τ-Equivalences and Refinement ⋆

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Abstract. The paper is devoted to the investigation of behavioural equivalences for Petri nets with silent transitions. Basic τ-equivalences and back-forth τ-bisimulation equivalences are supplemented by new ones, giving rise to complete set of equivalence notions in interleaving / true concurrency and linear / branching time semantics. Their interrelations are examined, and the preservation of all the equivalence notions by refinements is investigated.
Keywords: Petri nets with silent transitions, basic τ-equivalences, back-forth τ-bisimulation equivalences, refinement.

1 Introduction

The notion of equivalence is central in any theory of systems. It allows to compare systems taking into account particular aspects of their behaviour.

Petri nets became a popular formal model for design of concurrent and distributed systems. One of the main advantages of Petri nets is their ability for structural characterization of three fundamental features of concurrent computations: causality, nondeterminism and concurrency.

Silent transitions are transitions labelled by special *silent* action τ which represents an internal activity of a system to be modelled and it is invisible for an external observer.

Equivalences which abstract of silent actions are called *τ-equivalences* (these are labelled by the symbol 'τ' to distinguish them of relations not abstracting of silent actions). The following basic notions of τ-equivalences are known from the literature.

- *τ-trace equivalences* (they respect only protocols of behaviour of systems): interleaving ($\equiv_i^\tau$) [7], step ($\equiv_s^\tau$) [7], partial word ($\equiv_{pw}^\tau$) [10] and pomset ($\equiv_{pom}^\tau$) [7].
- *Usual τ-bisimulation equivalences* (they respect branching structure of behaviour of systems): interleaving ($\underline{\leftrightarrow}_i^\tau$) [7], step ($\underline{\leftrightarrow}_s^\tau$) [7], partial word ($\underline{\leftrightarrow}_{pw}^\tau$) [10] and pomset ($\underline{\leftrightarrow}_{pom}^\tau$) [7].

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- *ST-τ-bisimulation equivalences* (they respect the duration or maximality of events in behaviour of systems): interleaving ($\underline{\leftrightarrow}^{\tau}_{iST}$) [10], partial word ($\underline{\leftrightarrow}^{\tau}_{pwST}$) [10] and pomset ($\underline{\leftrightarrow}^{\tau}_{pomST}$) [10].
- *History preserving τ-bisimulation equivalences* (they respect the "past" or "history" of behaviour of systems): pomset ($\underline{\leftrightarrow}^{\tau}_{pomh}$) [7].
- *History preserving ST-τ-bisimulation equivalences* (they respect the "history" and the duration or maximality of events in behaviour of systems): pomset ($\underline{\leftrightarrow}^{\tau}_{pomhST}$) [7].
- *Usual branching τ-bisimulation equivalences* (they respect branching structure of behaviour of systems taking a special care for silent actions): interleaving ($\underline{\leftrightarrow}^{\tau}_{ibr}$) [7].
- *History preserving branching τ-bisimulation equivalences* (they respect "history" and branching structure of behaviour of systems taking a special care for silent actions): pomset history preserving ($\underline{\leftrightarrow}^{\tau}_{pomhbr}$) [7].
- *Isomorphism* ($\simeq$) (i.e. coincidence of systems up to renaming of their components).

Another important group of equivalences are back-forth bisimulation ones which are based on the idea that bisimulation relation do not only require systems to simulate each other behaviour in the forward direction but also when going back in history. They are closely connected with equivalences of logics with past modalities.

These equivalence notions were initially introduced in [6]. On transition systems with silent actions it was shown that back-forth variant ($\underline{\leftrightarrow}^{\tau}_{ibif}$) of interleaving τ-bisimulation equivalence coincide with $\underline{\leftrightarrow}^{\tau}_{ibr}$.

In [4] the new variants of step, partial word and pomset back-forth bisimulation equivalences were defined in the framework of event structures without silent actions.

In [8] the new idea of differentiating the kinds of back and forth simulations appeared. The set of all possible back-forth equivalence notions was proposed in interleaving, step, partial word and pomset semantics for event structures without silent actions. The new notion of τ-equivalence was proposed for event structures with silent actions: pomset back pomset forth ($\underline{\leftrightarrow}^{\tau}_{pombpomf}$) τ-bisimulation equivalence. Its coincidence with $\underline{\leftrightarrow}^{\tau}_{pomhbr}$ was proved.

To choose most appropriate behavioural viewpoint on systems to be modelled, it is very important to have a complete set of equivalence notions in all semantics and understand their interrelations. This branch of research is usually called *comparative concurrency semantics*. Treating equivalences for preservation by refinements allows one to decide which of them may be used for top-down design.

Working in the framework of Petri nets with silent transitions, in this paper we continue research of [9] and extend the set of basic notions of τ-equivalences by interleaving ST-branching τ-bisimulation one ($\underline{\leftrightarrow}^{\tau}_{iSTbr}$), pomset history preserving ST-branching τ-bisimulation one ($\underline{\leftrightarrow}^{\tau}_{pomhSTbr}$) and multi-event structure one ($\equiv^{\tau}_{mes}$). Let us note that an idea to introduce $\underline{\leftrightarrow}^{\tau}_{pomhSTbr}$ appeared initially in [8]. We complete back-forth τ-equivalences from [8] by 6 new notions in interleaving – pomset semantics. We compare all the τ-equivalences and obtain a diagram of their interelations.

In [3], SM-refinement operator for Petri nets was proposed, which "replaces" their transitions by SM-nets, a special subclass of state machine nets. We treat all the τ-equivalences for preservation by SM-refinements. We show that $\underline{\leftrightarrow}^{\tau}_{iSTbr}$, $\underline{\leftrightarrow}^{\tau}_{pomhSTbr}$ and $\equiv^{\tau}_{mes}$, i.e. all the new equivalences introduced in this paper, are preserved by SM-refinements. Thus, we have branching and conflict preserving equivalences which may be used for multilevel design. In the literature, a stability w.r.t. SM-refinements was proved only for $\underline{\leftrightarrow}^{\tau}_{pomhST}$ in [3] and for $\underline{\leftrightarrow}^{\tau}_{iST}$ in [5]. The preservation result for other ST-τ-bisimulation equivalences was proved in [10], but it was done on event structures and an other refinement operator was used. The preservation of trace τ-equivalences was not established before. Thus, our results for $\underline{\leftrightarrow}^{\tau}_{pwST}$, $\underline{\leftrightarrow}^{\tau}_{pomST}$, $\equiv^{\tau}_{pw}$ and $\equiv^{\tau}_{pom}$ are also new.

2 Basic definitions

In this section we give some basic definitions used further.

2.1 Labelled nets

Let $Act = \{a, b, \ldots\}$ be a set of *action names* or *labels*. The symbol $\tau \notin Act$ denotes a special *silent* action. We denote $Act_\tau = Act \cup \{\tau\}$.

