S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 16, стр. 826–862 (2019) УДК 519.217.2, 519.681.2, 519.681.3
DOI 10.33048/semi.2019.16.055 MSC 60J27, 60K15, 68Q85, 68Q87

LOGICAL CHARACTERIZATION OF FLUID EQUIVALENCES

I.V. TARASYUK, P. BUCHHOLZ

Abstract. We investigate fluid equivalences that allow one to compare and reduce behaviour of labeled fluid stochastic Petri nets (LFSPNs) with a single continuous place while preserving their discrete and continuous properties. We propose a linear-time relation of fluid trace equivalence and its branching-time counterpart, fluid bisimulation equivalence. Both fluid relations take into account the essential features of the LFSPNs behaviour, such as *functional activity*, *stochastic timing* and *fluid flow*. We consider the LFSPNs whose continuous markings have no influence to the discrete ones, i.e. every discrete marking determines completely both the set of enabled transitions, their firing rates and the fluid flow rates of the incoming and outgoing arcs for each continuous place. Moreover, we require that the discrete part of the LFSPNs should be continuous time stochastic Petri nets. The underlying stochastic model for the discrete part of the LFSPNs is continuous time Markov chains (CTMCs). The performance analysis of the continuous part of LFSPNs is accomplished via the associated stochastic fluid models (SFMs). We characterize logically fluid trace and bisimulation equivalences with two novel fluid modal logics HML_{flt} and HML_{flb}, constructed on the basis of the well-known Hennessy-Milner Logic (HML). These characterizations guarantee that two LFSPNs are fluid (trace or bisimulation) equivalent iff they satisfy the same formulas of the respective logic, i.e. they are logically equivalent. The results imply operational characterizations of the logical equivalences.

Keywords: labeled fluid stochastic Petri net, continuous time stochastic Petri net, continuous time Markov chain, stochastic fluid model, transient and stationary behaviour, fluid trace and bisimulation equivalences, fluid modal logic, logical and operational characterizations.

Tarasyuk I.V., Buchholz P., Logical characterization of fluid equivalences.

The work was supported in part by Deutsche Forschungsgemeinschaft (DFG) under grant BE 1267/14-1.
Received May, 10, 2018, published June, 11, 2019.

1. Introduction

An important scientific problem that has been often addressed in the last decades is the design and analysis of parallel systems, which takes into account both qualitative (functional) and quantitative (timed, probabilistic, stochastic) features of their behaviour. The main goal of the research on this topic is the development of models and methods respecting performance requirements for concurrent and distributed systems with time constraints (such as deterministic, nondeterministic and stochastic time delays) to construct, validate and optimize the performability of realistic large-scale applications: computing systems, networks and software, controllers for industrial devices, manufacturing lines, vehicle, aircraft and transportation engines. A fruitful approach to achieving progress in this direction appeared to be a combined application of the theories of Petri nets, stochastic processes and fluid flow systems to the specification and analysis of such time-dependent systems with inherent behavioural stochasticity [51].

1.1. **Fluid stochastic Petri nets.**

In the past, many extensions of stochastic Petri nets (SPNs) [64, 62, 63, 60, 61, 12, 13] have been developed to specify, model, simulate and analyze some particular classes of systems, such as computer systems, communication networks or manufacturing plants. These new formalisms have been constructed as a response to the needs for more expressive power in describing real-world systems, and to the requirements for compact models and efficient analysis techniques. One of the extensions are fluid stochastic Petri nets (FSPNs), capable of modeling hybrid systems that combine continuous state variables, corresponding to the fluid levels, with discrete state variables, specifying the token numbers. The continuous part of the FSPNs allows one to represent the fluid level in continuous places and fluid flow along continuous arcs. This part can naturally describe continuous variables in physical systems whose behaviour is commonly represented by differential equations. Continuous variables may also be used to describe a macroscopic view of discrete items that appear in large populations, e.g., packets in a computer network, molecules in a chemical reaction or people in a crowd. The discrete part of an FSPN is essentially its underlying SPN, obtained from the FSPN by removing all the fluid-related continuous elements. This part usually models the discrete control of the continuous process. The control may demonstrate some stochastic behavior that captures uncertainty about the detailed system behavior.

FSPNs have been proposed in [76, 36, 85] to model stochastic fluid flow systems [50, 46, 56]. To analyze FSPNs, simulation, numerical and matrix-geometric methods are widely used [54, 37, 25, 47, 48, 43, 44, 55, 49]. The major problem of FSPNs is the high complexity of computing their solution, resulting in huge memory and time requirements while analyzing realistic models. A positive feature of the FSPN formalism is that it hides from a modeler the technical difficulties with solving differential equations for the underlying stochastic processes and that it unifies in one framework the evolution equations for the discrete and continuous parts of systems.

For every FSPN, the discrete part of its marking is determined by the natural number of tokens contained in the discrete places. The continuous places of an FSPN are associated with the non-negative real-valued fluid levels that determine the continuous part of the FSPN marking. Thus, FSPNs have a hybrid (discrete-continuous) state space. The discrete part of every hybrid marking of FSPNs is called discrete marking while the continuous part is called continuous marking. The discrete part of each hybrid marking has an influence on the continuous part. For more general FSPNs, the reverse dependence is possible as well. As a basic model for constructing labeled FSPNs (LFSPNs), we consider only those FSPNs in which the continuous parts of markings have no influence on the discrete ones, i.e. such that every discrete part determines completely both the set of enabled transitions and the rates of incoming and outgoing arcs for each continuous place [43, 49]. We also require that the discrete part of LFSPNs should be labeled continuous time stochastic Petri nets (CTSPNs) [62, 60, 61, 12], to simplify the definitions of the behavioural equivalences for LFSPNs, as explained in the next subsection.

1.2. **Fluid equivalences.** In this paper, we investigate the behavioural relations of fluid trace and bisimulation equivalences that are useful for the comparison and reduction of the behaviour of LFSPNs with a single continuous place, since these relations preserve the functionality and performability of their discrete and continuous parts. The mentioned fluid equivalences have not been considered in the literature until they have been originally constructed by the authors in [74, 75]. Those papers also contain a survey of the non-behavioural (not respecting the action names) equivalence notions that have been defined on the related models. The models include Fluid Process Algebra (FPA) [78, 79], Fluid Extended Process Algebra (FEPA) [80, 57], heterogenous systems specified by ordinary differential equations (ODEs) [81], chemical reaction networks (CRNs) [30], Intermediate Drift Oriented Language (IDOL) [31] and product form queueing networks (QNs) [3]. The most recent results on this subject are forward and backward equivalences (FE and BE) on the polynomial ODE systems [32] and polynomial dynamical systems (PDSs) [77], approximate back and forth differential equivalences (ε-BDE and ε-FDE) for polynomial initial value problem (PIVP) over the ODE variables [34], syntactic Markovian bisimulation (SMB) equivalence on CRNs with stochastic CTMC-based semantics [33, 77], and also $\mathcal{L}$-bisimulation equivalence on the polynomials in the variables for systems of polynomial ODEs [27]. Again, all the mentioned relations are not traditional behavioural equivalences, since FE, BE, ε-BDE, ε-FDE and $\mathcal{L}$-bisimulation are defined for the ODE system specifications that include no action symbols while SMB considers species instead of actions.

The definitions of the fluid equivalences should be given at the level of LFSPNs, but they must use the transition rates of the extracted CTMC. These rates cannot be easily (i.e. with a simple expression) defined at the level of more general LFSPNs, whose discrete part is labeled GSPNs. In addition, the action labels of immediate transitions are lost and their individual probabilities are redistributed while GSPNs are transformed into CTSPNs. The individual probabilities of immediate transitions are "dissolved" in the total transition rates between tangible states when vanishing states are eliminated from SMCs while reducing them to CTMCs. Therefore, to make the definition of the fluid equivalences less intricate and complex, we have decided to consider only LFSPNs with labeled CTSPNs as their discrete part. Then the underlying stochastic process of the discrete part of LFSPNs will be that of CTSPNs, i.e. CTMCs.

First, we consider a linear-time relation of *fluid trace equivalence* on LFSPNs. Linear-time equivalences, unlike branching-time ones, do not respect the points of choice among several alternative continuations of the system's behavior. We require that fluid trace equivalence on discrete markings of two LFSPNs should be a standard (strong) Markovian trace equivalence. Moreover, the average sojourn times in (or the exit rates from) the respective discrete markings should be the same. Finally, for the two equivalent LFSPNs, the cumulative execution probabilities of all the paths corresponding to a particular sequence of actions, together with a concrete sequence of the average sojourn times (exit rates), should be equal. Therefore, our definition of the trace equivalence on the discrete markings of LFSPNs is similar to that of ordinary (that with the absolute time counter or with the countdown timer) Markovian trace equivalence [83] on transition-labeled CTMCs. Ordinary Markovian trace equivalence and its variants from [83] have been later investigated and enhanced on interactive Markov chains (IMCs) in [84], on sequential and concurrent Markovian process calculi SMPC and CMPC in [14, 18, 15, 16, 19], on Uniform Labeled Transition Systems (ULTraS) in [21, 22, 17], on continuous time Markov decision processes (CTMDPs) in [66] and on Markov automata (MAs) in [67]. As for the continuous markings of the two LFSPNs, we further select the paths with the same extracted action sequence and the same sequence of the extracted average sojourn times (exit rates) by counting the execution probabilities only of those paths additionally having the same sequence of extracted potential fluid flow rates of the respective continuous places (we assume that each compared LFSPN has only one continuous place) in the corresponding discrete markings.

Second, we consider a branching-time relation of *fluid bisimulation equivalence* on LFSPNs. We prove that it is strictly stronger than fluid trace equivalence, i.e. the former relation generally makes less identifications among the compared LFSPNs than the latter. We require the fluid bisimulation on the discrete markings of two LFSPNs to be a standard (strong) Markovian bisimulation. Thus, our definition of the bisimulation equivalence on the discrete markings of LFSPNs is similar to that of the performance bisimulation equivalences [28, 29] on labeled CTSPNs and labeled generalized SPNs (GSPNs) [60, 61, 12, 13], as well as the strong equivalence from [53] on stochastic process algebra PEPA. All these relations belong to the family of Markovian bisimulation equivalences, investigated on sequential and concurrent Markovian process calculi SMPC and CMPC in [14, 18, 15, 16, 19], as well as on Uniform Labeled Transition Systems (ULTraS) in [21, 22, 17]. As for the continuous markings, we require that, for every pair of the Markovian bisimilar discrete markings, the fluid flow rate of the continuous place in the first LFSPN should coincide with that of the continuous place in the second LFSPN (again, we compare only LFSPNs with a single continuous place each).

1.3. **Logical characterization.**

A characterization of equivalences via modal logics is used to change the operational reasoning on systems behaviour by the logical one that is more appropriate for verification. Moreover, such an interpretation elucidates the nature of the equivalences, defined in an operational manner. It is generally accepted that the natural and nice modal characterization of a behavioural equivalence justifies its relevance. On the other hand, we get an operational characterization of logical equivalences. The importance of modal logical characterization for behavioural equivalences has been explained in [1], in particular, the resulting capabilities to express *distinguishing formulas* for automatic verification of systems and *characteristic formulas* for the equivalence classes of processes [2], to demonstrate *finitariness and algebraicity* of behavioural preorders, as well as to give a *testing interpretation* of bisimulation equivalence [59]. Logical characterization of bisimulation equivalence guarantees that the validity of all logical formulas is preserved while quotienting (by the equivalence) the state space before model checking, thereby simplifying verification of behavioural properties [9, 69, 24].

In the literature, several logical characterizations of stochastic and Markovian equivalences have been proposed. In [38, 39], the characterization of strong equivalence has been presented with the logic PML_{μ}, which is a stochastic extension of Probabilistic Modal Logic (PML) [59] on probabilistic transitions systems to the stochastic process algebra PEPA [53]. In [45], a branching time temporal logic has been described which is an extension of Continuous Stochastic Logic (CSL) [6] on CTMCs to a wide class of SFMs. The CSL-based logical characterizations of various stochastic bisimulation equivalences have been reported in [8, 9, 10, 69, 11, 24] on labeled CTMCs, in [40] on labeled continuous time Markov processes (CTMPs), in [41] on analytic spaces, in [7] on labeled Markov reward models (MRMs) and in [68] on continuous time Markov decision processes (CTMDPs). In [65], simulation, bisimulation and simulation distance on semi-Markov decision processes have been characterized via timed Markovian logic (TML). In [18, 15], on sequential and concurrent Markovian process calculi SMPC (MPC) and CMPC, the logical characterizations of Markovian trace and bisimulation equivalences have been accomplished with the modal logics HML_{MTr} and HML_{MB}, based on Hennessy-Milner Logic (HML) [52]. In [19], on (sequential) Markovian process calculus MPC, the characterizations of Markovian trace and bisimulation equivalences have been constructed with the HML-based modal logics HML_{NPMTr} and HML_{MB}.

The main result of the paper is that we provide fluid trace and bisimulation equivalences with the logical characterizations, accomplished via formulas of the specially constructed novel fluid modal logics HML_{flt} and HML_{flb}, respectively. The new logics are based on Hennessy-Milner Logic (HML) [52]. The logical characterizations guarantee that two LFSPNs are fluid (trace or bisimulation) equivalent iff they satisfy the same formulas of the respective fluid modal logic, i.e. they are logically equivalent. Thus, instead of comparing LFSPNs operationally, one may only check the corresponding satisfaction relation. This provides one with the possibility for logical reasoning on fluid equivalences for LFSPNs. Such an approach is often more convenient for the purpose of verification. The obtained results may also be interpreted as operational characterizations of the corresponding logical equivalences. We have also explored how to adopt (if possible) the testing interpretations of probabilistic and Markovian equivalences (related to their logical characterizations) for fluid trace and bisimulation equivalences that are standardly defined in the operational manner.

The fluid modal logic HML_{flt} is used to characterize fluid trace equivalence. Therefore, the interpretation function of the logic has an additional argument, which is the sequence of the potential fluid flow rates for the single continuous place of an LFSPN (remember that in the definition of fluid trace equivalence we compare only LFSPNs, each having exactly one continuous place). In HML_{flt}, one can express the properties like “the execution probability of a sequence of actions starting from a state, with given average sojourn times and potential fluid flow rates in the initial, intermediate and final states, is equal to a particular value”.

The fluid modal logic HML_{flb} is intended to characterize fluid bisimulation equivalence. For this purpose, the logic has a new modality, decorated with the potential fluid flow rate value for the single continuous place of an LFSPN (again, remember that in the definition of fluid bisimulation equivalence we consider only

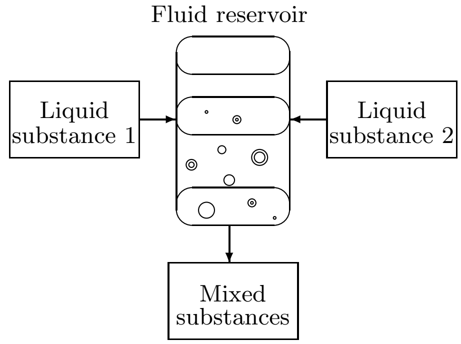

FIG. 1. The diagram of the production line

LFSPNs, each having a single continuous place). The resulting formula (i.e. the new modality with the flow rate value) is used to check whether the potential fluid flow rate in a discrete marking of an LFSPN coincides with a certain value, the fact that corresponds to a condition from the fluid bisimulation definition. Thus, HML_{flb} is able to describe the properties such as “an action can be executed with a given minimal rate in a state with a given potential fluid flow rate”.

Example 1. *For a production line in a food processing or a chemicals plant, we can verify in HML_{flt} the* probability *that the first liquid substance fills (this is specified by the action f_1) the fluid reservoir with the potential flow rate r_1 during the exponentially distributed time period with the average s_1;* then *the second liquid substance fills (the action f_2) the reservoir with the potential flow rate r_2 during the exponentially distributed time period with the average s_2;* finally*, the reservoir is emptied with the potential flow rate r_3 for the exponentially distributed time period with the average s_3.*

For the production line mentioned above, we can verify in HML_{flb} the validity *that the first liquid substance fills (the action f_1) the fluid reservoir with the potential flow rate r_1 during the exponentially distributed time period with the minimal rate λ_1* or *the second liquid substance fills (the action f_2) the reservoir with the same potential flow rate r_1 during the exponentially distributed time period with the minimal rate λ_2. Note that disjunction in HML_{flb} can be defined standardly, i.e. via conjunction and negation.*

The diagram of the production line is depicted in Figure 1.

