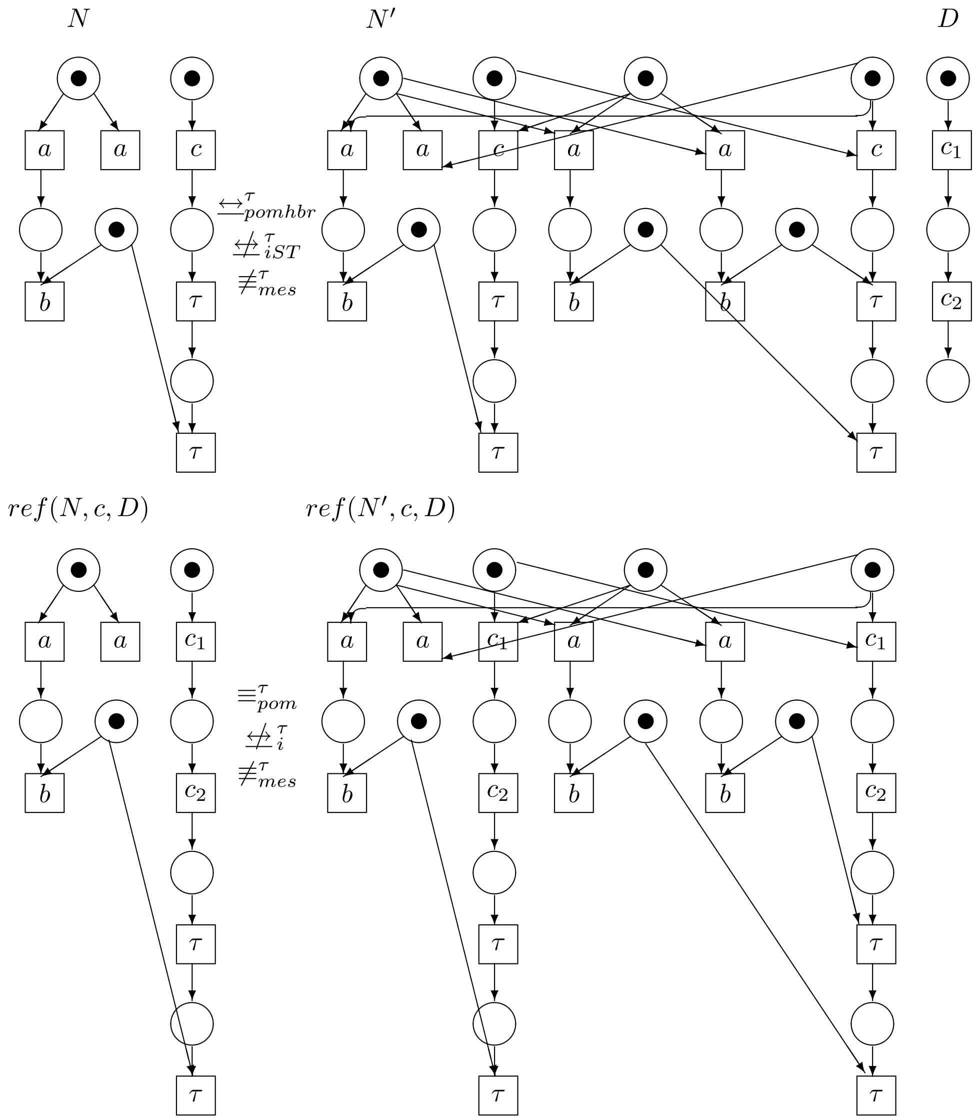

Figure 12: The τ-equivalences between $\underline{\leftrightarrow}_i^\tau$ and $\underline{\leftrightarrow}_{pomhbr}^\tau$ are not preserved by SM-refinements

Theorem 6.1 *Let* $\leftrightarrow\in\{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$ *and* $\star \in \{i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr,$ $pomhbr, mes, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. *For nets* $N,\ N'$ *s.t.* $a \in l_N(T_N) \cap l_{N'}(T_{N'}) \cap Act$ *and SM-net* D *the following holds:* $N \leftrightarrow_\star N' \;\Rightarrow\; ref(N, a, D) \leftrightarrow_\star ref(N', a, D)$ *iff the equivalence* $\leftrightarrow_\star$ *is in oval in Figure 14.*

Proof. Omitted. □

7 The τ-equivalences on some net subclasses

In this section we consider the τ-equivalences on nets without silent transitions and sequential nets.