Definition 1. *A* labelled net *is a quadruple* $N = \langle P_N, T_N, F_N, l_N \rangle$*, where:*

- $P_N = \{p, q, \ldots\}$ *is a set of* places*;*
- $T_N = \{t, u, \ldots\}$ *is a set of* transitions*;*
- $F_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbf{N}$ *is the* flow relation *with weights (*$\mathbf{N}$ *denotes a set of natural numbers);*
- $l_N : T_N \to Act_\tau$ *is a* labelling *of transitions with action names.*

Given labelled nets $N = \langle P_N, T_N, F_N, l_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', denoted by $\beta : N \simeq N'$, if:

1. β is a bijection such that $\beta(P_N) = P_{N'}$ and $\beta(T_N) = T_{N'}$;
2. $\forall p \in P_N\ \forall t \in T_N\ F_N(p, t) = F_{N'}(\beta(p), \beta(t))$ and $F_N(t, p) = F_{N'}(\beta(t), \beta(p))$;
3. $\forall t \in T_N\ l_N(t) = l_{N'}(\beta(t))$.

Labelled nets N and N' are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Given a labelled net N and some transition $t \in T_N$, the *precondition* and *postcondition* t, denoted by ${}^\bullet t$ and $t^\bullet$ respectively, are the multisets defined in such a way: $({}^\bullet t)(p) = F_N(p, t)$ and $(t^\bullet)(p) = F_N(t, p)$. Analogous definitions are introduced for places: $({}^\bullet p)(t) = F_N(t, p)$ and $(p^\bullet)(t) = F_N(p, t)$. Let ${}^\circ N = \{p \in P_N \mid {}^\bullet p = \emptyset\}$ is a set of *input* places of N and $N^\circ = \{p \in P_N \mid p^\bullet = \emptyset\}$ is a set of *output* places of N.

A labelled net N is *acyclic*, if there exist no transitions $t_0, \ldots, t_n \in T_N$ such that $t^\bullet_{i-1} \cap {}^\bullet t_i \neq \emptyset$ $(1 \leq i \leq n)$ and $t_0 = t_n$. A labelled net N is *ordinary* if $\forall p \in P_N$ ${}^\bullet p$ and $p^\bullet$ are proper sets (not multisets).

Let $N = \langle P_N, T_N, F_N, l_N \rangle$ be acyclic ordinary labelled net and $x, y \in P_N \cup T_N$. Let us introduce the following notions.

- $x \prec_N y \;\Leftrightarrow\; x F^+_N y$, where F^+_N is a transitive closure of F_N (*strict causal dependence* relation);
- $x \preceq_N y \;\Leftrightarrow\; (x \prec_N y) \vee (x = y)$ (a relation of *causal dependence*);
- $x \#_N y \;\Leftrightarrow\; \exists t, u \in T_N$ $(t \neq u,\; {}^\bullet t \cap {}^\bullet u \neq \emptyset,\; t \preceq_N x,\; u \preceq_N y)$ (a relation of *conflict*);
- $\downarrow_N x = \{y \in P_N \cup T_N \mid y \prec_N x\}$ (the set of *strict predecessors* of x).

A set $T \subseteq T_N$ is *left-closed* in N, if $\forall t \in T$ $(\downarrow_N t) \cap T_N \subseteq T$.

2.2 Marked nets

We denote the *set of all finite multisets* over a set X by $\mathcal{M}(X)$. A *marking* of a labelled net N is a multiset $M \in \mathcal{M}(P_N)$.

Definition 2. *A* marked net (net) *is a tuple* $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$*, where* $\langle P_N, T_N, F_N, l_N \rangle$ *is a labelled net and* $M_N \in \mathcal{M}(P_N)$ *is the* initial *marking.*

Given nets $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ and $N' = \langle P_{N'}, T_{N'}, F_{N'}, l_{N'}, M_{N'} \rangle$. A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', denoted by $\beta : N \simeq N'$, if $\beta : \langle P_N, T_N, F_N, l_N \rangle \simeq \langle P_{N'}, T_{N'}, F_{N'}, l_{N'} \rangle$ and $\forall p \in P_N$ $M_N(p) = M_{N'}(\beta(p))$. Nets N and N' are *isomorphic*, denoted by $N \simeq N'$, if $\exists \beta : N \simeq N'$.

Let $M \in \mathcal{M}(P_N)$ be a marking of a net N. A transition $t \in T_N$ is *firable* in M, if ${}^\bullet t \subseteq M$. If t is firable in M, firing it yields a new marking $\widetilde{M} = M - {}^\bullet t + t^\bullet$, denoted by $M \xrightarrow{t} \widetilde{M}$. $Mark(N)$ denotes a *set of all reachable* markings of a net N.

2.3 Partially ordered sets

Definition 3. *A* labelled partially ordered set (lposet) *is a triple* $\rho = \langle X, \prec, l \rangle$*, where:*

- $X = \{x, y, \ldots\}$ *is some set;*
- $\prec \subseteq X \times X$ *is a strict partial order (irreflexive transitive relation) over* X*;*
- $l : X \to Act_\tau$ *is a* labelling *function.*

Let $\rho = \langle X, \prec, l \rangle$ be lposet and $Y \subseteq X$. A *restriction* of ρ to the set Y is defined as follows: $\rho|_Y = \langle Y, \prec \cap (Y \times Y), l|_Y \rangle$.

Let $\rho = \langle X, \prec, l \rangle$ and $\rho' = \langle X', \prec', l' \rangle$ be lposets.

A mapping $\beta : X \to X'$ is a *label-preserving bijection* between ρ and ρ', denoted by $\beta : \rho \asymp \rho'$, if β is a bijection s.t. $\forall x \in X$ $l(x) = l'(\beta(x))$. We write $\rho \asymp \rho'$, if $\exists \beta : \rho \asymp \rho'$.

A mapping $\beta : X \to X'$ is a *homomorphism* between ρ and ρ', denoted by $\beta : \rho \sqsubseteq \rho'$, if $\beta : \rho \asymp \rho'$ and $\forall x, y \in X\ x \prec y \ \Rightarrow\ \beta(x) \prec' \beta(y)$. We write $\rho \sqsubseteq \rho'$, if $\exists \beta : \rho \sqsubseteq \rho'$.

A mapping $\beta : X \to X'$ is an *isomorphism* between ρ and ρ', denoted by $\beta : \rho \simeq \rho'$, if $\beta : \rho \sqsubseteq \rho'$ and $\beta^{-1} : \rho' \sqsubseteq \rho$. Lposets ρ and ρ' are *isomorphic*, denoted by $\rho \simeq \rho'$, if $\exists \beta : \rho \simeq \rho'$.

Definition 4. Partially ordered multiset (pomset) *is an isomorphism class of lposets.*

2.4 Event structures

Definition 5. *A* labelled event structure (LES) *is a quadruple* $\xi = \langle X, \prec, \#, l\rangle$, *where:*

- $X = \{x, y, \ldots\}$ *is a set of* events*;*
- $\prec \subseteq X \times X$ *is a strict partial order, a* causal dependence *relation, satisfying to the principle of* finite causes*:* $\forall x \in X\ |\downarrow x| < \infty$*;*
- $\# \subseteq X \times X$ *is an irreflexive symmetrical* conflict relation*, satisfying to the principle of* conflict heredity*:* $\forall x, y, z \in X\ x\#y \prec z \ \Rightarrow\ x\#z$*;*
- $l : X \to Act_\tau$ *is a* labelling *function.*

Let $\xi = \langle X, \prec, \#, l\rangle$ be LES and $Y \subseteq X$. A *restriction* of ξ to the set Y is defined as follows: $\xi|_Y = \langle Y, \prec \cap (Y \times Y), \# \cap (Y \times Y), l|_Y\rangle$.