1.4. **Previous works and outline of the paper.** The previous results on the fluid equivalences can be found in [74, 75], where we have proposed a class of LFSPNs, for which we have defined fluid trace and bisimulation equivalences and investigated their interrelations. We have shown that fluid trace equivalence preserves average potential fluid change volume for the transition sequences of every certain length. We have proved that fluid bisimulation equivalence preserves the aggregated (by such a bisimulation) probability functions and therefore guarantees identity of the discrete and continuous performance measures. Moreover, we have used fluid bisimulation equivalence to simplify the qualitative and quantitative analysis of LFSPNs by means of quotienting (by the equivalence) the discrete reachability graph, underlying CTMC and associated SFM. The present paper extends those works with the novel logical characterizations of fluid trace and bisimulation equivalences via two original fluid modal logics HML_{flt} and HML_{flb} that take into account the fluid flow in linear and branching time semantics, respectively.

The rest of the paper is organized as follows. In Section 2, we present the definition and behaviour of LFSPNs. Section 3 explores the discrete part of LFSPNs, i.e. the derived labeled CTSPNs and their underlying CTMCs. Section 4 investigates the continuous part of LFSPNs, which is the associated SFMs. In Section 5, we construct a linear-time relation of fluid trace equivalence for LFSPNs. In Section 6, we propose a branching-time relation of fluid bisimulation equivalence for LFSPNs and compare it with the fluid trace one. Section 7 is devoted to the logical characterization of fluid trace equivalence with the fluid modal logic HML_{flt}. Section 8 presents the logical characterization of fluid bisimulation equivalence with the fluid modal logic HML_{flb}. Section 9 summarizes the results obtained and outlines research perspectives in this area. The complex and long proofs are moved into Appendix A.

2. Basic concepts of LFSPNs

Let us introduce a class of labeled fluid stochastic Petri nets (LFSPNs), whose transitions are labeled with action names, used to specify different system activities. Without labels, LFSPNs are essentially a subclass of FSPNs [54, 43, 49], so that their discrete part describes CTSPNs [62, 60, 61, 12]. This means that LFSPNs have no inhibitor arcs, priorities and immediate transitions, which are used in the standard FSPNs, which are the continuous extension of GSPNs. However, in many practical applications, the performance analysis of GSPNs is simplified by transforming them into CTSPNs or reducing their underlying semi-Markov chains into CTMCs (which are the underlying stochastic process of CTSPNs) by eliminating vanishing states [61, 12, 13]. Transition labeling in LFSPNs is similar to the labeling, proposed for CTSPNs in [28]. We also suppose that the firing rates of transitions and flow rates of the continuous arcs do not depend on the continuous markings (fluid levels).

Let $\mathbb{N}$ be the set of *all natural numbers* and $\mathbb{N}_{\geq 1}$ be the set of *all positive natural numbers*. Further, let $\mathbb{R}$ be the set of *all real numbers*, $\mathbb{R}_{\geq 0}$ be the set of *all non-negative real numbers* and $\mathbb{R}_{>0}$ be the set of *all positive real numbers*. The set of *all row vectors of* $n \in \mathbb{N}_{\geq 1}$ *elements from a set* X is defined as $X^n = \{(x_1, \ldots, x_n) \mid x_i \in X\ (1 \leq i \leq n)\}$. The set of *all mappings from a set* X *to a set* Y is defined as $Y^X = \{f \mid f : X \to Y\}$. Let $Act = \{a, b, \ldots\}$ be the set of *actions*.

First, we present a formal definition of LFSPNs.

Definition 1. *A* labeled fluid stochastic Petri net (LFSPN) *is a tuple* $N = (P_N, T_N, W_N, C_N, R_N, \Omega_N, L_N, \mathcal{M}_N)$*, where*

- $P_N = Pd_N \uplus Pc_N$ *is a finite set of* discrete and continuous places *($\uplus$ is disjoint union);*
- T_N *is a finite set of* transitions*, such that* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$*;*
- $W_N : (Pd_N \times T_N) \cup (T_N \times Pd_N) \to \mathbb{N}$ *is a function providing the* weights of discrete arcs *between discrete places and transitions;*
- $C_N \subseteq (Pc_N \times T_N) \cup (T_N \times Pc_N)$ *is the* set of continuous arcs *between continuous places and transitions;*
- $R_N : C_N \times \mathbb{N}^{|Pd_N|} \to \mathbb{R}_{\geq 0}$ *is a function providing the* (fluid) flow rates of continuous arcs *in given discrete markings (the markings are defined later);*

- $\Omega_N : T_N \times \mathbb{N}^{|Pd_N|} \to \mathbb{R}_{>0}$ *is the* transition (firing) rate *function associating transitions with (firing) rates in given discrete markings;*
- $L_N : T_N \to Act$ *is the* transition labeling *function assigning actions to transitions;*
- $\mathcal{M}_N = (M_N, \mathbf{0})$*, where* $M_N \in \mathbb{N}^{|Pd_N|}$ *and* $\mathbf{0}$ *is a row vector of* $|Pc_N|$ *values* 0*, is the* initial (discrete-continuous) marking.

Consider in detail the tuple elements from the definition above. Let N be an LFSPN.

Every discrete place $p_i \in Pd_N$ may contain discrete tokens, whose number is represented by a natural number $M_i \in \mathbb{N}$ $(1 \leq i \leq |Pd_N|)$. Each continuous place $q_j \in Pc_N$ may contain continuous fluid, with the level represented by a non-negative real number $X_j \in \mathbb{R}_{\geq 0}$ $(1 \leq j \leq |Pc_N|)$. Then the complete hybrid (discrete-continuous) marking of N is a pair (M, X), where $M = (M_1, \dots, M_{|Pd_N|})$ is a discrete marking and $X = (X_1, \dots, X_{|Pc_N|})$ is a continuous marking. When needed, these vectors can also be seen as the mappings $M : Pd_N \to \mathbb{N}$ with $M(p_i) = M_i$ $(1 \leq i \leq |Pd_N|)$ and $X : Pc_N \to \mathbb{R}_{\geq 0}$ with $X(q_j) = X_j$ $(1 \leq j \leq |Pc_N|)$.

The set of *all reachable markings (reachability set)* of N is denoted by $RS(N)$. Then $DRS(N) = \{M \mid (M, X) \in RS(N)\}$ is the set of *all reachable discrete markings (discrete reachability set)* of N. $DRS(N)$ will be formally defined later. Further, $CRS(N) = \{X \mid (M, X) \in RS(N)\} \subseteq \mathbb{R}_{\geq 0}^{|Pc_N|}$ is the set of *all reachable continuous markings (continuous reachability set)* of N.

Every marking $(M, X) \in RS(N)$ evolves in time, hence, we can interpret it as a stochastic process $\{(M(\delta), X(\delta)) \mid \delta \geq 0\}$. Then the initial marking of N is that at the zero time moment, i.e. $\mathcal{M}_N = (M_N, \mathbf{0}) = (M(0), X(0))$, where $X(0) = \mathbf{0}$ means that all the continuous places are initially empty.

Every transition $t \in T_N$ has an associated positive real-valued instantaneous rate $\Omega_N(t, M) \in \mathbb{R}_{>0}$, which is a parameter of the exponential distribution governing the transition delay (being a random variable), when the current discrete marking is M. Transitions are labeled with actions, each representing a sort of activity that they model.

Every discrete arc $da = (p, t)$ or $da = (t, p)$, where $p \in Pd_N$ and $t \in T_N$, connects discrete places and transitions. It has a non-negative integer-valued weight $W_N(da) \in \mathbb{N}$ assigned, representing its multiplicity. The zero weight indicates that the corresponding discrete arc does not exist, since its multiplicity is zero in this case. In the discrete marking $M \in DRS(N)$, every continuous arc $ca = (q, t)$ or $ca = (t, q)$, where $q \in Pc_N$ and $t \in T_N$, connects continuous places and transitions. It has a non-negative real-valued flow rate $R_N(ca, M) \in \mathbb{R}_{\geq 0}$ of fluid through ca, when the current discrete marking is M. The zero flow rate indicates that the fluid flow along the corresponding continuous arc is stopped in some discrete marking.

The graphical representation of LFSPNs resembles that for standard labeled Petri nets, but supplemented with the rates or weights, written near the corresponding transitions or arcs. Discrete places are drawn with ordinary circles while double concentric circles correspond to the continuous ones. The multiplicity of each discrete place in a discrete marking is represented by the number of tokens, depicted as black dots within the place. Square boxes with the action names inside depict transitions and their labels. Discrete arcs are drawn as thin lines with arrows at the end while continuous arcs should represent pipes, so the latter are depicted by thick arrowed lines. If the rates are not given in the picture then they are assumed to be of no importance in the corresponding examples. The names of places and transitions are depicted near them when needed.

We now consider the behaviour of LFSPNs.

Let N be an LFSPN and M be a discrete marking of N. A transition $t \in T_N$ is *enabled* in M if $\forall p \in Pd_N\ W_N(p,t) \leq M(p)$. Let $Ena(M)$ be the set of *all transitions enabled in* M. Firings of transitions are atomic operations, and only single transitions are fired at once. Note that the enabling condition depends only on the discrete part of N and this condition is the same as for CTSPNs. Firing of a transition $t \in Ena(M)$ changes M to another discrete marking $\widetilde{M}$, such as $\forall p \in Pd_N\ \widetilde{M}(p) = M(p) - W_N(p,t) + W_N(t,p)$, denoted by $M \xrightarrow{t}_\lambda \widetilde{M}$, where $\lambda = \Omega_N(t,M)$. We write $M \xrightarrow{t} \widetilde{M}$ if $\exists \lambda\ M \xrightarrow{t}_\lambda \widetilde{M}$ and $M \to \widetilde{M}$ if $\exists t\ M \xrightarrow{t} \widetilde{M}$.

Let us formally define the discrete reachability set of N.

Definition 2. *Let N be an LFSPN. The* discrete reachability set *of N, denoted by $DRS(N)$, is the minimal set of discrete markings such that*

- $M_N \in DRS(N)$;
- *if* $M \in DRS(N)$ *and* $M \to \widetilde{M}$ *then* $\widetilde{M} \in DRS(N)$.

Let us now define the discrete reachability graph of N.

Definition 3. *Let N be an LFSPN. The* discrete reachability graph *of N is a labeled transition system $DRG(N) = (S_N, \mathcal{L}_N, \mathcal{T}_N, s_N)$, where*

- *the set of* states *is* $S_N = DRS(N)$;
- *the set of* labels *is* $\mathcal{L}_N = T_N \times \mathbb{R}_{>0}$;
- *the set of* transitions *is* $\mathcal{T}_N = \{(M,(t,\lambda),\widetilde{M}) \mid M, \widetilde{M} \in DRS(N),\ M \xrightarrow{t}_\lambda \widetilde{M}\}$;
- *the* initial state *is* $s_N = M_N$.

Example 2. *In Figure 2, the LFSPNs N and N' are presented. As we shall see in Section 5, they are fluid trace equivalent, denoted by $N \equiv_{fl} N'$. For instance, LFSPN N has the discrete places p_1 (with 1 token inside at the initial discrete marking) and p_2 (with 0 tokens inside at the initial discrete marking); continuous place q; transitions t_1 (with the firing rate 2 at any discrete marking), t_2 and t_3 (both with the same firing rate 1 at any discrete marking); discrete arcs (p_1,t_1), (t_1,p_2), (p_2,t_2), (p_2,t_3), (t_2,p_1) and (t_3,p_1) (all with the same weight 1); continuous arcs (t_1,q) (with the flow rate 5 at any discrete marking), (q,t_1) (with the flow rate 4 at any discrete marking), (t_2,q) (with the flow rate 1 at any discrete marking), (q,t_2), (t_3,q) (both with the same flow rate 2 at any discrete marking) and (q,t_3) (with the flow rate 3 at any discrete marking).*

We have $DRS(N) = \{M_1, M_2\}$, where $M_1 = (1,0)$, $M_2 = (0,1)$, and $DRS(N') = \{M_1', M_2', M_3'\}$, where $M_1' = (1,0,0)$, $M_2' = (0,1,0)$, $M_3' = (0,0,1)$. In Figure 3, the discrete reachability graphs $DRG(N)$ and $DRG(N')$ are depicted.

3. Discrete part of LFSPNs

We have restricted the class of FSPNs underlying LFSPNs to those whose discrete part is CTSPNs, since the performance analysis of standard FSPNs with GSPNs as the discrete part is finally based on the CTMCs which are extracted from the underlying semi-Markov chains (SMCs) of the GSPNs by removing vanishing states. Let us now consider the behaviour of the discrete part of LFSPNs, which is labeled CTSPNs.

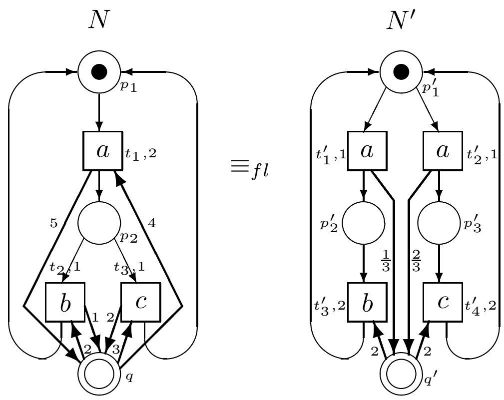

FIG. 2. Fluid trace equivalent LFSPNs

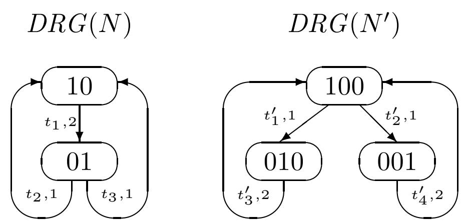

FIG. 3. The discrete reachability graphs of the fluid trace equivalent LFSPNs

For an LFSPN N, a continuous random variable $\xi(M)$ is associated with every discrete marking $M \in DRS(N)$. The variable captures a residence (sojourn) time in M. We adopt the *race* semantics, in which the fastest stochastic transition (i.e. that with the minimal exponentially distributed firing delay) fires first. Hence, the *probability distribution function (PDF)* of the sojourn time in M is that of the minimal firing delay of transitions from $Ena(M)$. Since exponential distributions are closed under minimum, the sojourn time in M is (again) exponentially distributed with a parameter that is called the *exit rate from the discrete marking* M, defined as

$$RE(M) = \sum_{t \in Ena(M)} \Omega_N(t, M).$$

Note that we may have $RE(M) = 0$, meaning that there is no exit from M, if it is a *terminal discrete marking*, i.e. there are no transitions from it to different discrete markings.

Hence, the PDF of the sojourn time in M (the probability of the residence time in M being less than δ) is $F_{\xi(M)}(\delta) = \mathsf{P}(\xi(M) < \delta) = 1 - e^{-RE(M)\delta}$ $(\delta \geq 0)$. Then the *probability density function* of the residence time in M (the limit probability of staying in M at the time δ) is $f_{\xi(M)}(\delta) = \lim_{\Delta \to 0} \frac{F_{\xi(M)}(\delta+\Delta) - F_{\xi(M)}(\delta)}{\Delta} = \frac{dF_{\xi(M)}(\delta)}{d\delta} = RE(M)e^{-RE(M)\delta}$ $(\delta \geq 0)$. The mean value (average, expectation) formula for the exponential distribution allows us to calculate the average sojourn time in M as $\mathsf{M}(\xi(M)) = \int_0^\infty \delta f_{\xi(M)}(\delta) d\delta = \frac{1}{RE(M)}$.

The *average sojourn time in the discrete marking* M is

$$SJ(M) = \frac{1}{\sum_{t \in Ena(M)} \Omega_N(t, M)} = \frac{1}{RE(M)}.$$

The *average sojourn time vector* SJ of N has the elements $SJ(M)$, $M \in DRS(N)$.

Note that we may have $SJ(M) = \infty$, meaning that we stay in M forever, if it is a terminal discrete marking.

To evaluate performance with the use of the discrete part of N, we should investigate the stochastic process associated with it. The process is the underlying continuous time Markov chain, denoted by $CTMC(N)$.

Let $M, \widetilde{M} \in DRS(N)$. The *rate of moving from* M *to* $\widetilde{M}$ *by firing any transition* is

$$RM(M, \widetilde{M}) = \sum_{\{t \mid M \xrightarrow{t} \widetilde{M}\}} \Omega_N(t, M).$$

Definition 4. *Let N be an LFSPN. The* underlying continuous time Markov chain (CTMC) *of N, denoted by $CTMC(N)$, has the state space $DRS(N)$, the initial state M_N and the transitions $M \to_\lambda \widetilde{M}$, if $M \to \widetilde{M}$, where $\lambda = RM(M, \widetilde{M})$.*

Isomorphism is a coincidence of systems up to renaming their components or states. Let $\simeq$ denote isomorphism between CTMCs that binds their initial states.