7.1 The τ-equivalences on nets without silent transitions

Let us consider the τ-equivalences on nets without silent transitions, where no transition is labelled by the action τ.

Proposition 7.1 *Let* $\leftrightarrow\in\{\equiv, \underline{\leftrightarrow}\}$, $\star \in \{i, s, pw, pom, iST, pwST, pomST, mes, sbsf, sbpwf, sbpomf\}$, $\star\star \in \{s, pw, pom\}$. *For nets without silent transitions* N *and* N'*:*

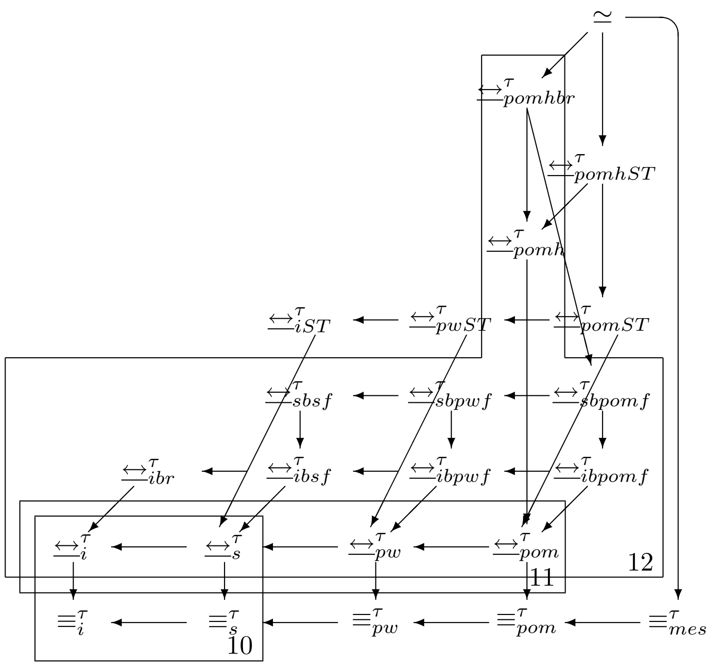

Figure 13: The τ-equivalences which are not preserved by SM-refinements

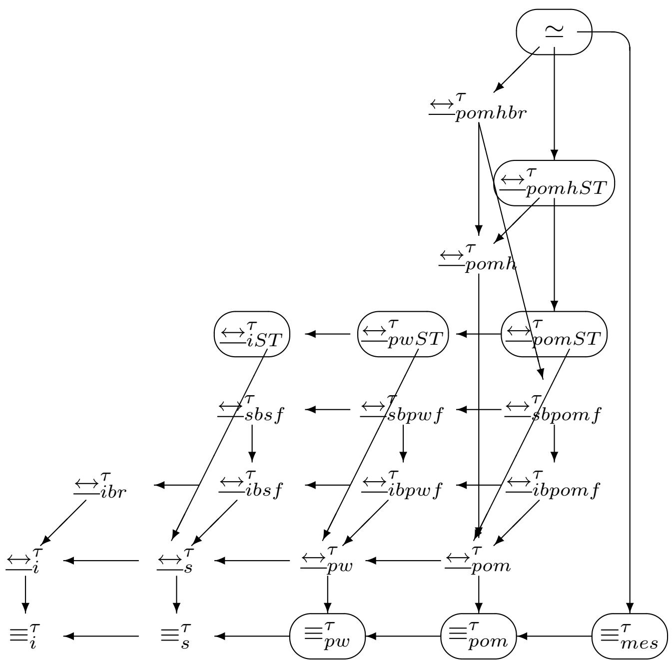

Figure 14: Preservation of the τ-equivalences by SM-refinements

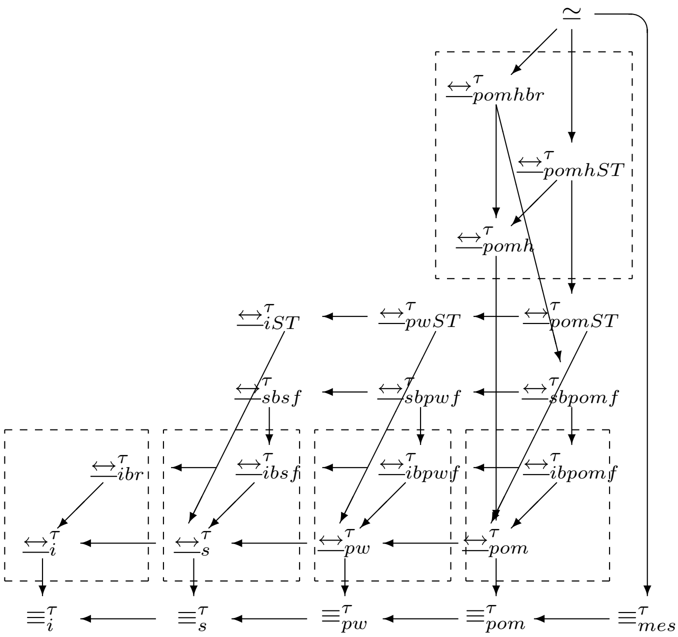

Figure 15: Merging of the τ-equivalences on nets without silent transitions

1. $N \leftrightarrow_{\star} N' \;\Leftrightarrow\; N \leftrightarrow^{\tau}_{\star} N'$;
2. $N \underline{\leftrightarrow}_{pomh} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{pomhST} N'$;
3. $N \underline{\leftrightarrow}_{i} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{ibr} N'$;
4. $N \underline{\leftrightarrow}_{pomh} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$;
5. $N \underline{\leftrightarrow}_{\star\star} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{ib\star\star f} N'$.

Proof. $(\Leftarrow)$

1. By definitions.
2. We prove with construction one the basis of the relation $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomhST} N'$ the new relation $\mathcal{S} : N \underline{\leftrightarrow}_{pomh} N$, defined as follows: $\mathcal{S} = \{(\pi, \pi', \beta) \mid ((\pi, \pi), (\pi', \pi'), \beta) \in \mathcal{R}\}$.