Let $\xi = \langle X, \prec, \#, l\rangle$ and $\xi' = \langle X', \prec', \#', l'\rangle$ be LES's. A mapping $\beta : X \to X'$ is an *isomorphism* between ξ and ξ', denoted by $\beta : \xi \simeq \xi'$, if:

1. β is a bijection;
2. $\forall x \in X\ l(x) = l'(\beta(x))$;
3. $\forall x, y \in X\ x \prec y \ \Leftrightarrow\ \beta(x) \prec' \beta(y)$;
4. $\forall x, y \in X\ x\#y \ \Leftrightarrow\ \beta(x)\#'\beta(y)$.

LES's ξ and ξ' are *isomorphic*, denoted by $\xi \simeq \xi'$, if $\exists \beta : \xi \simeq \xi'$.

Definition 6. *A* multi-event structure (MES) *is an isomorphism class of LES's.*

2.5 C-processes

Definition 7. *A* causal net *is an acyclic ordinary labelled net* $C = \langle P_C, T_C, F_C, l_C\rangle$*, s.t.:*

1. $\forall r \in P_C\ |{}^\bullet r| \leq 1$ *and* $|r^\bullet| \leq 1$*, i.e. places are unbranched;*
2. $\forall x \in P_C \cap T_C\ |\downarrow_C x| < \infty$*, i.e. a set of causes is finite.*

On the basis of any causal net $C = \langle P_C, T_C, F_C, l_C\rangle$ one can define lposet $\rho_C = \langle T_C, \prec_N \cap (T_C \times T_C), l_C\rangle$.

The fundamental property of causal nets is: if C is a causal net, then there exists a sequence of transition firings ${}^\circ C = L_0 \stackrel{v_1}{\to} \cdots \stackrel{v_n}{\to} L_n = C^\circ$ such that $L_i \subseteq P_C$ $(0 \leq i \leq n)$, $P_C = \cup_{i=0}^n L_i$ and $T_C = \{v_1, \ldots, v_n\}$. Such a sequence is called a *full execution* of C.

Definition 8. *Given a net N and a causal net C. A mapping $\varphi : P_C \cup T_C \to P_N \cup T_N$ is an* embedding *C into N, denoted by $\varphi : C \to N$, if:*

1. *$\varphi(P_C) \in \mathcal{M}(P_N)$ and $\varphi(T_C) \in \mathcal{M}(T_N)$, i.e. sorts are preserved;*
2. *$\forall v \in T_C$ ${}^\bullet\varphi(v) = \varphi({}^\bullet v)$ and $\varphi(v)^\bullet = \varphi(v^\bullet)$, i.e. flow relation is respected;*
3. *$\forall v \in T_C$ $l_C(v) = l_N(\varphi(v))$, i.e. labelling is preserved.*

Since embeddings respect the flow relation, if ${}^\circ C \xrightarrow{v_1} \cdots \xrightarrow{v_n} C^\circ$ is a full execution of C, then $M = \varphi({}^\circ C) \xrightarrow{\varphi(v_1)} \cdots \xrightarrow{\varphi(v_n)} \varphi(C^\circ) = \widetilde{M}$ is an sequence of transition firings in N.

Definition 9. *A* firable in marking M C-process (process) *of a net N is a pair $\pi = (C, \varphi)$, where C is a causal net and $\varphi : C \to N$ is an embedding such that $M = \varphi({}^\circ C)$. A firable in M_N process is a* process *of N.*

We write $\Pi(N, M)$ for a *set of all firable* in marking M processes of a net N and $\Pi(N)$ for the *set of all processes* of a net N. The *initial* process of a net N is $\pi_N = (C_N, \varphi_N) \in \Pi(N)$, such that $T_{C_N} = \emptyset$. If $\pi \in \Pi(N, M)$, then firing of this process transforms a marking M into $\widetilde{M} = M - \varphi({}^\circ C) + \varphi(C^\circ) = \varphi(C^\circ)$, denoted by $M \xrightarrow{\pi} \widetilde{M}$.

Let $\pi = (C, \varphi)$, $\tilde{\pi} = (\widetilde{C}, \tilde{\varphi}) \in \Pi(N)$, $\hat{\pi} = (\widehat{C}, \hat{\varphi}) \in \Pi(N, \varphi(C^\circ))$. A process $\tilde{\pi}$ is an *extension* of π *by process* $\hat{\pi}$, denoted by $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, if $T_C \subseteq T_{\widetilde{C}}$ is a left-closed set in $\widetilde{C}$ and $T_{\widehat{C}} = T_{\widetilde{C}} \setminus T_C$. We write $\pi \to \tilde{\pi}$, if $\exists \hat{\pi}\ \pi \xrightarrow{\hat{\pi}} \tilde{\pi}$.

A process $\tilde{\pi}$ is an extension of a process π *by one transition*, denoted by $\pi \xrightarrow{v} \tilde{\pi}$ or $\pi \xrightarrow{a} \tilde{\pi}$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, $T_{\widehat{C}} = \{v\}$ and $l_{\widehat{C}}(v) = a$.

A process $\tilde{\pi}$ is an extension of a process π *by sequence of transitions*, denoted by $\pi \xrightarrow{\sigma} \tilde{\pi}$ or $\pi \xrightarrow{\omega} \tilde{\pi}$, if $\exists \pi_i \in \Pi(N)$ $(1 \leq i \leq n)$ $\pi \xrightarrow{v_1} \pi_1 \xrightarrow{v_2} \ldots \xrightarrow{v_n} \pi_n = \tilde{\pi}$, $\sigma = v_1 \cdots v_n$ and $l_{\widehat{C}}(\sigma) = \omega$.

A process $\tilde{\pi}$ is an extension of a process π *by multiset of transitions*, denoted by $\pi \xrightarrow{V} \tilde{\pi}$ or $\pi \xrightarrow{A} \tilde{\pi}$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, $\prec_{\widehat{C}} = \emptyset$, $T_{\widehat{C}} = V$ and $l_{\widehat{C}}(V) = A$.

2.6 O-processes

Definition 10. *An* occurrence net *is an acyclic ordinary labelled net $O = \langle P_O, T_O, F_O, l_O \rangle$, s.t.:*

1. *$\forall r \in P_O$ $|{}^\bullet r| \leq 1$, i.e. there are no backwards conflicts;*
2. *$\forall x \in P_O \cup T_O$ $\neg(x \#_O x)$, i.e. conflict relation is irreflexive;*
3. *$\forall x \in P_O \cup T_O$ $|\downarrow_O x| < \infty$, i.e. set of causes is finite.*

Let $O = \langle P_O, T_O, F_O, l_O \rangle$ be occurrence net and $N = \langle P_N, T_N, F_N, l_N, M_N \rangle$ be some net. A mapping $\psi : P_O \cup T_O \to P_N \cup T_N$ is an *embedding* O into N, notation $\psi : O \to N$, if:

1. $\psi(P_O) \in \mathcal{M}(P_N)$ and $\psi(T_O) \in \mathcal{M}(T_N)$. i.e. sorts are preserved;
2. $\forall v \in T_O$ $l_O(v) = l_N(\psi(v))$, i.e. labelling is preserved;

3. $\forall v \in T_O\ {}^\bullet\psi(v) = \psi({}^\bullet v)$ and $\psi(v)^\bullet = \psi(v^\bullet)$, i.e. flow relation is respected;
4. $\forall v, w \in T_O\ ({}^\bullet v = {}^\bullet w) \wedge (\psi(v) = \psi(w)) \Rightarrow\ v = w$, i.e. there are no "superfluous" conflicts.