Definition 5. *Let N be an LFSPN. The elements $\mathcal{Q}_{ij}$ $(1 \le i, j \le n = |DRS(N)|)$ of the* transition rate matrix (TRM)*, also called* infinitesimal generator*, $\mathbf{Q}$ for $CTMC(N)$ are defined as*

$$\mathcal{Q}_{ij} = \begin{cases} RM(M_i, M_j), & i \neq j; \\ -\sum_{\{k \mid 1 \le k \le n,\ k \neq i\}} RM(M_i, M_k), & i = j. \end{cases}$$

The transient *probability mass function (PMF)* $\varphi(\delta) = (\varphi_1(\delta), \ldots, \varphi_n(\delta))$ for $CTMC(N)$ is calculated via matrix exponent as

$$\varphi(\delta) = \varphi(0) e^{\mathbf{Q}\delta},$$

where $\varphi(0) = (\varphi_1(0), \ldots, \varphi_n(0))$ is the initial PMF, defined as

$$\varphi_i(0) = \begin{cases} 1, & M_i = M_N; \\ 0, & \text{otherwise.} \end{cases}$$

The steady-state PMF $\varphi = (\varphi_1, \ldots, \varphi_n)$ for $CTMC(N)$ is a solution of the linear equation system

$$\begin{cases} \varphi \mathbf{Q} = \mathbf{0} \\ \varphi \mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{0}$ is a row vector of n values 0 and $\mathbf{1}$ is that of n values 1.

Note that the vector φ exists and is unique, if $CTMC(N)$ is ergodic. Then $CTMC(N)$ has a single steady state, and we have $\varphi = \lim_{\delta \to \infty} \varphi(\delta)$.

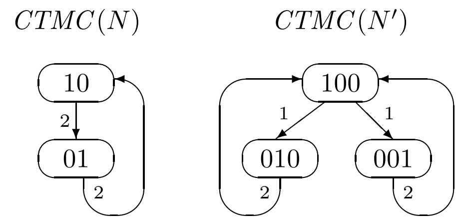

FIG. 4. The underlying CTMCs of the fluid trace equivalent LFSPNs

Example 3. *Consider the LFSPNs N and N' in Figure 2. In Figure 4, the underlying CTMCs $CTMC(N)$ and $CTMC(N')$ are drawn.*

The average sojourn time vectors of N and N' are

$$SJ = \left(\frac{1}{2}, \frac{1}{2}\right),\ SJ' = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

The TRMs $\mathbf{Q}$ and $\mathbf{Q}'$ for $CTMC(N)$ and $CTMC(N')$ are

$$\mathbf{Q} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},\quad \mathbf{Q}' = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}.$$

The steady-state PMFs for $CTMC(N)$ and $CTMC(N')$ are

$$\varphi = \left(\frac{1}{2}, \frac{1}{2}\right),\ \varphi' = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

4. Continuous part of LFSPNs

We now consider the impact the discrete part of LFSPNs has on their continuous part, which is stochastic fluid models (SFMs). We investigate LFSPNs with a single continuous place, since our subsequent definitions of the fluid equivalences assume that fact.

Let N be an LFSPN such that $Pc_N = \{q\}$ and $M(\delta) \in DRS(N)$ be its discrete marking at the time $\delta \geq 0$. Every continuous arc $ca = (q,t)$ or $ca = (t,q)$, where $t \in T_N$, changes the fluid level in the continuous place q at the time δ with the flow rate $R_N(ca, M(\delta))$. This means that in the discrete marking $M(\delta)$ fluid can leave q along the continuous arc (q,t) with the rate $R_N((q,t), M(\delta))$ and can enter q along the continuous arc (t,q) with the rate $R_N((t,q), M(\delta))$ for every transition $t \in Ena(M(\delta))$.

The *potential rate of the fluid level change (fluid flow rate) for the continuous place* q in the discrete marking $M(\delta)$ is

$$RP(M(\delta)) = \sum_{\{t\in Ena(M(\delta))|(t,q)\in C_N\}} R_N((t,q), M(\delta)) - \sum_{\{t\in Ena(M(\delta))|(q,t)\in C_N\}} R_N((q,t), M(\delta)).$$

Let $X(\delta)$ be the fluid level in q at the time δ. It is clear that the fluid level in a continuous place can never be negative. Therefore, $X(\delta)$ satisfies the following ordinary differential equation describing the *actual fluid flow rate for the continuous place* q in the marking $(M(\delta), X(\delta))$:

$$RA(M(\delta), X(\delta)) = \frac{dX(\delta)}{d\delta} = \begin{cases} \max\{RP(M(\delta)), 0\}, & X(\delta) = 0; \\ RP(M(\delta)), & X(\delta) > 0. \end{cases}$$

In the first case considered in the definition above, we have $X(\delta) = 0$. In this case, if $RP(M(\delta)) \geq 0$ then the fluid level is growing and the derivative is equal to the potential rate. Otherwise, if $RP(M(\delta)) < 0$ then we should prevent the fluid level from crossing the lower boundary (zero) by stopping the fluid flow. In the second case, $X(\delta) > 0$ and the derivative is assumed to be equal to the potential rate.

Note that $\frac{dX(\delta)}{d\delta}$ is a piecewise constant function of $X(\delta)$ during the time periods when $M(\delta)$ remains unchanged. Hence, for each different “constant” segment we have $\frac{dX(\delta)}{d\delta} = RP(M(\delta))$ or $\frac{dX(\delta)}{d\delta} = 0$ and, therefore, we can suppose that within each such segment $RP(M(\delta))$ or 0 are the *actual* fluid flow rates for the continuous place q in the marking $(M(\delta), X(\delta))$. While constructing differential equations that describe the behaviour of SFMs associated with LFSPNs, we are interested only in the segments where $\frac{dX(\delta)}{d\delta} = RP(M(\delta))$. The SFMs behaviour within the remaining segments, where $\frac{dX(\delta)}{d\delta} = 0$, is completely described by the buffer empty probability function that collects the probability mass at the lower boundary.

Definition 6. *Let N be an LFSPN. The elements $\mathcal{R}_{ij}$ $(1 \leq i, j \leq n = |DRS(N)|)$ of the* fluid rate matrix (FRM) $\mathbf{R}$ *for the continuous place q are defined as*

$$\mathcal{R}_{ij} = \begin{cases} RP(M_i), & i = j; \\ 0, & i \neq j. \end{cases}$$

The underlying SFMs of LFSPNs are the first order, infinite buffer, homogeneous Markov fluid models [43, 49]. The discrete part of the SFM derived from an LFSPN N is the CTMC $CTMC(N)$ with the TRM $\mathbf{Q}$. The evolution of the continuous part of the SFM (fluid flow drift) is described by the FRM $\mathbf{R}$.

Let us consider the *stationary behaviour* of the SFM associated with an LFSPN N. We do not discuss here in detail the conditions under which the steady state for the associated SFM *exists* and is *unique*, since this topic has been extensively explored in [54, 43, 49]. Particularly, according to [54, 49], the steady-state PDF *exists* (i.e. the transient functions approach their stationary values, as the time parameter δ tends to infinity in the transient equations), when the associated SFM is a Markov fluid model, whose fluid flow drift (described by the matrix $\mathbf{R}$) and transition rates (described by the matrix $\mathbf{Q}$) are fluid level independent, and the following *stability condition* holds:

$$FluidFlow(q) = \sum_{i=1}^{n} \varphi_i RP(M_i) = \varphi \mathbf{R} \mathbf{1}^T < 0,$$

stating that the steady-state *mean potential fluid flow rate for the continuous place q* is negative. Stable infinite buffer models usually converge, hence, the existing steady-state PDF is also *unique* in this case.

Definition 7. *Let N be an LFSPN and $(M(\delta), X(\delta)) \in RS(N)$ be its marking at the time $\delta \geq 0$. The following steady-state probability functions are obtained from the transient ones by taking the limit $\delta \to \infty$.*

- $\varphi_i = \lim_{\delta \to \infty} \mathsf{P}(M(\delta) = M_i)$ *is the* steady-state discrete marking probability*;*

- $\ell_i = \lim_{\delta \to \infty} \mathsf{P}(X(\delta) = 0,\ M(\delta) = M_i)$ *is the* steady-state buffer empty probability (probability mass at the lower boundary)*;*
- $F_i(x) = \lim_{\delta \to \infty} \mathsf{P}(X(\delta) < x,\ M(\delta) = M_i)$ *is the* steady-state fluid probability distribution function*;*
- $f_i(x) = \frac{dF_i(x)}{dx} = \lim_{h \to 0} \frac{F_i(x+h) - F_i(x)}{h} = \lim_{\delta \to \infty} \lim_{h \to 0} \frac{\mathsf{P}(x < X(\delta) < x+h,\ M(\delta)=M_i)}{h}$ *is the* steady-state fluid probability density function.

Let $\varphi, \ell, F(x), f(x)$ *be the row vectors with the elements* $\varphi_i, \ell_i, F_i(x), f_i(x)$*, respectively* $(1 \le i \le n)$.

By the total probability law for the stationary behaviour, we have

$$\ell + \int_{0+}^{\infty} f(x)dx = \varphi.$$

The ordinary differential equations describing the stationary behaviour are

$$\frac{dF(x)}{dx}\mathbf{R} = F(x)\mathbf{Q},\ x > 0;$$

$$\frac{df(x)}{dx}\mathbf{R} = f(x)\mathbf{Q},\ x > 0.$$

Note that we have $\frac{dF(x)}{dx} = f(x),\ F(0) = \ell,\ F(\infty) = \varphi$.

The ordinary differential equation for the steady-state buffer empty probabilities (stationary lower boundary conditions) are

$$f(0)\mathbf{R} = \ell\mathbf{Q}.$$

The stationary lower boundary constraint is: if $\mathcal{R}_{ii} = RP(M_i) > 0$ then $F_i(0) = \ell_i = 0$ $(1 \le i \le n)$.

The stationary normalizing condition is

$$\ell\mathbf{1}^T + \int_{0+}^{\infty} f(x)dx\mathbf{1}^T = 1,$$

where $\mathbf{1}$ is a row vector of n values 1.

The solutions of the equations for $F(x)$ and $f(x)$ in the form of *matrix exponent* are $F(x) = \ell e^{x\mathbf{QR}^{-1}}$ and $f(x) = \ell\mathbf{QR}^{-1}e^{x\mathbf{QR}^{-1}}$, respectively. Since the steady-state existence implies boundedness of the SFM associated with an LFSPN and we do not have a finite upper fluid level bound, the positive eigenvalues of $\mathbf{QR}^{-1}$ must be excluded. Moreover, $\mathbf{R}^{-1}$ does not exist if for some i $(1 \le i \le n)$ we have $\mathcal{R}_{ii} = 0$. These difficulties are avoided in the alternative solution method for $F(x)$, called *spectral decomposition* [76, 54, 43, 49, 46, 74, 75].

Example 4. *Consider the LFSPNs N and N' in Figure 2.*

The FRMs $\mathbf{R}$ and $\mathbf{R}'$ for the SFM of N and N' are

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \mathbf{R}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The stability conditions *for the SFMs of N and N' are fulfilled: $FluidFlow(q) = \sum_{i=1}^{2} \varphi_i RP(M_i) = \frac{1}{2}\cdot 1+\frac{1}{2}(-2) = -\frac{1}{2} < 0$ and $FluidFlow(q') = \sum_{j=1}^{3} \varphi'_j RP(M'_j) = \frac{1}{2}\cdot 1 + \frac{1}{4}(-2) + \frac{1}{4}(-2) = -\frac{1}{2} < 0$.*

The steady-state fluid PDFs for the SFMs of N and N' are

$$F(x) = \left(\frac{1}{2} - \frac{1}{2}e^{-x}, \frac{1}{2} - \frac{1}{4}e^{-x}\right), \; F'(x) = \left(\frac{1}{2} - \frac{1}{2}e^{-x}, \frac{1}{4} - \frac{1}{8}e^{-x}, \frac{1}{4} - \frac{1}{8}e^{-x}\right).$$

The steady-state fluid probability density functions for the SFMs of N and N' are

$$f(x) = \frac{dF(x)}{dx} = \left(\frac{1}{2}e^{-x}, \frac{1}{4}e^{-x}\right), \; f'(x) = \frac{dF'(x)}{dx} = \left(\frac{1}{2}e^{-x}, \frac{1}{8}e^{-x}, \frac{1}{8}e^{-x}\right).$$

The steady-state buffer empty probabilities for the SFMs of N and N' are

$$\ell = F(0) = \left(0, \frac{1}{4}\right), \; \ell' = F'(0) = \left(0, \frac{1}{8}, \frac{1}{8}\right).$$

One can see that $F(\infty) = \left(\frac{1}{2}, \frac{1}{2}\right) = \varphi$ and $F'(\infty) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) = \varphi'$. Note also that $f(0) = \left(\frac{1}{2}, \frac{1}{4}\right)$ and it holds $f(0)\mathbf{R} = \left(\frac{1}{2}, -\frac{1}{2}\right) = \ell\mathbf{Q}$. Analogously, $f'(0) = \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$ and we have $f'(0)\mathbf{R}' = \left(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}\right) = \ell'\mathbf{Q}'$.

5. Fluid trace equivalence

Trace equivalences are the least discriminating ones. In the trace semantics, the behavior of a system is associated with the set of all possible sequences of actions, i.e. the protocols of work or computations. The points of choice of an external observer between several extensions of a particular computation are not taken into account.

The formal definition of fluid trace equivalence resembles that of ordinary Markovian trace equivalence, proposed on transition-labeled CTMCs in [83], on sequential and concurrent Markovian process calculi SMPC and CMPC in [14, 18, 15, 16, 19] and on Uniform Labeled Transition Systems (ULTraS) in [21, 22, 17]. While defining fluid trace equivalence, we additionally have to take into account the fluid flow rates in the corresponding discrete markings of two compared LFSPNs. Hence, in order to construct fluid trace equivalence, we should determine how to calculate the cumulative execution probabilities of all the specific (selected) paths. A *path* in the discrete reachability graph of an LFSPN is a sequence of its discrete markings and transitions that is generated by some firing sequence in the LFSPN.

First, we should *multiply the transition firing probabilities* for all the transitions along the paths starting in the initial discrete marking of the LFSPN. The resulting product will be the *execution probability of the path*. Second, we should *sum the path execution probabilities* for all the selected paths corresponding to the same *sequence of actions*, moreover, to the same *sequence of the average sojourn times* and the same *sequence of the potential fluid flow rates* in all the discrete markings participating the paths. We suppose that each LFSPN has exactly one continuous place. The resulting sum will be the *cumulative execution probability of the selected paths* corresponding to some fluid stochastic trace. A *fluid stochastic trace* is a pair with the first element being the triple of the correlated sequences of actions, average sojourn times and potential fluid flow rates, and the second element being the execution probability of the triple. Each element of the triple guarantees that fluid trace equivalence respects the following important aspects of the LFSPNs behaviour: *functional activity*, *stochastic timing* and *fluid flow*.

Fluid trace equivalence can also be defined using Markovian trace machine (MTM) from [83] (featuring the action display, time display and reset button), enhanced with an additional display showing the potential fluid flow rate in the current state. Such an enhanced black box tester will be called fluid stochastic trace machine (FSTM), to be in disposal of the external observer. Remember that the action display shows the latest action whose execution (being instantaneous after an exponentially timed delay) has led to the current state. The time display shows either global time (absolute time counter) or an upper bound for the remaining local time (countdown timer) before the next action occurrence. Pressing the reset button terminates the current run and starts another one, so that the length of each run can be controlled.

In our setting, each such run corresponds to (can be extracted from) some sequence of transition firings started in the initial discrete marking of an LFSPN. After infinitely many runs of the FSTM we shall be able to calculate the probabilities of the correlated sequences of actions, time values and potential fluid flow rates. Then two LFSPNs are fluid trace equivalent if the mentioned probabilities coincide for all possible triples of that kind, called observations. As demonstrated in [83], implementing absolute or countdown timer results in the same equivalence. Moreover, it appeared to be enough collecting the average sojourn times in the states between which the actions occur, instead of using the timers. The latter approach gives an alternative to the testing with MTM. Such a viewpoint to the linear-time behaviour also substantially simplifies definitions and proofs related to the Markovian trace equivalences, so we have decided to adopt that approach for LFSPNs, as an alternative to the experiments with FSTM.

Note that $CTMC(N)$ can be interpreted as a semi-Markov chain (SMC) [58], denoted by $SMC(N)$, which is analyzed by extracting from it the embedded (absorbing) discrete time Markov chain (EDTMC) corresponding to N, denoted by $EDTMC(N)$. The construction of the latter is analogous to that applied in the context of GSPNs in [60, 61, 12, 13]. $EDTMC(N)$ only describes the state changes of $SMC(N)$ while ignoring its time characteristics. Thus, to construct the EDTMC, we should abstract from all time aspects of behaviour of the SMC, i.e. from the sojourn time in its states. It is well-known that every SMC is fully described by the EDTMC and the state sojourn time distributions (the latter can be specified by the vector of PDFs of residence time in the states) [51].