3. By definitions.
4. By definitions.
5. We prove with construction one the basis of the relation $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{ib\star\star f} N'$ the new relation $\mathcal{S} : N \underline{\leftrightarrow}_{\star\star} N'$, defined as follows: $\mathcal{S} = \{(\pi, \pi') \mid \exists \sigma, \sigma' \; ((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}\}$.

$(\Rightarrow)$ By Proposition 5.1, because nets without silent transitions are a subclass of that of with silent transitions. □

In Figure 15 dashed lines embrace the τ-equivalences coinciding on nets without silent transitions.

Theorem 7.1 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\!\!\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \simeq\}$, $\star, \star\star \in \{i, s, pw, pom, iST, pwST, pomST, pomh, ibr, mes, sbsf, sbpwf, sbpomf\}$. *For nets without silent transitions* N *and* N' $N \leftrightarrow_{\star} N' \;\Rightarrow\; N \leftrightarrow\!\!\!\!\!\leftrightarrow_{\star\star} N'$ *iff in the graph in Figure 16 there exists a directed path from* $\leftrightarrow_{\star}$ *to* $\leftrightarrow\!\!\!\!\!\leftrightarrow_{\star\star}$.

Proof. By Proposition 7.1 and Theorem 3.1 from [19]. □

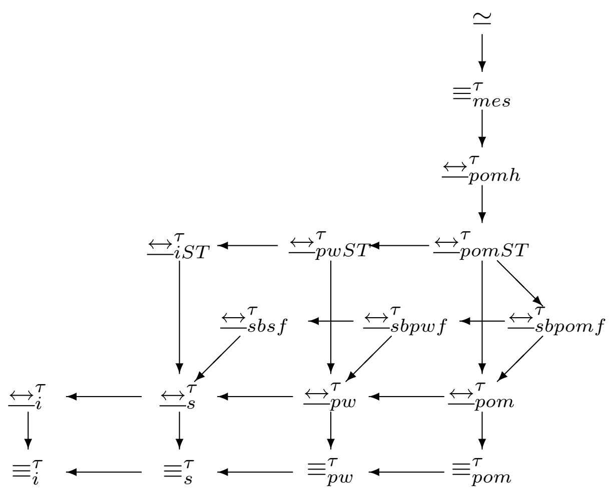

Figure 16: Interrelations of the τ-equivalences on nets without silent transitions

7.2 The τ-equivalences on sequential nets

Let us consider the τ-equivalences on sequential nets, where no two transitions can be fired concurrently.