Definition 11. *An* O-process *of a net N is a pair $\varpi = (O, \psi)$, where O is an occurrence net and $\psi : O \to N$ is an embedding s.t. $M_N = \psi(^\circ O)$.*

We write $\wp(N)$ for a set of *all O-processes* of a net N. The *initial* O-process of a net N coincides with its initial C-process, i.e. $\varpi_N = \pi_N$.

Let $\varpi = (O, \psi),\ \tilde{\varpi} = (\widetilde{O}, \tilde{\psi}) \in \wp(N),\ O = \langle P_O, T_O, F_O, l_O\rangle$, $\widetilde{O} = \langle P_{\widetilde{O}}, T_{\widetilde{O}}, F_{\widetilde{O}}, l_{\widetilde{O}}\rangle$. An O-process $\tilde{\varpi}$ is an *extension* of ϖ, denoted by $\varpi \to \tilde{\varpi}$, if $T_O \subseteq T_{\widetilde{O}}$ is a left-closed set in $\widetilde{O}$.

An O-process ϖ of a net N is *maximal*, if $\forall \varpi = (O, \psi)$ s.t. $\varpi \to \tilde{\varpi} :\ T_{\widetilde{O}} \setminus T_O = \emptyset$. A set of all maximal O-processes of a net N consists of the unique O-process $\varpi_{max} = (O_{max}, \psi_{max})$. In such a case an isomorphism class of occurrence net O_{max} is an *unfolding* of a net N, notation $\mathcal{U}(N)$.

Let us note that on the basis of any occurrence net O one can define LES $\xi_O = \langle T_O, \prec_O \cap (T_O \times T_O), \#_O \cap (T_O \times T_O), l_O\rangle$. Then on the basis of unfolding $\mathcal{U}(N)$ of a net N one can define MES $\mathcal{E}(N) = \xi_{\mathcal{U}(N)}$ which is an isomorphism class of LES ξ_O for $O \in \mathcal{U}(N)$.

3 Basic τ-equivalences

In this section we propose basic τ-equivalences: trace, bisimulation and conflict preserving.

3.1 τ-trace equivalences

We denote the empty string by the symbol ε.

Let $\sigma = a_1 \cdots a_n \in Act_\tau^*$. We define $vis(\sigma)$ as follows ($a \in Act_\tau$).

1. $vis(\varepsilon) = \varepsilon$;
2. $vis(\sigma a) = \begin{cases} vis(\sigma)a, & a \neq \tau; \\ vis(\sigma), & a = \tau. \end{cases}$

Definition 12. *A* visible interleaving trace *of a net N is a sequence $vis(a_1 \cdots a_n) \in Act^*$ s.t. $\pi_N \stackrel{a_1}{\to} \pi_1 \stackrel{a_2}{\to} \ldots \stackrel{a_n}{\to} \pi_n$, where $\pi_i \in \Pi(N)$ $(1 \leq i \leq n)$. We denote a set of* all visible interleaving traces *of a net N by $VisIntTraces(N)$. Two nets N and N' are* interleaving τ-trace equivalent*, denoted by $N \equiv_i^\tau N'$, if $VisIntTraces(N) = VisIntTraces(N')$.*

Let $\Sigma = A_1 \cdots A_n \in (\mathcal{M}(Act_\tau))^*$. We define $vis(\Sigma)$ as follows ($A \in \mathcal{M}(Act_\tau)$).

1. $vis(\varepsilon) = \varepsilon$;
2. $vis(\Sigma A) = \begin{cases} vis(\Sigma)(A \cap Act), & A \cap Act \neq \emptyset; \\ vis(\Sigma), & \text{otherwise.} \end{cases}$

Definition 13. *A* visible step trace *of a net* N *is a sequence* $vis(A_1 \cdots A_n) \in (\mathcal{M}(Act))^*$ *s.t.* $\pi_N \stackrel{A_1}{\to} \pi_1 \stackrel{A_2}{\to} \ldots \stackrel{A_n}{\to} \pi_n$, *where* $\pi_i \in \Pi(N)$ $(1 \leq i \leq n)$. *We denote a set of* all visible step traces *of a net* N *by* $VisStepTraces(N)$. *Two nets* N *and* N' *are* step τ-trace equivalent, *denoted by* $N \equiv_s^\tau N'$, *if* $VisStepTraces(N) = VisStepTraces(N')$.

Let $\rho = \langle X, \prec, l\rangle$ is lposet s.t. $l : X \to Act_\tau$. We denote $vis(X) = \{x \in X \mid l(x) \in Act\}$ and $vis(\rho) = \rho|_{vis(X)}$.

Definition 14. *A* visible pomset trace *of a net* N *is a pomset* $vis(\rho)$, *an isomorphism class of lposet* $vis(\rho_C)$ *for* $\pi = (C, \varphi) \in \Pi(N)$. *We denote a set of* all visible pomsets *of a net* N *by* $VisPomsets(N)$. *Two nets* N *and* N' *are* partial word τ-trace equivalent, *denoted by* $N \equiv_{pw}^\tau N'$, *if* $VisPomsets(N) \sqsubseteq VisPomsets(N')$ *and* $VisPomsets(N') \sqsubseteq VisPomsets(N)$.

Definition 15. *Two nets* N *and* N' *are* pomset τ-trace equivalent, *denoted by* $N \equiv_{pom}^\tau N'$, *if* $VisPomsets(N)$
$= VisPomsets(N')$.

3.2 τ-bisimulation equivalences

Let $C = \langle P_C, T_C, F_C, l_C\rangle$ be C-net. We denote $vis(T_C) = \{v \in T_C \mid l_C(v) \in Act\}$ and $vis(\prec_C) = \prec_C \cap (vis(T_C) \times vis(T_C))$.

Usual τ-bisimulation equivalences

Definition 16. *Let* N *and* N' *be some nets. A relation* $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ *is a* $\star$-τ-bisimulation *between* N *and* N', $\star \in$ {interleaving, step, partial word, pomset}, *denoted by* $\mathcal{R} : N \underline{\leftrightarrow}_\star^\tau N'$, $\star \in \{i, s, pw, pom\}$, *if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.
2. $(\pi, \pi') \in \mathcal{R}$, $\pi \stackrel{\hat{\pi}}{\to} \tilde{\pi}$,
   (a) $|vis(T_{\widehat{C}})| = 1$, *if* $\star = i$;
   (b) $vis(\prec_{\widehat{C}}) = \emptyset$, *if* $\star = s$;

   $\Rightarrow$ $\exists \tilde{\pi}' :$ $\pi' \stackrel{\hat{\pi}'}{\to} \tilde{\pi}'$, $(\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$ *and*

   (a) $vis(\rho_{\widehat{C'}}) \sqsubseteq vis(\rho_{\widehat{C}})$, *if* $\star = pw$;
   (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C'}})$, *if* $\star \in \{i, s, pom\}$.
3. *As item 2, but the roles of* N *and* N' *are reversed.*

Two nets N *and* N' *are* $\star$-τ-bisimulation equivalent, $\star \in$ {interleaving, step, partial word, pomset}, *denoted by* $N \underline{\leftrightarrow}_\star^\tau N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_\star^\tau N'$, $\star \in \{i, s, pw, pom\}$.