An LFSPN N is *live*, if $\forall M \in DRS(N)\ Ena(M) \neq \emptyset$, i.e. transitions can fire at every reachable discrete marking of it. In this section, we shall consider only live FSPNs, to avoid terminating sequences of transition firings.

We first propose some helpful definitions of the probability functions for the transition firings and discrete marking changes. Let N be an LFSPN, $M, \widetilde{M} \in DRS(N)$ be its discrete markings and $t \in Ena(M)$.

The (time-abstract) *probability that the transition t fires in M* is

$$PT(t, M) = \frac{\Omega_N(t, M)}{\sum_{u \in Ena(M)} \Omega_N(u, M)} = \frac{\Omega_N(t, M)}{RE(M)} = SJ(M)\Omega_N(t, M).$$

We have $\forall M \in \mathbb{N}^{|Pd_N|}\ \sum_{t\in Ena(M)} PT(t,M)=\sum_{t\in Ena(M)} \frac{\Omega_N(t,M)}{\sum_{u\in Ena(M)}\Omega_N(u,M)} = \frac{\sum_{t\in Ena(M)}\Omega_N(t,M)}{\sum_{u\in Ena(M)}\Omega_N(u,M)} = 1$, i.e. $PT(t,M)$ defines a probability distribution.

The *probability to move from M to $\widetilde{M}$ by firing any transition* is

$$PM(M,\widetilde{M}) = \sum_{\{t|M\xrightarrow{t}\widetilde{M}\}} PT(t,M) = \frac{\sum_{\{t|M\xrightarrow{t}\widetilde{M}\}}\Omega_N(t)}{RE(M)} = SJ(M)\cdot \sum_{\{t|M\xrightarrow{t}\widetilde{M}\}}\Omega_N(t).$$

We write $M \to_{\mathcal{P}} \widetilde{M}$, if $M \to \widetilde{M}$, where $\mathcal{P} = PM(M,\widetilde{M})$. We have $\forall M \in \mathbb{N}^{|Pd_N|}$ $\sum_{\{\widetilde{M}|M\to\widetilde{M}\}} PM(M,\widetilde{M}) = \sum_{\{\widetilde{M}|M\to\widetilde{M}\}}\sum_{\{t|M\xrightarrow{t}\widetilde{M}\}} PT(t,M) =$
$\sum_{t\in Ena(M)} PT(t,M) = 1$, i.e. $PM(M,\widetilde{M})$ defines a probability distribution.

Definition 8. *Let N be an LFSPN. The* embedded (absorbing) discrete time Markov chain (EDTMC) *of N, denoted by $EDTMC(N)$, has the state space $DRS(N)$, the initial state M_N and the transitions $M \to_{\mathcal{P}} \widetilde{M}$, if $M \to \widetilde{M}$, where $\mathcal{P} = PM(M,\widetilde{M})$.*

The underlying SMC *of N, denoted by $SMC(N)$, has the EDTMC $EDTMC(N)$ and the sojourn time in every $M \in DRS(N)$ is exponentially distributed with the parameter $RE(M)$.*

Since the sojourn time in every $M \in DRS(N)$ is exponentially distributed, we have $SMC(N) = CTMC(N)$.

Definition 9. *Let N be an LFSPN. The elements $\mathcal{P}_{ij}$ $(1 \le i,j \le n = |DRS(N)|)$ of the* (one-step) transition probability matrix (TPM) $\mathbf{P}$ *for $EDTMC(N)$ are defined as*

$$\mathcal{P}_{ij} = \begin{cases} PM(M_i,M_j), & M_i \to M_j;\\ 0, & \textit{otherwise}.\end{cases}$$

Let X be a set, $n \in \mathbb{N}_{\ge 1}$ and $x_i \in X$ $(1 \le i \le n)$. Then $\chi = x_1\cdots x_n$ is a finite sequence over X of *length* $|\chi| = n$. When X is a set on numbers, we usually write $\chi = x_1 \circ\cdots\circ x_n$, to avoid confusion because of mixing up the operations of concatenation of sequences $(\circ)$ and multiplication of numbers $(\cdot)$. The *empty sequence* ε of length $|\varepsilon| = 0$ is an extra case. Let X^* denote the *set of all finite sequences* (including the empty one) over X.

Let $M_N = M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n$ $(n \in \mathbb{N})$ be a finite sequence of transition firings starting in the initial discrete marking M_N and called *firing sequence* in N. The firing sequence generates the *path* $M_0t_1M_1t_2\cdots t_nM_n$ in the discrete reachability graph $DRG(N)$. Since the *first discrete marking* $M_N = M_0$ of the path is fixed, one can see that the (finite) *transition sequence* $\vartheta = t_1\cdots t_n$ in N uniquely determines the *discrete marking sequence* $M_0\cdots M_n$, ending with the *last discrete marking* M_n of the mentioned path in $DRG(N)$. Hence, to refer the paths, one can simply use the transition sequences extracted from them as shown above. The *empty transition sequence* ε refers to the path M_0, consisting of one discrete marking (which is then the first and last one of the path).

Definition 10. *Let N be an LFSPN. The set of* all (finite) transition sequences *in N is defined as*

$$TranSeq(N) = \{\vartheta \mid \vartheta = \varepsilon \ or\ \vartheta = t_1\cdots t_n,\ M_N = M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n\}.$$

Let $\vartheta = t_1 \cdots t_n \in TranSeq(N)$ and $M_N = M_0 \stackrel{t_1}{\to} M_1 \stackrel{t_2}{\to} \cdots \stackrel{t_n}{\to} M_n$. The *probability to execute the transition sequence* ϑ is

$$PT(\vartheta) = \prod_{i=1}^{n} PT(t_i, M_{i-1}).$$

For $\vartheta = \varepsilon$ we define $PT(\varepsilon) = 1$. The following lemma shows that $PT(\vartheta)$ defines a probability distribution.

Lemma 1. [75] *Let N be an LFSPN. Then $\forall n \in \mathbb{N}$*

$$\sum_{\{\vartheta \in TranSeq(N) \mid |\vartheta| = n\}} PT(\vartheta) = 1.$$

Proof. See Appendix A.1. □

Let $\vartheta = t_1 \cdots t_n \in TranSeq(N)$ be a transition sequence in N and $M_N = M_0 \stackrel{t_1}{\to} M_1 \stackrel{t_2}{\to} \cdots \stackrel{t_n}{\to} M_n$. The *action sequence* of ϑ is $L_N(\vartheta) = L_N(t_1) \cdots L_N(t_n) \in Act^*$, i.e. it is the sequence of actions which label the transitions of that transition sequence. For $\vartheta = \varepsilon$ we define $L_N(\varepsilon) = \varepsilon$. Further, the *average sojourn time sequence* of $\vartheta = t_1 \cdots t_n$ is $SJ(\vartheta) = SJ(M_0) \circ \cdots \circ SJ(M_n) \in \mathbb{R}^*_{>0}$, i.e. it is the sequence of average sojourn times in the discrete markings of the path to which ϑ refers. For $\vartheta = \varepsilon$ we define $SJ(\varepsilon) = SJ(M_0)$. Similarly, the *(potential) fluid flow rate sequence* of $\vartheta = t_1 \cdots t_n$ is $RP(\vartheta) = RP(M_0) \circ \cdots \circ RP(M_n) \in \mathbb{R}^*$, i.e. it is the sequence of (potential) fluid flow rates in the discrete markings of the path to which ϑ refers. For $\vartheta = \varepsilon$ we define $RP(\varepsilon) = RP(M_0)$.

Definition 11. *Let N be an LFSPN and $(\sigma, \varsigma, \varrho) \in Act^* \times \mathbb{R}^*_{>0} \times \mathbb{R}^*$. The set of* $(\sigma, \varsigma, \varrho)$-selected (finite) transition sequences *in N is defined as*

$$TranSeq(N, M, \sigma, \varsigma, \varrho) = \left\{ \vartheta \in TranSeq(N, M) \;\middle|\; \begin{array}{l} L_N(\vartheta) = \sigma,\ SJ(M, \vartheta) = \varsigma, \\ RP(M, \vartheta) = \varrho \end{array} \right\}.$$

Let $TranSeq(N, \sigma, \varsigma, \varrho) \neq \emptyset$. Then the triple $(\sigma, \varsigma, \varrho)$, together with its execution probability, which is the cumulative execution probability of all the paths from which the triple is extracted (as described above), constitute a *fluid stochastic trace* of the LFSPN N. Fluid stochastic traces are formally introduced below, followed by the (first) definition of fluid stochastic trace equivalence.

Definition 12. *A* (finite) fluid stochastic trace *of an LFSPN N is a pair $((\sigma, \varsigma, \varrho), PT(\sigma, \varsigma, \varrho))$, where $TranSeq(N, \sigma, \varsigma, \varrho) \neq \emptyset$ and the (cumulative)* probability to execute $(\sigma, \varsigma, \varrho)$-selected transition sequences *is*

$$PT(\sigma, \varsigma, \varrho) = \sum_{\vartheta \in TranSeq(N, \sigma, \varsigma, \varrho)} PT(\vartheta).$$

We denote the set of all fluid stochastic traces *of an LFSPN N by $FluStochTraces(N)$. Two LFSPNs N and N' are* fluid trace equivalent*, denoted by $N \equiv_{fl} N'$, if*

$$FluStochTraces(N) = FluStochTraces(N').$$

By Lemma 1, we have
$\forall n \in \mathbb{N}\ \sum_{\{(\sigma,\varsigma,\varrho)\mid|\sigma|=n\}} PT(\sigma,\varsigma,\varrho) = \sum_{\{(\sigma,\varsigma,\varrho)\mid|\sigma|=n\}} \sum_{\vartheta\in TranSeq(N,\sigma,\varsigma,\varrho)} PT(\vartheta) = \sum_{(\sigma,\varsigma,\varrho)} \sum_{\{\vartheta\in TranSeq(N,\sigma,\varsigma,\varrho)\mid|\vartheta|=n\}} PT(\vartheta) = \sum_{\{\vartheta\in TranSeq(N)\mid|\vartheta|=n\}} PT(\vartheta) = 1$, i.e. $PT(\sigma,\varsigma,\varrho)$ defines a probability distribution.

The following (second) definition of fluid stochastic trace equivalence does not use fluid stochastic traces.

Definition 13. *Two LFSPNs N and N' are* fluid trace equivalent*, denoted by $N \equiv_{fl} N'$, if $\forall(\sigma,\varsigma,\varrho) \in Act^* \times \mathbb{R}^*_{>0} \times \mathbb{R}^*$ we have*

$$\sum_{\vartheta\in TranSeq(N,\sigma,\varsigma,\varrho)} PT(\vartheta) = \sum_{\vartheta'\in TranSeq(N',\sigma,\varsigma,\varrho)} PT(\vartheta').$$

Note that in the definition of $TranSeq(N,\sigma,\varsigma,\varrho)$, as well as in Definitions 12 and 13, for $\vartheta \in T_N^*$, we may use the *exit rate sequences* $RE(\vartheta) = RE(M_0) \circ \cdots \circ RE(M_n) \in \mathbb{R}^*_{\geq 0}$ instead of average sojourn time sequences $\varsigma = SJ(\vartheta) = SJ(M_0) \circ \cdots \circ SJ(M_n) \in \mathbb{R}^*_{>0}$, since we have $\forall M \in DRS(N)\ SJ(M) = \frac{1}{RE(M)}$ and $\forall M \in DRS(N)\ \forall M' \in DRS(N')\ SJ(M) = SJ(M') \Leftrightarrow RE(M) = RE(M')$.

Note also that our notion of fluid trace equivalence is based rather on that of Markovian trace equivalence from [83], since there the average sojourn times in the states "surrounding" the actions of the corresponding traces of the equivalent processes should *coincide* while in the definition of the mentioned equivalence from [14, 18, 15, 16, 19], the shorter average sojourn time may simulate the longer one. If we would adopt such a simulation then the smaller average potential fluid change volume (the product of the average sojourn time and potential fluid flow rate) would model the bigger one, since the potential fluid flow rate remains constant while residing in a discrete marking. Since we observe no intuition behind that modeling, we do not use it.

In [21, 22, 17], the following two types of Markovian trace equivalence have been proposed. The *state-to-state* Markovian trace equivalence requires coincidence of average sojourn times in all corresponding discrete markings of the paths. The *end-to-end* Markovian trace equivalence demands that only the sums of average sojourn times for all corresponding discrete markings of the paths should be equal. As a basis for constructing fluid trace equivalence, we have taken the state-to-state relation, since the constant potential fluid flow rate in the discrete markings may differ with their change (moreover, the actual fluid flow rate function may become discontinuous when the lower fluid boundary for a continuous place is reached in some discrete marking). Then, while summing the potential fluid flow rates for all discrete markings of a path, an important information is lost.

Example 5. *Consider the LFSPNs N and N' in Figure 2, such that $N \equiv_{fl} N'$. In Figure 5, the EDTMCs $EDTMC(N)$ and $EDTMC(N')$ are presented.*

The TPMs $\mathbf{P}$ and $\mathbf{P}'$ for $EDTMC(N)$ and $EDTMC(N')$ are

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{P}' = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have $t_1t_2 \in TranSeq\left(N, ab, \frac{1}{2}\circ\frac{1}{2}\circ\frac{1}{2}, 1\circ(-2)\circ 1\right)$ and $t_1t_3 \in TranSeq\left(N, ac, \frac{1}{2}\circ\frac{1}{2}\circ\frac{1}{2}, 1\circ(-2)\circ 1\right)$. We also get $t_1't_3' \in TranSeq\left(N', ab, \frac{1}{2}\circ\frac{1}{2}\circ\frac{1}{2}, 1\circ(-2)\circ 1\right)$

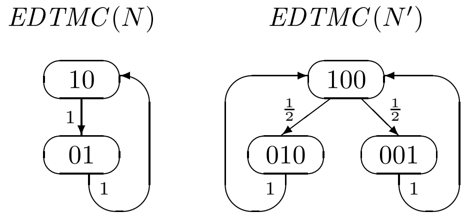

FIG. 5. The EDTMCs of the fluid trace equivalent LFSPNs

and $t_2't_4' \in TranSeq\left(N', ac, \frac{1}{2} \circ \frac{1}{2} \circ \frac{1}{2}, 1 \circ (-2) \circ 1\right)$. *It holds that* $PT(t_1t_2) = PT(t_1t_3) = 1 \cdot \frac{1}{2} = \frac{1}{2}$ *and* $PT(t_1't_3') = PT(t_2't_4') = \frac{1}{2} \cdot 1 = \frac{1}{2}$. *Then we have the equality* $FluStochTraces(N) = \{\left(\left(\varepsilon, \frac{1}{2}, 1\right), 1\right), \left(\left(a, \frac{1}{2} \circ \frac{1}{2}, 1 \circ (-2)\right), 1\right), \left(\left(ab, \frac{1}{2} \circ \frac{1}{2} \circ \frac{1}{2}, \ 1 \circ (-2) \circ 1\right), \frac{1}{2}\right), \left(\left(ac, \frac{1}{2} \circ \frac{1}{2} \circ \frac{1}{2}, 1 \circ (-2) \circ 1\right), \frac{1}{2}\right), \ldots\} = FluStochTraces(N')$.

6. Fluid bisimulation equivalence

Bisimulation equivalences respect particular points of choice in the behavior of a system. To define fluid bisimulation equivalence, we have to consider a bisimulation being an *equivalence* relation that partitions the states of the *union* of the discrete reachability graphs $DRG(N)$ and $DRG(N')$ of the LFSPNs N and N'. For N and N' to be bisimulation equivalent the initial states M_N and $M_{N'}$ of their discrete reachability graphs should be related by a bisimulation having the following transfer property: if two states are related then in each of them the same actions can occur, leading with the identical overall rate from each of the two states to *the same equivalence class* for every such action.

The novelty of the fluid bisimulation definition with respect to that of the Markovian bisimulations from [28, 53, 14, 18, 15, 16, 19, 21, 22, 17] is that, for each pair of bisimilar discrete markings of N and N', we require coincidence of the fluid flow rates of the continuous places of N and N' in these two discrete markings. Thus, fluid bisimulation equivalence takes into account *functional activity*, *stochastic timing* and *fluid flow*, like fluid trace equivalence does.

In [59, 82], it has been shown that probabilistic bisimulation equivalence coincides with probabilistic testing one on reactive probabilistic transition systems (for each state, the probabilities of its outgoing transitions by the same action are summed to one) under the image-finiteness or the minimal probability assumption. The probabilistic testing there is based on collecting the probabilities of observing or not observing actions while applying to a reactive probabilistic process each observation (execution experience) of a (possibly branching) test process with a goal to calculate the observation probability.