Definition 7.1 *A net $N = \langle P_N, T_N, F_N, l_N, M_N\rangle$ is* sequential*, if $\forall M \in Mark(N)\ \neg\exists t, u \in T_N :\ {}^{\bullet}t + {}^{\bullet}u \subseteq M$.*

Proposition 7.2 *For sequential nets N and N':*

1. $N \equiv^{\tau}_{i} N' \ \Leftrightarrow\ N \equiv^{\tau}_{pom} N'$;
2. $N \underline{\leftrightarrow}^{\tau}_{i} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}^{\tau}_{pomh} N'$;
3. $N \underline{\leftrightarrow}^{\tau}_{iST} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}^{\tau}_{pomhST} N'$;
4. $N \underline{\leftrightarrow}^{\tau}_{ibr} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$.

Proof.

1. See [18].
2. See [4].
3. Similar to the item 2.
4. Similar to the item 2. □

In Figure 17 dashed lines embrace the τ-equivalences coinciding on sequential nets.

Theorem 7.2 *Let $\leftrightarrow, \twoheadleftrightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$, $\star, \star\star \in \{i, iST, ibr, mes\}$. For sequential nets N and N' $N \leftrightarrow_{\star} N' \ \Rightarrow\ N \twoheadleftrightarrow_{\star\star} N'$ iff in the graph in Figure 18 there exists a directed path from $\leftrightarrow_{\star}$ to $\twoheadleftrightarrow_{\star\star}$.*

Proof. ($\Leftarrow$) By Theorem 4.1.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 18 is proved by the following examples on sequential nets.

- In Figure 4(a) $N \equiv^{\tau}_{mes} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{i} N'$.
- In Figure 4(c) $N \underline{\leftrightarrow}^{\tau}_{i} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ibr} N'$.
- In Figure 4(b) $N \underline{\leftrightarrow}^{\tau}_{i} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{iST} N'$.
- In Figure 3(c) $N \underline{\leftrightarrow}^{\tau}_{ibr} N'$, but $N \not\equiv^{\tau}_{mes} N'$.
- In Figure 3(c) $N \underline{\leftrightarrow}^{\tau}_{iST} N'$, but $N \not\equiv^{\tau}_{mes} N'$. □

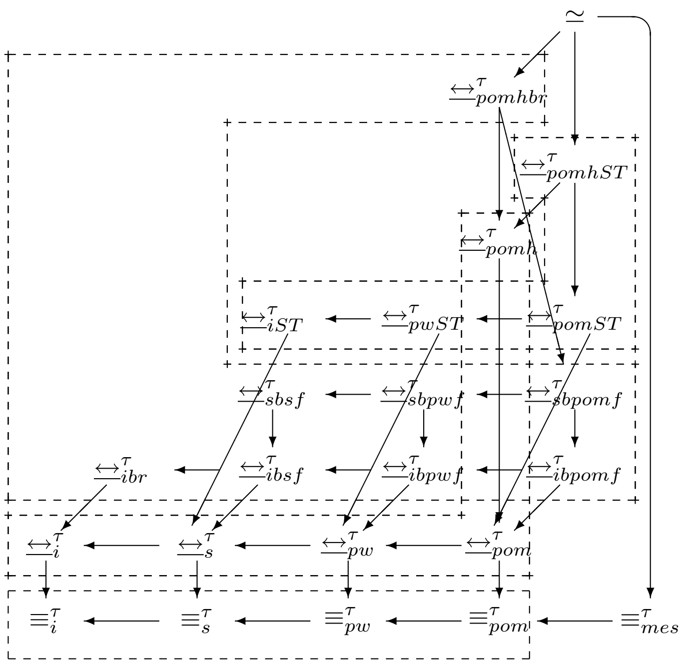

Figure 17: Merging of the τ-equivalences on sequential nets

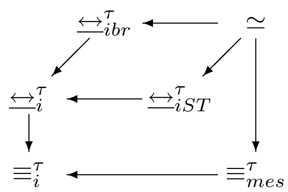

Figure 18: Interrelations of the τ-equivalences on sequential nets

8 Conclusion

In this paper, we supplemented by new ones and examined a group of basic τ-equivalences and back-forth τ-bisimulation equivalences. We compared them with relations which do not abstract of silent actions. We also compared them on the whole class of Petri nets as well as on their subclasses of nets without silent transitions and sequential nets. All the considered τ-equivalences were checked for preservation by SM-refinements. So, we can use the τ-equivalence notions that are preserved by SM-refinements, for top-down design of concurrent systems.

Further research may consist in the investigation of τ-variants of place bisimulation equivalences [2] which are used for effective semantically correct reduction of nets. In [3, 1] a notion of interleaving place τ-bisimulation equivalence was proposed, and its usefulness for simplification of concurrent systems was demonstrated.

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