ST-τ-bisimulation equivalences

Definition 17. *An* ST-τ-process *of a net N is a pair (π_E, π_P) s.t. $\pi_E, \pi_P \in \Pi(N)$, $\pi_P \to \pi_E$ and $\forall v, w \in T_{C_E}$ $(v \prec_{C_E} w) \vee (l_{C_E}(v) = \tau) \Rightarrow v \in T_{C_P}$.*

We denote a set of *all ST-τ-processes* of a net N by $ST^\tau - \Pi(N)$. (π_N, π_N) is the *initial ST-τ-process* of a net N. Let (π_E, π_P), $(\tilde{\pi}_E, \tilde{\pi}_P) \in ST^\tau - \Pi(N)$. We write $(\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \to \tilde{\pi}_E$ and $\pi_P \to \tilde{\pi}_P$.

Definition 18. *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$ is a* $\star$-ST-τ-bisimulation *between N and N',* $\star \in$\{interleaving, partial word, pomset\}*, denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star ST} N'$, $\star \in \{i, pw, pom\}$, if:*

1. *$((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.*
2. *$((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_{C_E}) \asymp vis(\rho_{C'_E})$ and $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.*
3. *$((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \Rightarrow \exists \tilde{\beta},\ (\tilde{\pi}'_E, \tilde{\pi}'_P) : (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta,\ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$, and if $\pi_P \xrightarrow{\pi} \tilde{\pi}_E,\ \pi'_P \xrightarrow{\pi'} \tilde{\pi}'_E,\ \gamma = \tilde{\beta}|_{T_C}$, then:*
   (a) $\gamma^{-1} : vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$, if $\star = pw$;
   (b) $\gamma : vis(\rho_C) \simeq vis(\rho_{C'})$, if $\star = pom$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are $\star$-ST-τ-bisimulation equivalent*,* $\star \in$\{interleaving, partial word, pomset\}*, denoted by $N \underline{\leftrightarrow}^\tau_{\star ST} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star ST} N'$, $\star \in \{i, pw, pom\}$.*

History preserving τ-bisimulation equivalences

Definition 19. *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, is a* pomset history preserving τ-bisimulation *between N and N', denoted by $N \underline{\leftrightarrow}^\tau_{pomh} N'$, if:*

1. *$(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.*
2. *$(\pi, \pi', \beta) \in \mathcal{R} \Rightarrow \beta : vis(\rho_C) \simeq vis(\rho_{C'})$.*
3. *$(\pi, \pi', \beta) \in \mathcal{R},\ \pi \to \tilde{\pi} \Rightarrow \exists \tilde{\beta},\ \tilde{\pi}' : \pi' \to \tilde{\pi}',\ \tilde{\beta}|_{vis(T_C)} = \beta,\ (\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.*
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{pomh} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomh} N'$.*

History preserving ST-τ-bisimulation equivalences

Definition 20. *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, is a* pomset history preserving ST-τ-bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$, if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \simeq vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \ \Rightarrow\ \exists \tilde{\beta},\ (\tilde{\pi}'_E, \tilde{\pi}'_P) : (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta,\ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving ST-τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{pomhST} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$.*

Usual branching τ-bisimulation equivalences For some net N and $\pi, \tilde{\pi} \in \Pi(N)$ we write $\pi \Rightarrow \tilde{\pi}$ when $\exists \hat{\pi} = (\widehat{C}, \hat{\varphi})$ s.t. $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$ and $vis(T_{\widehat{C}}) = \emptyset$.

Definition 21. *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ is an* interleaving branching τ-bisimulation *between N and N', denoted by $N \underline{\leftrightarrow}^\tau_{ibr} N'$, if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.
2. $(\pi, \pi') \in \mathcal{R},\ \pi \xrightarrow{a} \tilde{\pi} \ \Rightarrow$
   (a) $a = \tau$ *and* $(\tilde{\pi}, \pi') \in \mathcal{R}$ *or*
   (b) $a \neq \tau$ *and* $\exists \bar{\pi}',\ \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \xrightarrow{a} \tilde{\pi}',\ (\pi, \bar{\pi}') \in \mathcal{R},\ (\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$.
3. *As item 2, but the roles of N and N' are reversed.*

Two nets N and N' are interleaving branching τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{ibr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{ibr} N'$.*

History preserving branching τ-bisimulation equivalences

Definition 22. *Let N and N' be some nets. A relation $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : T_C \to T_{C'},\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, is a* pomset history preserving branching τ-bisimulation *between N and N', denoted by $N \underline{\leftrightarrow}^\tau_{pomhbr} N'$, if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.
2. $(\pi, \pi', \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_C) \simeq vis(\rho_{C'})$.
3. $(\pi, \pi', \beta) \in \mathcal{R},\ \pi \to \tilde{\pi} \ \Rightarrow$
   (a) $(\tilde{\pi}, \pi', \beta) \in \mathcal{R}$ *or*
   (b) $\exists \tilde{\beta},\ \bar{\pi}',\ \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \to \tilde{\pi}',\ \tilde{\beta}|_{vis(T_C)} = \beta,\ (\pi, \bar{\pi}', \beta) \in \mathcal{R},\ (\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving branching τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{pomhbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhbr} N'$.*

ST-branching τ-bisimulation equivalences Let $(\pi_E, \pi_P),\ (\tilde{\pi}_E, \tilde{\pi}_P) \in ST^\tau - \Pi(N)$. We write $(\pi_E, \pi_P) \Rightarrow (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \Rightarrow \tilde{\pi}_E$ and $\pi_P \Rightarrow \tilde{\pi}_P$.

Definition 23. *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$ is an* interleaving ST-branching τ-bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{iSTbr} N'$, if:*

1. *$((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.*
2. *$((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \asymp vis(\rho_{C'_E})$ and $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.*
3. *$((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)\ \Rightarrow$*
   (a) $((\tilde{\pi}_E, \tilde{\pi}_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ or
   (b) $\exists \tilde{\beta},\ (\bar{\pi}'_E, \bar{\pi}'_P),\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \Rightarrow (\bar{\pi}'_E, \bar{\pi}'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta$,
   $((\pi_E, \pi_P), (\bar{\pi}'_E, \bar{\pi}'_P), \beta) \in \mathcal{R},\ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are interleaving ST-branching τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{iSTbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{iSTbr} N'$.*

History preserving ST-branching τ-bisimulation equivalences

Definition 24. *Let N and N' be some nets. A relation $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, where $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$ is a* pomset history preserving ST-branching τ-bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$, if:*

1. *$((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.*
2. *$((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \simeq vis(\rho_{C'_E})$ and $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.*
3. *$((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)\ \Rightarrow$*
   (a) $((\tilde{\pi}_E, \tilde{\pi}_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ or
   (b) $\exists \tilde{\beta},\ (\bar{\pi}'_E, \bar{\pi}'_P),\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \Rightarrow (\bar{\pi}'_E, \bar{\pi}'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta$,
   $((\pi_E, \pi_P), (\bar{\pi}'_E, \bar{\pi}'_P), \beta) \in \mathcal{R},\ ((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

Two nets N and N' are pomset history preserving ST-branching τ-bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$.*

3.3 Conflict preserving τ-equivalences

Let $\xi = \langle X, \prec, \#, l \rangle$ be a LES s.t. $l : X \to Act_\tau$. We denote $vis(X) = \{x \in X \mid l(x) \in Act\}$ and $vis(\xi) = \xi|_{vis(X)}$.