For Markovian bisimulation equivalence, such a testing characterization does not yet exist. Two identical variants of Markovian testing equivalence have been proposed in [20] on $EMPA_{ct}$, a sublanguage for continuous time Markovian processes of the Markovian process algebra $EMPA$, and in [14] on (sequential) Markovian process calculus MPC. The Markovian testing there is based on summing either the average sojourn times or actual (exponentially distributed) delays in the states of each computation to calculate its duration, which should be not greater than a given amount of time. It has been proved that Markovian testing equivalence is strictly coarser than Markovian bisimulation one.

Fluid bisimulation equivalence on LFSPNs is a natural enhancement of Markovian bisimulation equivalence by adding the identity condition for the fluid flow rates in the related states. The same condition may be imposed on Markovian testing equivalence to get fluid testing equivalence on LFSPNs. Then, following [20, 14], it will be easy to prove that fluid testing equivalence is strictly weaker than fluid bisimulation equivalence. Thus, unlike fluid trace equivalence, fluid bisimulation equivalence cannot be tested by an external observer using the mentioned fluid testing approach and it should be defined in an operational manner.

We first propose some helpful extensions of the rate functions for the discrete marking changes and for the fluid flow in continuous places. Let N be an LFSPN and $\mathcal{H} \subseteq DRS(N)$. Then, for each $M \in DRS(N)$ and $a \in Act$, we write $M \xrightarrow{a}_\lambda \mathcal{H}$, where $\lambda = RM_a(M, \mathcal{H})$ is the *overall rate to move from M into the set of discrete markings $\mathcal{H}$ by action a*, defined as

$$RM_a(M, \mathcal{H}) = \sum_{\{t \mid \exists \widetilde{M} \in \mathcal{H}\ M \xrightarrow{t} \widetilde{M},\ L_N(t)=a\}} \Omega_N(t, M).$$

We write $M \xrightarrow{a} \mathcal{H}$ if $\exists \lambda\ M \xrightarrow{a}_\lambda \mathcal{H}$. Further, we write $M \rightarrow_\lambda \mathcal{H}$ if $\exists a\ M \xrightarrow{a} \mathcal{H}$, where $\lambda = RM(M, \mathcal{H})$ is the *overall rate to move from M into the set of discrete markings $\mathcal{H}$ by any actions*, defined as

$$RM(M, \mathcal{H}) = \sum_{\{t \mid \exists \widetilde{M} \in \mathcal{H}\ M \xrightarrow{t} \widetilde{M}\}} \Omega_N(t, M).$$

To construct a fluid bisimulation between LFSPNs N and N', we should consider the "composite" set of their discrete markings $DRS(N) \cup DRS(N')$, since we have to identify the rates to come from any two equivalent discrete markings into the same "composite" equivalence class (with respect to the fluid bisimulation). Note that, for $N \neq N'$, transitions starting from the discrete markings of $DRS(N)$ (or $DRS(N')$) always lead to those from the same set, since $DRS(N) \cap DRS(N') = \emptyset$, and this allows us to "mix" the sets of discrete markings in the definition of fluid bisimulation.

Let $Pc_N = \{q\}$ and $Pc_{N'} = \{q'\}$. Then for $M \in DRS(N)$ (or for $M' \in DRS(N')$) we denote by $RP(M)$ (or by $RP(M')$) the fluid level change rate for the continuous place q (or for the corresponding one q'), i.e. the argument discrete marking determines for which of the two continuous places, q or q', the flow rate function RP is taken.

Definition 14. *Let N and N' be LFSPNs such that $Pc_N = \{q\},\ Pc_{N'} = \{q'\}$. An* equivalence *relation $\mathcal{R} \subseteq (DRS(N) \cup DRS(N'))^2$ is a* fluid bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{fl} N'$, if:*

(1) $(M_N, M_{N'}) \in \mathcal{R}$.
(2) $(M_1, M_2) \in \mathcal{R} \ \Rightarrow\ RP(M_1) = RP(M_2),\ \forall \mathcal{H} \in (DRS(N) \cup DRS(N'))/_{\mathcal{R}}$, $\forall a \in Act$

$$M_1 \xrightarrow{a}_\lambda \mathcal{H} \ \Leftrightarrow\ M_2 \xrightarrow{a}_\lambda \mathcal{H}.$$

Two LFSPNs N and N' are fluid bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_{fl} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{fl} N'$.*

Let $\mathcal{R}_{fl}(N, N') = \bigcup\{\mathcal{R} \mid \mathcal{R} : N \underline{\leftrightarrow}_{fl} N'\}$ be the *union of all fluid bisimulations* between N and N'. The following proposition proves that $\mathcal{R}_{fl}(N, N')$ is also an *equivalence* and $\mathcal{R}_{fl}(N, N') : N \underline{\leftrightarrow}_{fl} N'$.

Proposition 1. [74, 75] *Let N and N' be LFSPNs and $N \underline{\leftrightarrow}_{fl} N'$. Then $\mathcal{R}_{fl}(N, N')$ is the largest fluid bisimulation between N and N'.*

Proof. Analogous to that of Proposition 8.2.1 from [53], which establishes the result for strong equivalence. □

We now intend to compare the introduced fluid equivalences to discover their interrelations. The following theorem demonstrates that fluid bisimulation equivalence is strictly stronger than fluid trace one.

Theorem 1. [75] *For LFSPNs N and N' the following* strict *implication holds:*

$$N \underline{\leftrightarrow}_{fl} N' \;\Rightarrow\; N \equiv_{fl} N'.$$

Proof. Let us check the validity of the implication.

- The implication $\underline{\leftrightarrow}_{fl} \to \equiv_{fl}$ is valid by Proposition 2 from [75].

Let us see that the implication is strict, i.e. the reverse one does not work, by the following counterexample.

- In Figure 2, $N \equiv_{fl} N'$, but $N \underline{\not\leftrightarrow}_{fl} N'$, since only in the LFSPN N' an action a can be executed so (by firing the transition t_2') that no action b can occur afterwards. □

Example 6. *In Figure 6, the LFSPNs N and N' are presented with $N \underline{\leftrightarrow}_{fl} N'$. The only difference between the* respective *LFSPNs in Figure 2 and in Figure 6 is that the transitions t_3 and t_4' are labeled with action c in the former, instead of action b in the latter.*

Therefore, the following notions coincide for the respective *LFSPNs in Figure 2 and those in Figure 6: the discrete reachability sets $DRS(N)$ and $DRS(N')$; the discrete reachability graphs $DRG(N)$ and $DRG(N')$; the underlying CTMCs $CTMC(N)$ and $CTMC(N')$; the average sojourn time vectors SJ and SJ' of N and N'; the TRMs $\mathbf{Q}$ and $\mathbf{Q}'$, the steady-state PMFs φ and φ' for $CTMC(N)$ and $CTMC(N')$; the TPMs $\mathbf{P}$ and $\mathbf{P}'$ for $EDTMC(N)$ and $EDTMC(N')$; the FRMs $\mathbf{R}$ and $\mathbf{R}'$, the steady-state fluid PDFs $F(x)$ and $F'(x)$, the steady-state fluid probability density functions $f(x)$ and $f'(x)$, the steady-state buffer empty probabilities ℓ and ℓ' for the SFMs of N and N'.*

We have $(DRS(N) \cup DRS(N'))/_{\mathcal{R}_{fl}(N,N')} = \{\mathcal{H}_1, \mathcal{H}_2\}$, where $\mathcal{H}_1 = \{M_1, M_1'\}$, $\mathcal{H}_2 = \{M_2, M_2', M_3'\}$.

7. Logic HML_{flt}

The modal logic HML_{NPMTr} has been introduced in [18, 15, 19] (called HML_{MTr} in [18, 15]) on (sequential) and concurrent Markovian process calculi SMPC (called MPC in [18, 19]) and CMPC for logical interpretation of Markovian trace equivalence. HML_{NPMTr} is based on the logic HML [52], to which a new interpretation function has been added that takes as arguments a process state and a sum or a sequence of the average sojourn times.

We now propose a novel fluid modal logic HML_{flt} for the characterization of fluid trace equivalence. For this, we extend the interpretation function of

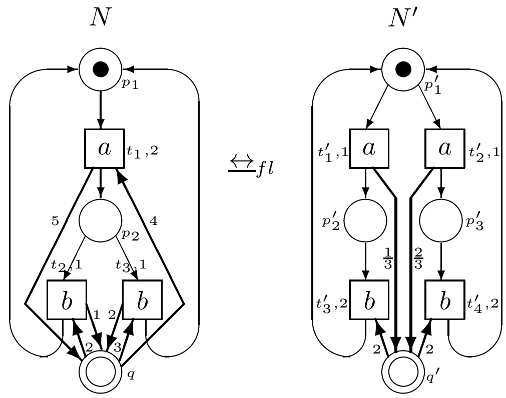

Fig. 6. Fluid bisimulation equivalent LFSPNs

HML_{NPMTr} with an additional argument, which is the sequence of the potential fluid flow rates for the single continuous place of an LFSPN (remember that in the definition of fluid trace equivalence we compare only LFSPNs, each having exactly one continuous place).

Note that Markovian trace equivalence and the corresponding interpretation function for HML_{MTr} in [18] are defined by summing up the average sojourn times in the process states. In our definition of fluid trace equivalence, we consider sequences of the average sojourn times in the discrete markings of LFSPNs. Hence, our fluid extension of HML_{NPMTr} is based rather on the definitions from [15, 19], where the latter approach (i.e. the sequences instead of sums) has been presented.

Definition 15. *Let* $\top$ *denote the truth and* $a \in Act$. *A* formula *of* HML_{flt} *is defined as follows:*

$$\Phi ::= \ \top \mid \langle a\rangle\Phi.$$

$\mathbf{HML}_{flt}$ denotes the set of *all formulas of the logic* HML_{flt}.

The interpretation function measures the probability with which a formula of HML_{flt} is satisfied in a discrete marking during the exponentially distributed time periods with given averages when the potential fluid flow rates have particular values.

Definition 16. *Let* N *be an LFSPN and* $M \in DRS(N)$. *The* interpretation function $[\![\,]\!]_{flt} : \mathbf{HML}_{flt} \to (DRS(N) \times \mathbb{R}^*_{>0} \times \mathbb{R}^* \to [0;1])$ *is defined as follows:*

(1) $$[\![\top]\!]_{flt}(M,\varsigma,\varrho) = \begin{cases} 0, & (\varsigma \neq SJ(M)) \vee (\varrho \neq RP(M)); \\ 1, & (\varsigma = SJ(M)) \wedge (\varrho = RP(M)); \end{cases}$$

(2) $$[\![\langle a\rangle\Phi]\!]_{flt}(M,\varsigma,\varrho) = \begin{cases} 0, & (\varsigma = \varepsilon) \vee (\varrho = \varepsilon) \vee \\ & ((\varsigma = s \circ \hat{\varsigma}) \wedge (SJ(M) \neq s)) \vee \\ & ((\varrho = r \circ \hat{\varrho}) \wedge (RP(M) \neq r)); \\ \sum_{\{t \mid M \stackrel{t}{\to} \widetilde{M},\ L_N(t)=a\}} PT(t,M)[\![\Phi]\!]_{flt}(\widetilde{M},\hat{\varsigma},\hat{\varrho}), & (\varsigma = s \circ \hat{\varsigma}) \wedge \\ & (\varrho = r \circ \hat{\varrho}) \wedge \\ & (SJ(M) = s) \wedge (RP(M) = r). \end{cases}$$

Thus, the interpretation is formally defined as a function from the formulas (essentially specifying the sequences of actions) to the functions assigning a probability to each triple consisting of a discrete marking (from which a given action sequence starts), together with the coordinated sequences of the average sojourn times and potential fluid flow rates (both starting in that discrete marking). As a result, the interpretation gives a probability to each set of $(\sigma, \varsigma, \varrho)$-selected transition sequences starting in a (possibly non-initial) discrete marking (such sets will be formalized later).

Note that the item 1 in the definition above describes the situation when only the empty transition sequence should start in the discrete marking M to reach the state (which is M itself), described by the identically true formula. Since we have just a single (mentioned) true state, it remains to check that second and third arguments of the interpretation function are the sequences of length one, as well as that they are equal to the average sojourn time and fluid flow rate in M, respectively.

Definition 17. *Let N be an LFSPN. Then we define* $[\![\Phi]\!]_{flt}(N, \varsigma, \varrho) = [\![\Phi]\!]_{flt}(M_N, \varsigma, \varrho)$. *Two LFSPNs N and N' are* logically equivalent *in HML_{flt}, denoted by $N =_{HML_{flt}} N'$, if* $\forall \Phi \in \mathbf{HML}_{flt}\ \forall \varsigma \in \mathbb{R}^*_{>0}\ \forall \varrho \in \mathbb{R}^*\ [\![\Phi]\!]_{flt}(N, \varsigma, \varrho) = [\![\Phi]\!]_{flt}(N', \varsigma, \varrho)$.

Let N be an LFSPN and $M \in DRS(N),\ a \in Act$. The set of discrete markings reached from M by execution of action a, called the *image set*, is defined as $Image(M, a) = \{\widetilde{M} \mid M \xrightarrow{t} \widetilde{M},\ L_N(t) = a\}$. An LFSPN N is an *image-finite* one, if $\forall M \in DRS(N)\ \forall a \in Act\ |Image(M, a)| < \infty$.

The following lemma states that all LFSPNs (whose transition sets are finite by definition) are image-finite.

Lemma 2. *Every LFSPN is image-finite.*

Proof. Follows from the inherent image-finiteness of (discrete) Petri nets. □

In order to get the intended logical characterization, we need in some auxiliary definitions considering the transition sequences starting not just in the initial discrete marking of an LFSPN, but in any reachable one.

Definition 18. *Let N be an LFSPN and $M \in DRS(N)$. The set of* all (finite) transition sequences *in N* starting in the discrete marking M *is defined as*

$$TranSeq(N, M) = \{\vartheta \mid \vartheta = \varepsilon \text{ or } \vartheta = t_1 \cdots t_n,\ M = M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n\}.$$

Let $\vartheta = t_1 \cdots t_n \in TranSeq(N, M)$ and $M = M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n$. The *probability to execute the transition sequence ϑ starting in the discrete marking M* is

$$PT(M, \vartheta) = \prod_{i=1}^{n} PT(t_i, M_{i-1}).$$

For $\vartheta = \varepsilon$ we define $PT(M, \varepsilon) = 1$.

Let $\vartheta = t_1 \cdots t_n \in TranSeq(N, M)$ and $M = M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n$. The *action sequence* of ϑ is $L_N(\vartheta) = L_N(t_1) \cdots L_N(t_n) \in Act^*$. We also define $L_N(\varepsilon) = \varepsilon$. The *average sojourn time sequence* of $\vartheta = t_1 \cdots t_n$ is $SJ(M, \vartheta) = SJ(M_0) \circ \cdots \circ SJ(M_n) \in \mathbb{R}^*_{>0}$. We also define $SJ(M, \varepsilon) = SJ(M_0)$. The *(potential) fluid flow rate sequence* of $\vartheta = t_1 \cdots t_n$ is $RP(M, \vartheta) = RP(M_0) \circ \cdots \circ RP(M_n) \in \mathbb{R}^*$. We also define $RP(M, \varepsilon) = RP(M_0)$.

Definition 19. *Let N be an LFSPN, $M \in DRS(N)$ and $(\sigma, \varsigma, \varrho) \in Act^* \times \mathbb{R}^*_{>0} \times \mathbb{R}^*$. The set of* $(\sigma, \varsigma, \varrho)$-selected (finite) transition sequences *in N* starting in the discrete marking M *is defined as*

$$TranSeq(N, M, \sigma, \varsigma, \varrho) = \left\{ \vartheta \in TranSeq(N, M) \,\middle|\, \begin{array}{l} L_N(\vartheta) = \sigma,\ SJ(M, \vartheta) = \varsigma, \\ RP(M, \vartheta) = \varrho \end{array} \right\}.$$

The (cumulative) *probability to execute* $(\sigma, \varsigma, \varrho)$*-selected transition sequences starting in the discrete marking* M is

$$PT(M, \sigma, \varsigma, \varrho) = \sum_{\vartheta \in TranSeq(N, M, \sigma, \varsigma, \varrho)} PT(M, \vartheta).$$

The following lemma provides a recursive definition of $PT(M, \sigma, \varsigma, \varrho)$ that will be used later in the proofs.

Lemma 3. *Let* N *be an LFSPN and* $M \in DRS(N)$*. Then for all* $(\sigma, \varsigma, \varrho) \in Act^* \times \mathbb{R}^*_{>0} \times \mathbb{R}^*$ *such that* $\sigma = a \cdot \hat{\sigma},\ \varsigma = s \circ \hat{\varsigma},\ \varrho = r \circ \hat{\varrho}$*, where* $a \in Act,\ s \in \mathbb{R}_{>0},\ r \in \mathbb{R}$*, we have*

$$PT(M, \sigma, \varsigma, \varrho) = \sum_{\{t \mid M \xrightarrow{t} \widetilde{M},\ L_N(t) = a,\ SJ(M) = s,\ RP(M) = r\}} PT(t, M) PT(\widetilde{M}, \hat{\sigma}, \hat{\varsigma}, \hat{\varrho}).$$

Proof. See Appendix A.2. □

The following propositions show that there exists a bijective correspondence between fluid stochastic traces of LFSPNs and formulas of $\mathbf{HML}_{flt}$, by proving that the probabilities of the triples $(\sigma, \varsigma, \varrho) \in Act^* \times \mathbb{R}^*_{>0} \times \mathbb{R}^*$ coincide in the net and logical frameworks.

Proposition 2. *Let* N *be an LFSPN. Then for each* $\sigma \in Act^*$ *there exists* $\Phi_\sigma \in \mathbf{HML}_{flt}$ *such that* $\forall M \in DRS(N)\ \forall \varsigma \in \mathbb{R}^*_{>0}\ \forall \varrho \in \mathbb{R}^*$

$$[\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho) = PT(M, \sigma, \varsigma, \varrho).$$

Proof. See Appendix A.3. □

Proposition 3. *Let* N *be an LFSPN. Then for each* $\Phi \in \mathbf{HML}_{flt}$ *there exists* $\sigma_\Phi \in Act^*$ *such that* $\forall M \in DRS(N)\ \forall \varsigma \in \mathbb{R}^*_{>0}\ \forall \varrho \in \mathbb{R}^*$

$$PT(M, \sigma_\Phi, \varsigma, \varrho) = [\![\Phi]\!]_{flt}(M, \varsigma, \varrho).$$

Proof. See Appendix A.4. □

The following theorem provides fluid trace equivalence with the logical characterization within $\mathbf{HML}_{flt}$.

Theorem 2. *For LFSPNs* N *and* N'

$$N \equiv_{fl} N' \ \Leftrightarrow\ N =_{HML_{flt}} N'.$$

Proof. The result follows from Proposition 2 and Proposition 3, which establish a bijective correspondence between fluid stochastic traces of LFSPNs and formulas of $\mathbf{HML}_{flt}$. □

Thus, in the trace semantics, we obtained a logical characterization of the fluid behavioural equivalence or, symmetrically, an operational characterization of the fluid modal logic equivalence.

Example 7. *Consider the LFSPNs N and N' in Figure 2, for which it holds $N \equiv_{fl} N'$, hence, $N =_{HML_{flt}} N'$. In particular, for $\Phi = \langle\{a\}\rangle\langle\{b\}\rangle\top$ we have $\sigma_\Phi = a \cdot b$ and $[\![\Phi]\!]_{flt}(M_N, \frac{1}{2} \circ \frac{1}{2} \circ \frac{1}{2}, 1 \circ (-2) \circ 1) = PT(t_1 t_2) = 1 \cdot \frac{1}{2} = \frac{1}{2} = 1 \cdot \frac{1}{2} = PT(t_1' t_3') = [\![\Phi]\!]_{flt}(M_{N'}, \frac{1}{2} \circ \frac{1}{2} \circ \frac{1}{2}, 1 \circ (-2) \circ 1)$. Thus, N and N' have the same* probability $\frac{1}{2}$ *of the following evolution from their initial discrete markings: while the action a is ready for execution, the single continuous place of each LFSPN is filled with the potential flow rate 1 during the exponentially distributed time period with the average $\frac{1}{2}$;* then, *while the action b is ready for execution, the continuous place of each LFSPN is filled with the potential flow rate -2 (i.e. the place is actually emptied with the potential flow rate 2) during the exponentially distributed time period with the average $\frac{1}{2}$;* finally, *the continuous place of each LFSPN is filled with the potential flow rate 1 for the exponentially distributed time period with the average $\frac{1}{2}$.*

8. Logic HML_{flb}

The modal logic HML_{MB} has been introduced in [15, 19] on sequential and concurrent Markovian process calculi SMPC (called MPC in [19]) and CPMC for logical interpretation of Markovian bisimulation equivalence. HML_{MB} is based on the logic HML [52], in which the diamond operator was decorated with the rate lower bound. Hence, HML_{MB} can also be seen as a modification of the logic PML [59], where the probability lower bound that decorates the diamond operator was replaced with the rate lower bound.

We now propose a novel fluid modal logic HML_{flb} for the characterization of fluid bisimulation equivalence. For this, we add to HML_{MB} a new modality $\wr_r$, where $r \in \mathbb{R}$ is the potential fluid flow rate value for the single continuous place of an LFSPN (remember that in the definition of fluid bisimulation equivalence we compare only LFSPNs, each having exactly one continuous place). The formula $\wr_r$ is used to check whether the potential fluid flow rate in a discrete marking of an LFSPN equals r. Finding this fact refers to a particular condition from the fluid bisimulation definition. Thus, $\wr_r$ can be seen as a supplement to the PML and HML_{MB} formula ∇_a, where $a \in Act$, since ∇_a is used to check whether the transitions labeled with the action a cannot be fired in a state (discrete marking). Finding this fact violates the bisimulation transfer property.

Definition 20. *Let $\top$ denote the truth and $a \in Act,\ r \in \mathbb{R},\ \lambda \in \mathbb{R}_{>0}$. A* formula *of HML_{flb} is defined as follows:*

$$\Phi ::= \ \top \mid \neg\Phi \mid \Phi \wedge \Phi \mid \nabla_a \mid \wr_r \mid \langle a\rangle_\lambda \Phi.$$

We define $\langle a\rangle\Phi = \exists\lambda\ \langle a\rangle_\lambda\Phi$ and $\Phi \vee \Psi = \neg(\neg\Phi \wedge \neg\Psi)$.

$\mathbf{HML}_{flb}$ denotes the set of *all formulas of the logic HML_{flb}*.

The satisfaction relation is used to verify the validity of a formula of HML_{flb} in a discrete marking.

Definition 21. *Let N be a LFSPN and $M \in DRS(N)$. The* satisfaction relation *$\models_{flb} \subseteq DRS(N) \times \mathbf{HML}_{flb}$ is defined as follows:*

(1) $M \models_{flb} \top$ *– always;*
(2) $M \models_{flb} \neg\Phi$*, if* $M \not\models_N \Phi$*;*
(3) $M \models_{flb} \Phi \wedge \Psi$*, if* $M \models_N \Phi$ *and* $M \models_N \Psi$*;*

(4) $M \models_{flb} \nabla_a$, *if it does not hold that* $M \xrightarrow{a} DRS(N)$;
(5) $M \models_{flb} \wr_r$, *if* $RP(M) = r$;
(6) $M \models_{flb} \langle a \rangle_\lambda \Phi$, *if* $\exists \mathcal{H} \subseteq DRS(N)\ M \xrightarrow{a}_\mu \mathcal{H},\ \mu \geq \lambda$ *and* $\forall \widetilde{M} \in \mathcal{H}\ \widetilde{M} \models_{flb} \Phi$.

Note that $\langle a \rangle_\mu \Phi$ implies $\langle a \rangle_\lambda \Phi$, if $\mu \geq \lambda$.

Definition 22. *Let N be an LFSPN. Then we write $N \models_{flb} \Phi$, if $M_N \models_{flb} \Phi$. LFSPNs N and N' are* logically equivalent *in HML_{flb}, denoted by $N =_{HML_{flb}} N'$, if* $\forall \Phi \in \mathbf{HML}_{flb}\ N \models_{flb} \Phi \ \Leftrightarrow\ N' \models_{flb} \Phi$.

The following theorem provides fluid bisimulation equivalence with the logical characterization within $\mathbf{HML}_{flb}$.

Theorem 3. *For LFSPNs N and N'*

$$N \underline{\leftrightarrow}_{fl} N' \ \Leftrightarrow\ N =_{HML_{flb}} N'.$$

Proof. See Appendix A.5. □

Thus, in the bisimulation semantics, we obtained a logical characterization of the fluid behavioural equivalence or, symmetrically, an operational characterization of the fluid modal logic equivalence.

Example 8. *Consider the LFSPNs N and N' in Figure 2, for which $N \underline{\not\leftrightarrow}_{fl} N'$, hence, $N \neq_{HML_{flb}} N'$. Indeed, for $\Phi = \langle a \rangle_2 \langle b \rangle_1 \top$ we have $N \models_{flb} \Phi$, but $N' \not\models_{flb} \Phi$, since only in N' action a can occur so that action b cannot occur afterwards.*

Take now the LFSPNs N and N' in Figure 6, for which $N \underline{\leftrightarrow}_{fl} N'$, hence, $N =_{HML_{flb}} N'$. In particular, for $\Psi = \wr_1 \wedge \langle a \rangle_2 (\wr_{-2} \wedge \langle b \rangle_2 \top)$ we have $N \models_{flb} \Psi$ and $N' \models_{flb} \Psi$. Thus, for N and N' the following evolution from their initial discrete markings is valid*: while the action a is ready for execution, the single continuous place of each LFSPN is filled with the potential flow rate 1 during the exponentially distributed time period with the minimal rate 2;* then*, while the action b is ready for execution, the continuous place of each LFSPN is filled with the potential flow rate -2 (i.e. the place is actually emptied with the potential flow rate 2) during the exponentially distributed time period with the minimal rate 2.*

9. Conclusion

In this paper, we have investigated two behavioural equivalences that preserve the qualitative and quantitative behavior of LFSPNs with a single continuous place, related to both their discrete part (labeled CTSPNs and the underlying CTMCs) and continuous part (the associated SFMs). We have considered on LFSPNs a linear-time relation of fluid trace equivalence and a branching-time relation of fluid bisimulation equivalence. Both equivalences respect *functional activity*, *stochastic timing* and *fluid flow* in the behaviour of LFSPNs.

9.1. **Fluid logical characterizations.** As the main result, we have characterized logically fluid trace and bisimulation equivalences with two novel fluid modal logics HML_{flt} and HML_{flb}. The characterizations give rise to better understanding of the basic features of the equivalences. In [18, 19], the local and global approaches to the temporal aspects of computations were explained. The local approach considers such aspects at the level of the individual actions in the computations while the global approach does it at the level of global computations. In the local case, the temporal parameters should be in the modal operators and the interpretation of

TABLE 1. Behavioural aspects of LFSPNs in the logical modalities and interpretations

Fluid modal logic	Semantics type (lin./branch. time)	Functional activity (action occurrences)	Stochastic timing (transition rates)	Fluid flow (fluid rates)
HML_{flt}	$\top$	$\langle a \rangle$	$[\![\cdot]\!]_{flt}(M, \boldsymbol{\varsigma}, \varrho)$	$[\![\cdot]\!]_{flt}(M, \varsigma, \boldsymbol{\varrho})$
HML_{flb}	$\top$, $\neg$, $\wedge$	∇_a, $\langle \boldsymbol{a} \rangle_\lambda$	$\langle a \rangle_{\boldsymbol{\lambda}}$	$\wr_r$

the formulas should be qualitative, i.e. it should return the truth value if a formula is satisfied. In the global case, the temporal parameters should not be present in the syntax and the interpretation of the formulas should be quantitative, i.e. it should give a value that measures how much (in which degree) a formula is satisfied. We have used the global approach in HML_{flt} and the local approach in HML_{flb}. Table 1 demonstrates how the modalities and interpretation functions of the logics HML_{flt} and HML_{flb} respect the following behavioural aspects of LFSPNs: *semantics type* (linear or branching time), *functional activity* (consisting in the action occurrences), *stochastic timing* (specified by the transition rates) and *fluid flow* (defined by the fluid rates). In case of the composite constructions, the variables describing particular aspects of behaviour are printed in bold font.

According to [1], we have demonstrated that the fluid equivalences are reasonable notions, by constructing their natural and mathematically elegant modal characterizations. In addition, the characterizations offer a possibility for the logical reasoning on resemblance of the fluid behaviour, while before it was only possible in the operational manner, as the following example shows.

Example 9. *Consider the LFSPNs N and N' in Figure 7 that model the production line from Example 1, for which $N \underline{\leftrightarrow}_{fl} N'$. Since LFSPNs have an* interleaving *semantics due to the* continuous *time approach and the* race *condition applied to transition firings, the parallel execution of actions (here in N) is modeled by the sequential non-determinism (in N'). Fluid bisimulation equivalence is an interleaving relation constructed in conformance with the LFSPNs semantics. In the initial discrete marking M_N, we now can specify and verify formally the properties described there: the* probability *given by the interpretation $[\![\langle f_1 \rangle \langle f_2 \rangle \top]\!]_{flt}(M_N, s_1 s_2 s_3, r_1 r_2(-r_3)) = \frac{\lambda_1}{\lambda_1+\lambda_2}$ in HML_{flt} and the* validity *of the satisfaction $M_N \models_{flb} \wr_{r_1} \wedge (\langle f_1 \rangle_{\lambda_1} \top \vee \langle f_2 \rangle_{\lambda_2} \top)$ in HML_{flb}, where $s_1 = \frac{1}{\lambda_1+\lambda_2}$, $s_2 = \frac{1}{\lambda_2}$, $s_3 = \frac{1}{\lambda_3}$, $r_1 = w_1 + w_2$, $r_2 = w_1$, $r_3 = w_3$. The same holds for the LFSPN N', since we have $N \underline{\leftrightarrow}_{fl} N'$, hence, also $N \equiv_{fl} N'$.*

A possible continuation of the presented work may be characterization of the fluid equivalences via more expressive logics, such as the CSL fluid extensions resembling the temporal logic for SFMs [45].

Moreover, by applying the theory from [50, 56], we can consider fluid equivalences of the LFSPNs with the level-dependent functions specifying transition firing rates and (piecewise-constant) fluid flow rates of continuous arcs. The associated SFM of each such extended LFSPN will be described by the TRM $\mathbf{Q}(x)$ (variable x denotes a fluid level value) with the non-diagonal elements $RM(M_i, M_j, x)$ ($1 \leq i, j \leq n$, $x \in [0; +\infty)$), and also by the FRM $\mathbf{R}(x)$ with the diagonal elements $RP(M_i, x)$ ($1 \leq i \leq n$, $x \in [0; +\infty)$), the latter being fluid level-independent functions within intervals between the boundaries, where fluid probability mass

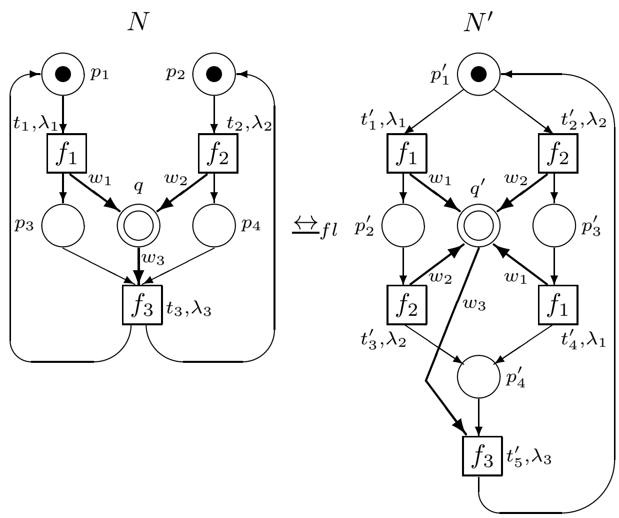

FIG. 7. The LFSPNs of the production line

may be created. For the model stability, the fluid flow rate must be negative in the last interval that is also the only infinite range. The matrices $\mathbf{Q}(x)$ and $\mathbf{R}(x)$ are used to build the ODE systems in those intervals for the probability density functions, so that the lumping approach may be applied. Then our results on the interrelations and quotienting will be also valid for the "level-sensitive" versions of the fluid equivalences. The corresponding fluid modal logics may be constructed for their characterization, by imposing fluid level dependence to the interpretation function of HML_{flt} and those modalities of HML_{flb}, which respect the transition and flow rates.

9.2. **Fluid place bisimulations.** In the future, we also plan to define a fluid place bisimulation relation that connects "similar" continuous places of LFSPNs, like place bisimulations [5, 4, 71, 72, 73] relate discrete places of (standard) Petri nets. The *lifting* of the relation to the discrete-continuous LFSPN markings (with discrete markings treated as the multisets of places) will respect both the fluid distribution among the related continuous places and the rates of fluid flow through them. For this purpose, we should introduce a novel notion of the multiset analogue with non-negative real-valued multiplicities of the elements. While multiset is a mapping from a countable set to all natural numbers, we need a more sophisticated mapping from the set of continuous places to all non-negative real numbers, corresponding to the associated fluid levels. Such an extension of the multiset notion may use the results of [23, 70], concerning hybrid sets (the multiplicities of the elements are arbitrary integers) and fuzzy multisets (the multiplicities belong to the interval [0;1]). In this way, both the initial amount of fluid and its transit flow rate in each discrete marking may be distributed among several continuous places of an LFSPN, such that all of them are bisimilar to a particular continuous place of the equivalent LFSPN.