Definition 25. *Two nets N and N' are* MES-τ-conflict preserving equivalent*, denoted by $N \equiv^\tau_{mes} N'$, if $vis(\mathcal{E}(N)) = vis(\mathcal{E}(N'))$.*

4 Back-forth τ-bisimulation equivalences

In this section we propose back-forth τ-bisimulation equivalences.

Definition 26. *A* sequential run *of a net* N *is a pair* (π, σ)*, where:*

- *a process* $\pi \in \Pi(N)$ *contains the information about causal dependencies of transitions which brought to this state;*
- *a sequence* $\sigma \in T_C^*$ *s.t.* $\pi_N \xrightarrow{\sigma} \pi$*, contains the information about the order in which the transitions occur which brought to this state.*

Let us denote the *set of all sequential runs* of a net N by $Runs(N)$.

The *initial* sequential run of a net N is a pair (π_N, ε). Let (π, σ), $(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$. We write $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$, $\exists \hat{\sigma} \in T^*_{\widetilde{C}}\ \pi \xrightarrow{\hat{\sigma}} \tilde{\pi}$ and $\tilde{\sigma} = \sigma\hat{\sigma}$.

Definition 27. *Let* N *and* N' *be some nets. A relation* $\mathcal{R} \subseteq Runs(N) \times Runs(N')$ *is a* $\star$-back $\star\star$-forth τ-bisimulation *between* N *and* N', $\star$, $\star\star \in${interleaving, step, partial word, pomset}*, denoted by* $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom\}$*, if:*

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$
   - *(back)*
     $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$,
     (a) $|vis(T_{\widehat{C}})| = 1$*, if* $\star = i$*;*
     (b) $vis(\prec_{\widehat{C}}) = \emptyset$*, if* $\star = s$*;*

     $\Rightarrow\ \exists(\tilde{\pi}', \tilde{\sigma}') :\ (\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma')$, $((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*
     (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$*, if* $\star = pw$*;*
     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$*, if* $\star \in \{i, s, pom\}$*;*
   - *(forth)*
     $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$,
     (a) $|vis(T_{\widehat{C}})| = 1$*, if* $\star\star = i$*;*
     (b) $vis(\prec_{\widehat{C}}) = \emptyset$*, if* $\star\star = s$*;*

     $\Rightarrow\ \exists(\tilde{\pi}', \tilde{\sigma}') :\ (\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}')$, $((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*
     (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$*, if* $\star\star = pw$*;*
     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$*, if* $\star\star \in \{i, s, pom\}$.
3. *As item 2, but the roles of* N *and* N' *are reversed.*

Two nets N *and* N' *are* $\star$-back $\star\star$-forth τ-bisimulation equivalent, $\star, \star\star \in$ {interleaving, step, partial word, pomset}*, denoted by* $N \underline{\leftrightarrow}^{\tau}_{\star b \star\star f} N'$*, if* $\exists \mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom\}$.

5 Interrelations of the τ-equivalences

Let us consider interrelations of all the introduced τ-equivalences.

Proposition 1. *Let $\star \in \{i, s, pw, pom\}$. For nets N and N':*

1. $N \underline{\leftrightarrow}^{\tau}_{pwb\star f} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{pomb\star f} N'$;
2. $N \underline{\leftrightarrow}^{\tau}_{\star bif} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{\star b\star f} N'$;
3. $N \underline{\leftrightarrow}^{\tau}_{ibif} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{ibr} N'$ *[6]*;
4. $N \underline{\leftrightarrow}^{\tau}_{pombpomf} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$ *[8]*;
5. $N \underline{\leftrightarrow}^{\tau}_{iSTbr} N' \;\Rightarrow\; N \underline{\leftrightarrow}^{\tau}_{ibsf} N'$.

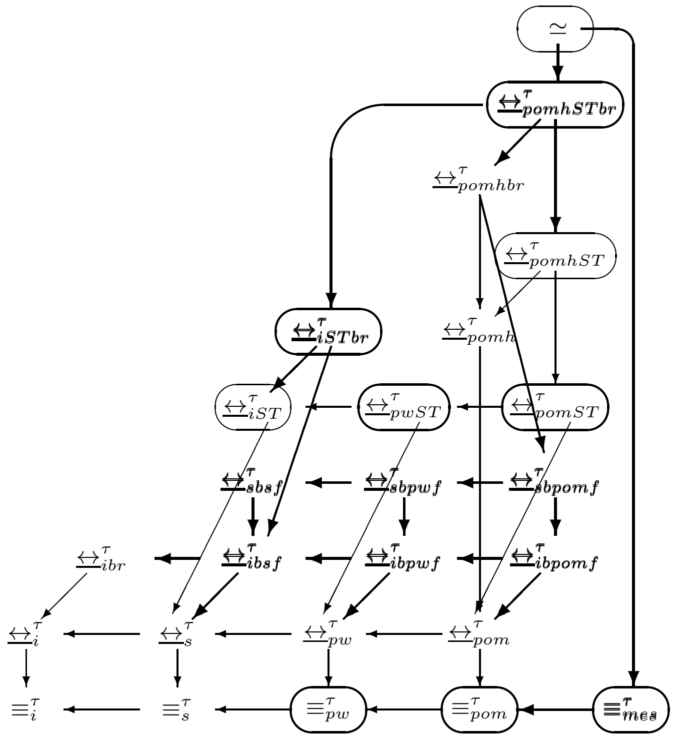

Fig. 1. Interrelations of the τ-equivalences and their preservation by SM-refinements

In the following, the symbol '_' will denote an empty alternative, and signs that equivalences subscribed by it are considered as that of without any subscription.

Theorem 1. *Let* $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$ *and* $\star, \star\star \in \{\_, i, s, pw, pom, iST, pwST,$ $pomST, pomh, pomhST, ibr, pomhbr, iSTbr, pomhSTbr, mes, ibsf, ibpwf,$

ibpomf, *sbsf*, *sbpwf*, *sbpomf*}. *For nets* N *and* N' : $N \leftrightarrow_\star N' \Rightarrow N \twoheadleftrightarrow_{\star\star} N'$ *iff there exists a directed path from* $\leftrightarrow_\star$ *to* $\twoheadleftrightarrow_{\star\star}$ *in the graph in Figure 1.*

Proof. ($\Leftarrow$) By Proposition 28 and the definitions of the equivalences.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 1 is proved by the following examples.