The interesting point here is that fluid distributed among several bisimilar continuous places should be taken as the fluid contained in a single continuous place, resulting from aggregating those "constituent" continuous places with the use of fluid place bisimulation. Then the fluid level in the "aggregate" continuous place will be a sum of the fluid levels in the "constituent" continuous places. The probability density function for the sum of random variables representing the fluid levels in the "constituent" continuous places is defined via *convolution* operation. In this approach, we should avoid or treat correctly the situations when the fluid flow in the "aggregate" continuous place becomes suddenly non-continuous. This happens when some of the "constituent" continuous places are emptied while the others still contain a positive amount of fluid. Obviously, such a discontinuity is a result of applying the aggregation since it is not caused by either reaching the lower fluid boundary (zero fluid level) or change of the current discrete marking.

After doing this, it would be rather interesting to provide fluid place bisimulation equivalences with logical characterizations by constructing new fluid "place" logics, whose modalities are capable to specify "aggregate" fluid flow rates, i.e. the sums of the rates for the equivalent continuous places.

Appendix A. Proofs

A.1. **Proof of Lemma 1.** We prove by induction on the transition sequences length n.

- $n = 0$
  By definition, $\sum_{\{\vartheta\in TranSeq(N)\mid |\vartheta|=0\}} PT(\vartheta)=PT(\varepsilon)=1$.
- $n \to n+1$
  By distributivity law for multiplication and addition, and since $\forall M\in\mathbb{N}^{|Pd_N|}\ \sum_{t\in Ena(M)} PT(t,M)=1$, we get $\sum_{\{\vartheta\in TranSeq(N)\mid |\vartheta|=n+1\}} PT(\vartheta)=$
  $\sum_{\{t_1,\dots,t_n,t_{n+1}\mid M_N=M_0\stackrel{t_1}{\to}M_1\stackrel{t_2}{\to}\cdots\stackrel{t_n}{\to}M_n\stackrel{t_{n+1}}{\to}M_{n+1}\}} \prod_{i=1}^{n+1} PT(t_i,M_{i-1})=$
  $\sum_{\{t_1,\dots,t_n\mid M_N=M_0\stackrel{t_1}{\to}M_1\stackrel{t_2}{\to}\cdots\stackrel{t_n}{\to}M_n\}}\sum_{\{t_{n+1}\mid M_n\stackrel{t_{n+1}}{\to}M_{n+1}\}} \prod_{i=1}^{n} PT(t_i,M_{i-1})PT(t_{n+1},M_n)=$
  $\sum_{\{t_1,\dots,t_n\mid M_N=M_0\stackrel{t_1}{\to}M_1\stackrel{t_2}{\to}\cdots\stackrel{t_n}{\to}M_n\}}\left(\prod_{i=1}^{n} PT(t_i,M_{i-1})\sum_{\{t_{n+1}\mid M_n\stackrel{t_{n+1}}{\to}M_{n+1}\}} PT(t_{n+1},M_n)\right)=$
  $\sum_{\{t_1,\dots,t_n\mid M_N=M_0\stackrel{t_1}{\to}M_1\stackrel{t_2}{\to}\cdots\stackrel{t_n}{\to}M_n\}} \prod_{i=1}^{n} PT(t_i,M_{i-1})\cdot 1=1.$ □

A.2. **Proof of Lemma 3.** We have $PT(M,\sigma,\varsigma,\varrho)=\sum_{\vartheta\in TranSeq(N,M,\sigma,\varsigma,\varrho)} PT(M,\vartheta)=$

$$\sum_{\{t_1,\dots,t_n\mid M=M_0\stackrel{t_1}{\to}M_1\stackrel{t_2}{\to}\cdots\stackrel{t_n}{\to}M_n,\ L_N(t_1\cdots t_n)=\sigma,\ SJ(M_0,t_1\cdots t_n)=\varsigma,\ RP(M_0,t_1\cdots t_n)=\varrho\}} \prod_{i=1}^{n} PT(t_i,M_{i-1})=$$

$$\sum_{\{t_1\mid M=M_0\stackrel{t_1}{\to}M_1,\ L_N(t_1)=a,\ SJ(M_0)=s,\ RP(M_0)=r\}}$$

$$\sum_{\{t_2,\dots,t_n\mid M_1\stackrel{t_2}{\to}M_2\stackrel{t_3}{\to}\cdots\stackrel{t_n}{\to}M_n,\ L_N(t_2\cdots t_n)=\hat{\sigma},\ SJ(M_1,t_2\cdots t_n)=\hat{\varsigma},\ RP(M_1,t_2\cdots t_n)=\hat{\varrho}\}} PT(t_1,M_0)\prod_{i=2}^{n} PT(t_i,M_{i-1})=$$

$$\sum_{\{t_1\mid M=M_0\stackrel{t_1}{\to}M_1,\ L_N(t_1)=a,\ SJ(M_0)=s,\ RP(M_0)=r\}} PT(t_1,M_0)$$

$$\left(\sum_{\{t_2,\dots,t_n\mid M_1\stackrel{t_2}{\to}M_2\stackrel{t_3}{\to}\cdots\stackrel{t_n}{\to}M_n,\ L_N(t_2\cdots t_n)=\hat{\sigma},\ SJ(M_1,t_2\cdots t_n)=\hat{\varsigma},\ RP(M_1,t_2\cdots t_n)=\hat{\varrho}\}} \prod_{i=2}^{n} PT(t_i,M_{i-1})\right)=$$

$$\sum_{\{t_1 \mid M=M_0 \xrightarrow{t_1} M_1,\ L_N(t_1)=a,\ SJ(M_0)=s,\ RP(M_0)=r\}} PT(t_1, M_0)PT(M_1, \hat{\sigma}, \hat{\varsigma}, \hat{\varrho}).$$

Let us now take $t = t_1$ and $\widetilde{M} = M_1$. Then we have proved the lemma. □

A.3. **Proof of Proposition 2.** We prove by induction on the length n of the action sequence σ.

- $n = 0$

  We have $|\sigma| = 0$, hence, $\sigma = \varepsilon$. In this case, we take $\Phi_\sigma = \top$. Let $M \in DRS(N),\ \varsigma \in \mathbb{R}^*_{>0},\ \varrho \in \mathbb{R}^*$.

  If $(\varsigma \neq SJ(M)) \vee (\varrho \neq RP(M))$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) = \emptyset$ and

  $$[\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho) = 0 = PT(M, \sigma, \varsigma, \varrho).$$

  Otherwise, if $(\varsigma = SJ(M)) \wedge (\varrho = RP(M))$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) = \{\varepsilon\}$ and

  $$[\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho) = 1 = PT(M, \sigma, \varsigma, \varrho).$$

- $n \to n + 1$

  We have $|\sigma| = n+1$, hence, $\sigma = a \cdot \hat{\sigma}$, where $a \in Act$ and $|\hat{\sigma}| = n$. In this case, we take $\Phi_\sigma = \langle a \rangle \Phi_{\hat{\sigma}}$, where the induction hypothesis holds for $\hat{\sigma}$ and $\Phi_{\hat{\sigma}}$. Let $M \in DRS(N),\ \varsigma \in \mathbb{R}^*_{>0},\ \varrho \in \mathbb{R}^*$.

  If no transition labeled with action a is enabled in M or $(\varsigma = \varepsilon) \vee (\varrho = \varepsilon) \vee ((\varsigma = s \circ \hat{\varsigma}) \wedge (SJ(M) \neq s)) \vee ((\varrho = r \circ \hat{\varrho}) \wedge (RP(M) \neq r))$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) = \emptyset$ and

  $$[\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho) = 0 = PT(M, \sigma, \varsigma, \varrho).$$

  Otherwise, if transitions labeled with action a are enabled in M and $(\varsigma = s \circ \hat{\varsigma}) \wedge (SJ(M) = s) \wedge (\varrho = r \circ \hat{\varrho}) \wedge (RP(M) = r)$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) \neq \emptyset$ and

  $$[\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho) = \sum_{\{t \mid M \xrightarrow{t} \widetilde{M},\ L_N(t)=a\}} PT(t, M)[\![\Phi_{\hat{\sigma}}]\!]_{flt}(\widetilde{M}, \hat{\varsigma}, \hat{\varrho}),$$

  as well as

  $$PT(M, \sigma, \varsigma, \varrho) = \sum_{\{t \mid M \xrightarrow{t} \widetilde{M},\ L_N(t)=a\}} PT(t, M)PT(\widetilde{M}, \hat{\sigma}, \hat{\varsigma}, \hat{\varrho}).$$

  By the induction hypothesis, for all discrete markings $\widetilde{M}$ reachable from M by firing transitions labeled with action a we have

  $$[\![\Phi_{\hat{\sigma}}]\!]_{flt}(\widetilde{M}, \hat{\varsigma}, \hat{\varrho}) = PT(\widetilde{M}, \hat{\sigma}, \hat{\varsigma}, \hat{\varrho}),$$

  thus, we have proved the proposition. □

A.4. **Proof of Proposition 3.** We prove by induction on the syntactical structure of the logical formula Φ.

- $\Phi = \top$

  In this case, we take $\sigma_\Phi = \varepsilon$. Let $M \in DRS(N),\ \varsigma \in \mathbb{R}^*_{>0},\ \varrho \in \mathbb{R}^*$.
  If $(\varsigma \neq SJ(M)) \vee (\varrho \neq RP(M))$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) = \emptyset$ and

$$PT(M, \sigma, \varsigma, \varrho) = 0 = [\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho).$$

  Otherwise, if $(\varsigma = SJ(M)) \wedge (\varrho = RP(M))$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) = \{\varepsilon\}$ and

$$PT(M, \sigma, \varsigma, \varrho) = 1 = [\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho).$$

- $\Phi = \langle a \rangle \Phi$

  In this case, we take $\sigma_\Phi = a \cdot \sigma_{\widehat{\Phi}}$, where the induction hypothesis holds for $\widehat{\Phi}$ and $\sigma_{\widehat{\Phi}}$. Let $M \in DRS(N),\ \varsigma \in \mathbb{R}^*_{>0},\ \varrho \in \mathbb{R}^*$.
  If no transition labeled with action a is enabled in M or $(\varsigma = \varepsilon) \vee (\varrho = \varepsilon) \vee ((\varsigma = s \circ \hat{\varsigma}) \wedge (SJ(M) \neq s)) \vee ((\varrho = r \circ \hat{\varrho}) \wedge (RP(M) \neq r))$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) = \emptyset$ and

$$PT(M, \sigma, \varsigma, \varrho) = 0 = [\![\Phi_\sigma]\!]_{flt}(M, \varsigma, \varrho).$$

  Otherwise, if transitions labeled with action a are enabled in M and $(\varsigma = s \circ \hat{\varsigma}) \wedge (SJ(M) = s) \wedge (\varrho = r \circ \hat{\varrho}) \wedge (RP(M) = r)$ then $TranSeq(N, M, \sigma, \varsigma, \varrho) \neq \emptyset$ and

$$PT(M, \sigma_\Phi, \varsigma, \varrho) = \sum_{\{t \mid M \xrightarrow{t} \widetilde{M},\ L_N(t) = a\}} PT(t, M) PT(\widetilde{M}, \sigma_{\widehat{\Phi}}, \hat{\varsigma}, \hat{\varrho}),$$

  as well as

$$[\![\Phi]\!]_{flt}(M, \varsigma, \varrho) = \sum_{\{t \mid M \xrightarrow{t} \widetilde{M},\ L_N(t) = a\}} PT(t, M) [\![\widehat{\Phi}]\!]_{flt}(\widetilde{M}, \hat{\varsigma}, \hat{\varrho}).$$

  By the induction hypothesis, for all discrete markings $\widetilde{M}$ reachable from M by firing transitions labeled with action a we have

$$PT(\widetilde{M}, \sigma_{\widehat{\Phi}}, \hat{\varsigma}, \hat{\varrho}) = [\![\widehat{\Phi}]\!]_{flt}(\widetilde{M}, \hat{\varsigma}, \hat{\varrho}),$$

  thus, we have proved the proposition. □

A.5. **Proof of Theorem 3.** Our reasoning is based on the proofs of Theorem 6.4 from [59] about characterization of probabilistic bisimulation equivalence for probabilistic transition systems and Theorem 1 from [38] about characterization of strong equivalence for PEPA. The differences are the LFSPNs context, and that we also respect the fluid flow rates in the discrete markings with the satisfaction check for the formulas $\wr_r,\ r \in \mathbb{R}$, as presented below.

($\Leftarrow$) Let us define the equivalence relation $\mathcal{R} = \{(M_1, M_2) \in (DRS(N) \cup DRS(N'))^2 \mid \forall \Phi \in \mathbf{HML}_{flb}\ M_1 \models_{flb} \Phi \Leftrightarrow M_2 \models_{flb} \Phi\}$. We have $(M_N, M_{N'}) \in \mathcal{R}$. Let us prove that $\mathcal{R}$ is a fluid bisimulation.

Assume that $M_N \xrightarrow{a}_\lambda \mathcal{H} \in (DRS(N) \cup DRS(N'))/_\mathcal{R}$. Let $M_{N'} \xrightarrow{a}_{\lambda'_1} M'_1, \ldots, M_{N'} \xrightarrow{a}_{\lambda'_i} M'_i,\ M_{N'} \xrightarrow{a}_{\lambda'_{i+1}} M'_{i+1}, \ldots, M_{N'} \xrightarrow{a}_{\lambda'_n} M'_n$ be the changes of the discrete marking $M_{N'}$ as a result of executing the action a. Since the LFSPN N' is image-finite one by Lemma 2, the number of such changes is finite. The discrete marking changes are ordered so that $M'_1, \ldots, M'_i \in \mathcal{H}$ and $M'_{i+1}, \ldots, M'_n \notin \mathcal{H}$.

Then $\exists \Phi_{i+1}, \ldots, \Phi_n \in \mathbf{HML}_{flb}$ such that $\forall j\ (i+1 \leq j \leq n)\ \forall M \in \mathcal{H}\ M \models_{flb} \Phi_j$, but $M'_j \not\models_{flb} \Phi_j$. We have $M_N \models_{flb} \langle a \rangle_\lambda (\wedge_{j=i+1}^n \Phi_j)$ and $M_{N'} \models_{flb} \langle a \rangle_{\lambda'} (\wedge_{j=i+1}^n \Phi_j)$, where $\lambda' = \sum_{j=1}^i \lambda'_j$.

Assume that $\lambda > \lambda'$. Then $M_{N'} \not\models_{flb} \langle a \rangle_\lambda (\wedge_{j=i+1}^n \Phi_j)$, which contradicts to $(M_N, M_{N'}) \in \mathcal{R}$. Hence, $\lambda \leq \lambda'$. Consequently, $M_{N'} \xrightarrow{a}_{\lambda'} \mathcal{H}$, where $\lambda \leq \lambda'$. By symmetry of $\mathcal{R}$, we have $\lambda \geq \lambda'$. Thus, $\lambda = \lambda'$, and $\mathcal{R}$ is a fluid bisimulation.

($\Rightarrow$) Let for LFSPNs N and N' we have $N \underline{\leftrightarrow}_{fl} N'$. Then $\exists \mathcal{R} : N \underline{\leftrightarrow}_{fl} N'$ and $(M_N, M_{N'}) \in \mathcal{R}$. It is sufficient to consider only the cases $\nabla_a, \wr_r$ and $\langle a \rangle_\lambda \Phi$, since the remaining cases are trivial.

The case ∇_a.

Assume that $M_N \models_{flb} \nabla_a$. Then it does not hold that $M_N \xrightarrow{a} DRS(N)$. Hence, there exist no t and $\widetilde{M}$ such that $M_N \xrightarrow{t} \widetilde{M}$ and $L_N(t) = a$. Since summing by the empty index set produces zero, the transitions from each discrete marking always lead to the discrete markings of the discrete reachability set to which that discrete marking belongs and $(M_N, M_{N'}) \in \mathcal{R}$, we get

$0 = \sum_{\{t \mid \exists \widetilde{M} \in DRS(N)\ M_N \xrightarrow{t} \widetilde{M},\ L_N(t)=a\}} \Omega_N(t, M_N) = RM_a(M_N, DRS(N)) = RM_a(M_N, DRS(N) \cup DRS(N')) = \sum_{\mathcal{H} \in (DRS(N) \cup DRS(N'))/_\mathcal{R}} RM_a(M_N, \mathcal{H}) = \sum_{\mathcal{H} \in (DRS(N) \cup DRS(N'))/_\mathcal{R}} RM_a(M_{N'}, \mathcal{H}) = RM_a(M_{N'}, DRS(N) \cup DRS(N')) = RM_a(M_{N'}, DRS(N')) = \sum_{\{t' \mid \exists \widetilde{M}' \in DRS(N')\ M_{N'} \xrightarrow{t'} \widetilde{M}',\ L_{N'}(t')=a\}} \Omega_{N'}(t', M_{N'})$.