- In Figure 2(a) $N\underline{\leftrightarrow}^\tau_{ibr}N'$, but $N \not\equiv^\tau_s N'$, since only in the net N' actions a and b cannot happen concurrently.
- In Figure 2(c) $N\underline{\leftrightarrow}^\tau_{iSTbr}N'$, but $N \not\equiv^\tau_{pw} N'$, since for the pomset corresponding to the net N there is no even less sequential pomset in N'.
- In Figure 2(b) $N\underline{\leftrightarrow}^\tau_{pwST}N'$, but $N \not\equiv^\tau_{pom} N'$, since only in the net N' action b can depend on action a.
- In Figure 4(a) $N \equiv^\tau_{mes} N'$, but $N\underline{\not\leftrightarrow}^\tau_i N'$, since only in the net N' action τ can happen so that in the corresponding initial state of the net N action a cannot happen.
- In Figure 3(a) $N\underline{\leftrightarrow}^\tau_{pom}N'$, but $N\underline{\not\leftrightarrow}^\tau_{iST}N'$, since only in the net N' action a can start so that no action b can begin to work until finishing a.
- In Figure 3(b) $N\underline{\leftrightarrow}^\tau_{pomST}N'$, but $N\underline{\not\leftrightarrow}^\tau_{pomh}N'$, since only in the net N' after action a action b can happen so that action c must depend on a.
- In Figure 4(b) $N\underline{\leftrightarrow}^\tau_{pomh}N'$, but $N\underline{\not\leftrightarrow}^\tau_{iST}N'$, since only in the net N' action a can start so that the action b can never occur.
- In Figure 4(c) $N\underline{\leftrightarrow}^\tau_{pomhST}N'$, but $N\underline{\not\leftrightarrow}^\tau_{ibr}N'$, since in the net N' an action a can happen so that it will be simulated by sequence of actions τa in N. Then the state of the net N reached after τ must be related with the initial state of a net N, but in such a case the occurrence of action b from the initial state of N' cannot be imitated from the corresponding state of N.
- In Figure 4(e) $N\underline{\leftrightarrow}^\tau_{pomhbr}N'$, but $N\underline{\not\leftrightarrow}^\tau_{iST}N'$, since in the net N' an action c may start so that during work of the corresponding action c in the net N an action a may happen in such a way that the action b never occur.
- In Figure 3(c) $N\underline{\leftrightarrow}^\tau_{pomhSTbr}N'$, but $N \not\equiv^\tau_{mes} N'$, since only the MES corresponding to the net N' has two conflict actions a.
- In Figure 3(d) $N \equiv^\tau_{mes} N'$, but $N \not\simeq N'$, since unfireable transitions of the nets N and N' are labelled by different actions (a and b).
- In Figure 2(c) $N\underline{\leftrightarrow}^\tau_{sbsf}N'$, but $N \not\equiv^\tau_{pw} N'$.
- In Figure 4(d) $N\underline{\leftrightarrow}^\tau_{sbpwf}N'$, but $N \not\equiv^\tau_{pom} N'$.
- In Figure 3(a) $N\underline{\leftrightarrow}^\tau_{ibpomf}N'$, but $N\underline{\not\leftrightarrow}^\tau_{sbsf}N'$.
- In Figure 2(b) $N\underline{\leftrightarrow}^\tau_{iSTbr}N'$, but $N\underline{\not\leftrightarrow}^\tau_{sbsf}N'$. □

In Figure 1, the new equivalence notions and new interrelations are printed in bold font.

6 Transition refinement

In this section we treat the considered τ-equivalences for preservation by transition refinements.

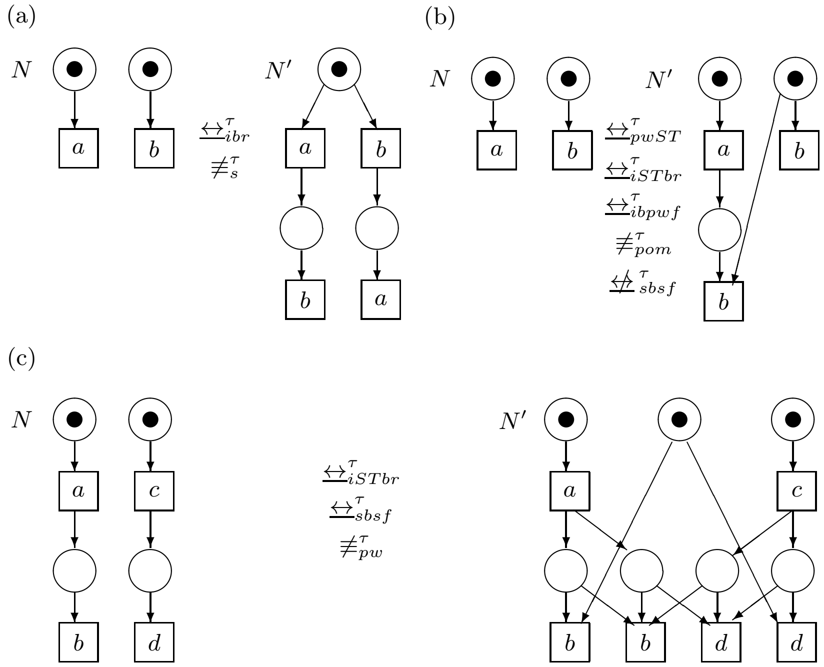

Fig. 2. Examples of the τ-equivalences

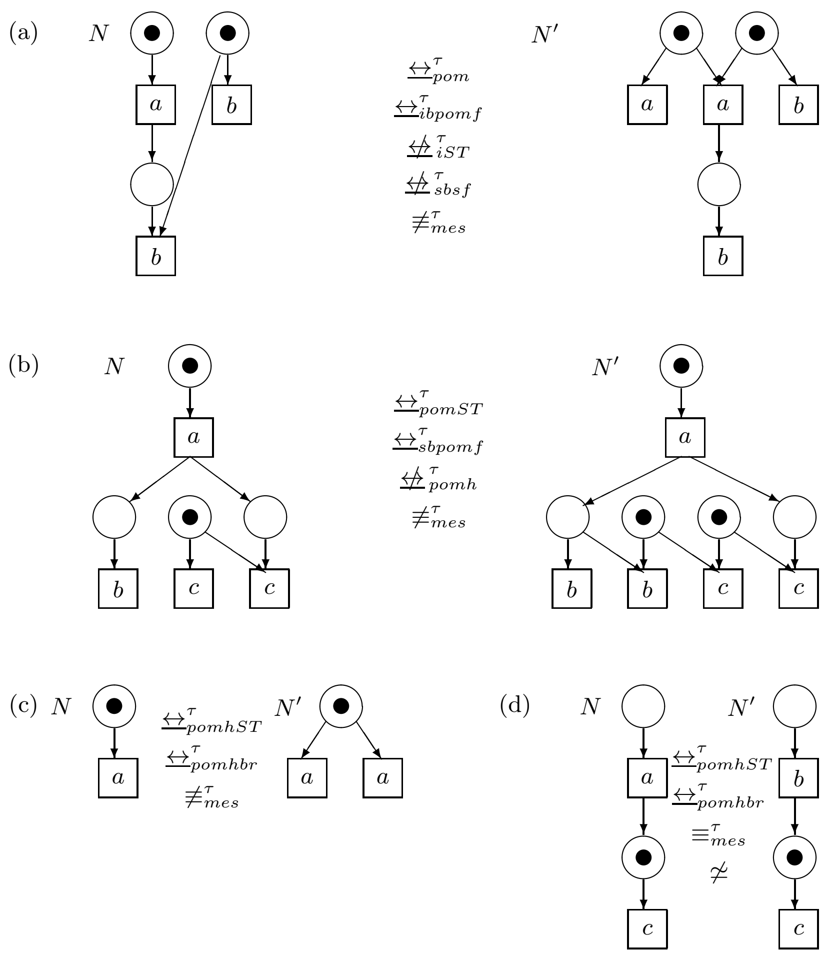

Fig. 3. Examples of the τ-equivalences (continued)

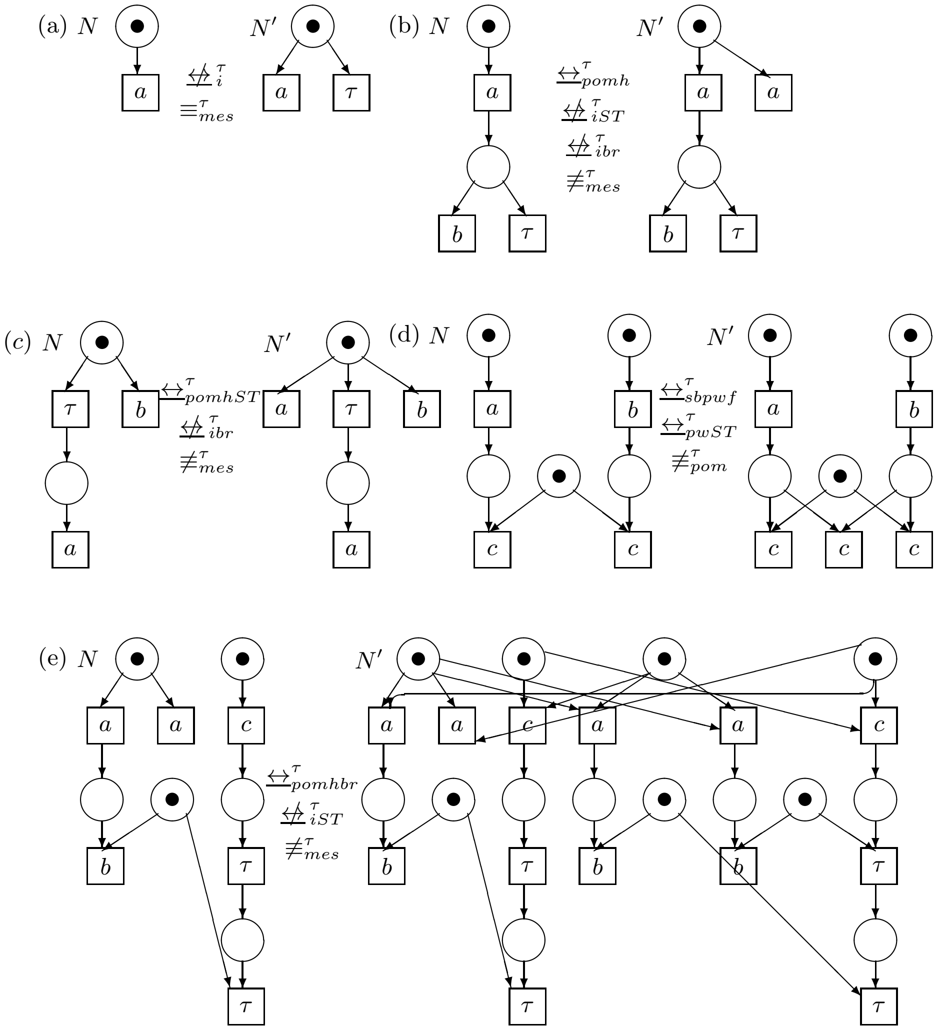

Fig. 4. Examples of the τ-equivalences (continued 2)

Definition 28. *An* SM-net *is a net* $D = \langle P_D, T_D, F_D, l_D, M_D\rangle$ *s.t.:*

1. $\forall t \in T_D\ |{}^\bullet t| = |t^\bullet| = 1$, *i.e. each transition has exactly one input and one output place;*
2. $\exists p_{in},\ p_{out} \in P_D$ *s.t.* $p_{in} \neq p_{out}$ *and* ${}^\circ D = \{p_{in}\},\ D^\circ = \{p_{out}\}$, *i.e. net D has unique input and unique output place.*
3. $M_D = \{p_{in}\}$, *i.e. at the beginning there is unique token in* p_{in}.

Definition 29. *Let* $N = \langle P_N, T_N, F_N, l_N, M_N\rangle$ *be some net,* $a \in l_N(T_N)$ *and* $D = \langle P_D, T_D, F_D, l_D, M_D\rangle$ *be SM-net. An* SM-refinement*, denoted by* $ref(N, a, D)$, *is a net* $\overline{N} = \langle P_{\overline{N}}, T_{\overline{N}}, F_{\overline{N}}, l_{\overline{N}}, M_{\overline{N}}\rangle$, *where:*

- $P_{\overline{N}} = P_N \cup \{\langle p, u\rangle \mid p \in P_D \setminus \{p_{in}, p_{out}\},\ u \in l_N^{-1}(a)\}$;
- $T_{\overline{N}} = (T_N \setminus l_N^{-1}(a)) \cup \{\langle t, u\rangle \mid t \in T_D,\ u \in l_N^{-1}(a)\}$;
- $F_{\overline{N}}(\bar{x}, \bar{y}) = \begin{cases} F_N(\bar{x}, \bar{y}),\ \bar{x}, \bar{y} \in P_N \cup (T_N \setminus l_N^{-1}(a)); \\ F_D(x, y),\ \bar{x} = \langle x, u\rangle,\ \bar{y} = \langle y, u\rangle,\ u \in l_N^{-1}(a); \\ F_N(\bar{x}, u),\ \bar{y} = \langle y, u\rangle,\ \bar{x} \in {}^\bullet u,\ u \in l_N^{-1}(a),\ y \in p_{in}^\bullet; \\ F_N(u, \bar{y}),\ \bar{x} = \langle x, u\rangle,\ \bar{y} \in {}^\bullet u,\ u \in l_N^{-1}(a),\ x \in {}^\bullet p_{out}; \\ 0, \qquad \textit{otherwise}; \end{cases}$
- $l_{\overline{N}}(\bar{u}) = \begin{cases} l_N(\bar{u}),\ \bar{u} \in T_N \setminus l_N^{-1}(a); \\ l_D(t),\ \bar{u} = \langle t, u\rangle,\ t \in T_D,\ u \in l_N^{-1}(a); \end{cases}$
- $M_{\overline{N}}(p) = \begin{cases} M_N(p),\ p \in P_N; \\ 0, \qquad \textit{otherwise.} \end{cases}$

An equivalence is *preserved by refinements*, if equivalent nets remain equivalent after applying any refinement operator to them.

Theorem 2. *Let* $\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$ *and* $\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr, iSTbr, pomhSTbr, pomhbr, mes, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. *For nets* $N,\ N'$ *s.t.* $a \in l_N(T_N) \cap l_{N'}(T_{N'})$ *and SM-net* $D:\ N \leftrightarrow_\star N' \ \Rightarrow\ ref(N, a, D) \leftrightarrow_\star ref(N', a, D)$ *iff the equivalence* $\leftrightarrow_\star$ *is in oval in Figure 1.*

7 Conclusion

In this paper, we supplemented by new ones and examined a group of basic τ-equivalences and back-forth τ-bisimulation equivalences. We also compared them on the whole class of Petri nets as well as on subclass of sequential nets. All the considered τ-equivalences were checked for preservation by SM-refinements.

Further research may consist in the investigation of τ-variants of place bisimulation equivalences [2] which are used for effective semantically correct reduction of nets. In [1] a notion of interleaving place τ-bisimulation equivalence was proposed, and its usefulness for simplification of concurrent systems was demonstrated.

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