Hence, there exist no t' and $\widetilde{M}'$ such that $M_{N'} \xrightarrow{t'} \widetilde{M}'$ and $L_{N'}(t') = a$. Thus, it does not hold that $M_{N'} \xrightarrow{a} DRS(N')$ and we have $M_{N'} \models_{flb} \nabla_a$.

The case $\wr_r$.

Assume that $M_N \models_{flb} \wr_r$. Then, respecting that $(M_N, M_{N'}) \in \mathcal{R}$, we get $r = RP(M_N) = RP(M_{N'})$, hence, $M_{N'} \models_{flb} \wr_r$.

The case $\langle a \rangle_\lambda \Phi$.

Assume that $M_N \models_{flb} \langle a \rangle_\lambda \Phi$. Then $\exists \mathcal{H} \subseteq DRS(N)$ such that $M_N \xrightarrow{a}_\mu \mathcal{H},\ \mu \geq \lambda$ and $\forall M \in \mathcal{H}\ M \models_{flb} \Phi$. Let us define $\widetilde{\mathcal{H}} = \bigcup \{\overline{\mathcal{H}} \in (DRS(N) \cup DRS(N'))/_\mathcal{R} \mid \overline{\mathcal{H}} \cap \mathcal{H} \neq \emptyset\}$. Then $\forall \widetilde{M} \in \widetilde{\mathcal{H}}\ \exists M \in \mathcal{H}\ (M, \widetilde{M}) \in \mathcal{R}$. Since $\forall M \in \mathcal{H}\ M \models_{flb} \Phi$, we have $\forall \widetilde{M} \in \widetilde{\mathcal{H}} \widetilde{M} \models_{flb} \Phi$ by the induction hypothesis.

Since $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$, we get $M_N \xrightarrow{a}_{\tilde{\mu}} \widetilde{\mathcal{H}},\ \tilde{\mu} \geq \mu$. Since $\widetilde{\mathcal{H}}$ is the union of the equivalence classes with respect to $\mathcal{R}$, we have $(M_N, M_{N'}) \in \mathcal{R}$ implies $M_{N'} \xrightarrow{a}_{\tilde{\mu}} \widetilde{\mathcal{H}}$. Since $\tilde{\mu} \geq \mu \geq \lambda$, we get $M_{N'} \models_{flb} \langle a \rangle_\lambda \Phi$. Therefore, N' satisfies all the formulas which N does. By symmetry of $\mathcal{R}$, N satisfies all the formulas which N' does. Thus, the sets of satisfiable formulas for N and N' coincide. □

References

[1] L. Aceto, *Some of my favourite results in classic process algebra*, Bulletin of the EATCS, **81** (2003), 90–108. Zbl 1169.68533

[2] L. Aceto, D. Della Monica, I. Fábregas, A. Ingólfsdóttir, *When are prime formulae characteristic?* Lecture Notes in Computer Science, **9234** (2015), 76–88. Zbl 06482726

[3] A. Angius, A. Horváth, S.M. Halawani, O. Barukab, A.R. Ahmad, G. Balbo, *Use of flow equivalent servers in the transient analysis of product form queueing networks*, Lecture Notes in Computer Science, **9081** (2015), 15–29. Zbl 1392.68113

[4] C. Autant, W. Pfister, Ph. Schnoebelen, *Place bisimulations for the reduction of labeled Petri nets with silent moves*, Proc. 6^{th} Int. Conf. on Computing and Information (ICCI) 1994, Trent University, Peterborough, Ontario, Canada, May 1994, 230–246, 1994. http://www.lsv.ens-cachan.fr/Publis/PAPERS/PS/APS-icci94.ps

[5] C. Autant, Ph. Schnoebelen, *Place bisimulations in Petri nets*, Lecture Notes in Computer Science, **616** (1992), 45–61. MR1253088

[6] A. Aziz, K. Sanwal, V. Singhal, R. Brayton, *Model checking continuous time Markov chains*, ACM Transactions on Computational Logic, **1**:1 (2000), 162–170. Zbl 1365.68313

[7] C. Baier, E.M. Hahn, B.R. Haverkort, H. Hermanns, J.-P. Katoen, *Model checking for performability*, Mathematical Structures in Computer Science, **23**:4 (2013), 751–795. Zbl 1358.68181

[8] C. Baier, B.R. Haverkort, H. Hermanns, J.-P. Katoen, *Model checking continuous-time Markov chains by transient analysis*, Lecture Notes in Computer Science, **1855** (2000), 358–372. Zbl 0974.68017

[9] C. Baier, B.R. Haverkort, H. Hermanns, J.-P. Katoen, *Model-checking algorithms for continuous-time Markov chains*, IEEE Transactions on Software Engineering, **29**:6 (2003), 524–541.

[10] C. Baier, H. Hermanns, J.-P. Katoen, V. Wolf, *Comparative branching-time semantics for Markov chains*, Lecture Notes in Computer Science, **2761** (2003), 492–507. Zbl 1274.68264

[11] C. Baier, J.-P. Katoen, H. Hermanns, V. Wolf, *Comparative branching-time semantics for Markov chains*, Information and Computation, **200**:2 (2005), 149–214. Zbl 1101.68053

[12] G. Balbo, *Introduction to stochastic Petri nets*, Lecture Notes in Computer Science, **2090** (2001), 84–155. Zbl 0990.68092

[13] G. Balbo, *Introduction to generalized stochastic Petri nets*, Lecture Notes in Computer Science, **4486** (2007), 83–131. Zbl 1323.68400

[14] M. Bernardo, *Markovian testing and trace equivalences exactly lump more than Markovian bisimilarity*, Electronic Notes in Theoretical Computer Science, **162** (2006), 87–99. Zbl 1316.68086

[15] M. Bernardo, *A survey of Markovian behavioral equivalences*, Lecture Notes in Computer Science, **4486** (2007), 180–219. Zbl 1323.68402

[16] M. Bernardo, *Non-bisimulation-based Markovian behavioral equivalences*, Journal of Logic and Algebraic Programming, **72** (2007), 3–49. Zbl 1121.68077

[17] M. Bernardo, *ULTraS at work: compositionality metaresults for bisimulation and trace semantics*, Journal of Logical and Algebraic Methods in Programming, **94** (2018), 150–182. Zbl 1381.68195

[18] M. Bernardo, S. Botta, *Modal logic characterization of Markovian testing and trace equivalences*, Electronic Notes in Theoretical Computer Science, **169** (2006), 7–18. Zbl 1276.68119

[19] M. Bernardo, S. Botta, *A survey of modal logics characterizing behavioural equivalences for non-deterministic and stochastic systems*, Mathematical Structures in Computer Science, **18** (2008), 29–55. MR2459612

[20] M. Bernardo, R. Cleaveland, *A theory of testing for Markovian processes*, Lecture Notes in Computer Science, **1877** (2000), 305–319. Zbl 0999.68144

[21] M. Bernardo, R. De Nicola, M. Loreti, *A uniform framework for modeling nondeterministic, probabilistic, stochastic, or mixed processes and their behavioral equivalences*, Information and Computation, **225** (2013), 29–82. Zbl 1358.68210

[22] M. Bernardo, L. Tesei, *Encoding timed models as uniform labeled transition systems*, Lecture Notes in Computer Science, **8168** (2013), 104–118.

[23] W.D. Blizard, *The development of multiset theory*, The Review of Modern Logic, **1**:4 (1991), 319–352. MR1112352

[24] S. Blom, B.R. Haverkort, M. Kuntz, J. van de Pol, *Distributed Markovian bisimulation reduction aimed at CSL model checking*, Electronic Notes in Theoretical Computer Science, **220**:2 (2008), 35–50. Zbl 1286.68306

[25] A. Bobbio, S. Garg, M. Gribaudo, A. Horváth, M. Sereno, M. Telek, *Modeling software systems with rejuvenation, restoration and checkpointing through fluid stochastic Petri nets*, Proc. 8^{th} Int. Workshop on Petri Nets and Performance Models (PNPM) 1999, Zaragoza,

Spain, September 1999, 82–91, IEEE Computer Society Press, 1999. http://www.di.unito.it/~marcog/Downloads/PNPM99-BGGHST.pdf

[26] A. Bobbio, A. Puliafito, M. Telek, K.S. Trivedi, *Recent developments in non-Markovian stochastic Petri nets*, Journal of Circuits, Systems and Computers, **8**:1 (1998), 119–158. MR1740521

[27] M. Boreale, *Algebra, coalgebra, and minimization in polynomial differential equations*, Lecture Notes in Computer Science, **10203** (2017), 71–87. Zbl 06720984

[28] P. Buchholz, *A notion of equivalence for stochastic Petri nets*, Lecture Notes in Computer Science, **935** (1995), 161–180. MR1461026

[29] P. Buchholz, *Iterative decomposition and aggregation of labeled GSPNs*, Lecture Notes in Computer Science, **1420** (1998), 226–245.

[30] L. Cardelli, M. Tribastone, M. Tschaikowski, A. Vandin, *Forward and backward bisimulations for chemical reaction networks*, Proc. 26^{th} Int. Conf. on Concurrency Theory (CONCUR) 2015, Madrid, Spain, September 2015, Leibniz International Proceedings in Informatics (LIPIcs), **42** (2015), 226–239. Zbl 1374.68204

[31] L. Cardelli, M. Tribastone, M. Tschaikowski, A. Vandin, *Symbolic computation of differential equivalences*, Proc. 43^{rd} Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL) 2016, St. Petersburg, Florida, USA, January 2016, 137–150, ACM Press, 2016. http://sysma.imtlucca.it/wp-content/uploads/2015/11/z3-popl16.pdf Zbl 1347.68258

[32] L. Cardelli, M. Tribastone, M. Tschaikowski, A. Vandin, *Maximal aggregation of polynomial dynamical systems*, Proceedings the National Academy of Sciences of the United States of America, **114**:38 (2017), 10029–10034.

[33] L. Cardelli, M. Tribastone, M. Tschaikowski, A. Vandin, *Syntactic Markovian bisimulation for chemical reaction networks*, Lecture Notes in Computer Science, **10460** (2017), 466–483.

[34] L. Cardelli, M. Tribastone, M. Tschaikowski, A. Vandin, *Guaranteed error bounds on approximate model abstractions through reachability analysis*, Lecture Notes in Computer Science, **11024** (2018), 104–121.

[35] G. Ciardo, J.K. Muppala, K.S. Trivedi, *On the solution of GSPN reward models*, Performance Evaluation, **12**:4 (1991), 237–253. Zbl 0754.60097

[36] G. Ciardo, D. Nicol, K.S. Trivedi, *Discrete-event simulation of fluid stochastic Petri nets*, Report, **97-24**, 15 p., Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, Virginia, USA, 1997. http://historical.ncstrl.org/tr/pdf/icase/TR-97-24.pdf

[37] G. Ciardo, D. Nicol, K.S. Trivedi, *Discrete-event simulation of fluid stochastic Petri nets*, IEEE Transactions on Software Engineering, **25**:2 (1999), 207–217.

[38] G. Clark, S. Gimore, J. Hillston, *Specifying performance measures for PEPA*, Lecture Notes in Computer Science, **1601** (1999), 211–227.

[39] G. Clark, S. Gimore, J. Hillston, M. Ribaudo, *Exploiting modal logic to express performance measures*, Lecture Notes in Computer Science, **1786** (2000), 247–261. Zbl 0962.68120

[40] J. Desharnais, P. Panangaden, *Continuous stochastic logic characterizes bisimulation of continuous-time Markov processes*, Journal of Logic and Algebraic Programming, **56**:1–2 (2003), 99–115. Zbl 1053.68065

[41] E.-E. Doberkrat, *Congruences and bisimulations for continuous-time stochastic logic*, Lecture Notes in Computer Science, **3722** (2005), 409–423. Zbl 1169.68503

[42] A.I. Elwalid, D. Mitra, *Statistical multiplexing with loss priorities in rate-based congestion control of high-speed networks*, IEEE Transactions on Communications, **42**:11 (1994), 2989–3002.

[43] M. Gribaudo, *Hybrid formalism for performance evaluation: theory and applications*, Ph.D. thesis, 198 p., Department of Computer Science, University of Turin, Turin, Italy, 2002. http://www.di.unito.it/~marcog/Downloads/PhDThesis.pdf

[44] M. Gribaudo, A. Horváth, *Fluid stochastic Petri nets augmented with flush-out arcs: a transient analysis technique*, IEEE Transactions on Software Engineering, **28**:10 (2002), 944–955.

[45] M. Gribaudo, A. Horváth, *Model checking functional and performability properties of stochastic fluid models*, Electronic Notes in Theoretical Computer Science, **128**:6 (2005), 295–310. Zbl 1272.68251

[46] M. Gribaudo, D. Manini, B. Sericola, M. Telek, *Second order fluid models with general boundary behaviour*, Annals of Operations Research, **160** (2008), 69–82. Zbl 1138.76012

[47] M. Gribaudo, M. Sereno, *Simulation of fluid stochastic Petri nets*, Proc. 8^{th} Int. Symposium on Modeling, Analysis and Simulation of Computer and Telecommunication Systems (MASCOTS) 2000, San Francisco, USA, August 2000, 231–239, 2000. http://www.di.unito.it/~marcog/Downloads/MASCOTS00.pdf

[48] M. Gribaudo, M. Sereno, A. Horváth, A. Bobbio, *Fluid stochastic Petri nets augmented with flush-out arcs: modelling and analysis*, Discrete Event Dynamic Systems: Theory and Applications, **11**:1–2 (2001), 97–117. Zbl 0972.93040

[49] M. Gribaudo, M. Telek, *Fluid models in performance analysis*, Lecture Notes in Computer Science, **4486** (2007), 271–317. Zbl 1323.68051

[50] M. Gribaudo, M. Telek, *Stationary analysis of fluid level dependent bounded fluid models*, Performance Evaluation, **65**:3–4 (2008), 241–261.

[51] B.R. Haverkort, *Markovian models for performance and dependability evaluation*, Lecture Notes in Computer Science, **2090** (2001), 38–83. Zbl 0990.68020

[52] M.C.B. Hennessy, R.A.J. Milner, *Algebraic laws for non-determinism and concurrency*, Journal of the ACM, **32**:1 (1985), 137–161. MR0832336

[53] J. Hillston, *A compositional approach to performance modelling*, Cambridge University Press, Cambridge, UK, 1996. http://www.dcs.ed.ac.uk/pepa/book.pdf MR1427945

[54] G. Horton, V.G. Kulkarni, D.M. Nicol, K.S. Trivedi, *Fluid stochastic Petri nets: theory, applications, and solution techniques*, European Journal of Operations Research, **105**:1 (1998), 184–201. Zbl 0957.90011

[55] A. Horváth, M. Gribaudo, *Matrix geometric solution of fluid stochastic Petri nets*, Proc. 4^{th} Int. Conf. on Matrix-Analytic Methods in Stochastic Models (MAM) 2002, Adelaide, Australia, July 2002, 163–182, World Scientific, 2002. http://www.di.unito.it/~marcog/Downloads/HoGr02.pdf Zbl 1162.93382

[56] G. Horváth, M. Telek, *Matrix-analytic solution of infinite, finite and level-dependent second-order fluid models*, Queueing Systems — Theory and Applications, **87**:3–4 (2017), 325–343. Zbl 1377.60086

[57] G. Iacobelli, M. Tribastone, A. Vandin, *Differential bisimulation for a Markovian process algebra*, Lecture Notes in Computer Science, **9234** (2015), 293–306. Zbl 06482743

[58] V.G. Kulkarni, *Modeling and analysis of stochastic systems. Second edition*, Texts in Statistical Science, **84**, Chapman and Hall / CRC Press, 2009. Zbl 1191.60003

[59] K.G. Larsen, A. Skou, *Bisimulation through probabilistic testing*, Information and Computation, **94**:1 (1991), 1–28. Zbl 0756.68035

[60] M.A. Marsan, *Stochastic Petri nets: an elementary introduction*, Lecture Notes in Computer Science, **424** (1990), 1–29.

[61] M.A. Marsan, G. Balbo, G. Conte, S. Donatelli, G. Franceschinis, *Modelling with generalised stochastic Petri nets*, Wiley Series in Parallel Computing, John Wiley and Sons, 1995. http://www.di.unito.it/~greatspn/GSPN-Wiley/ Zbl 0843.68080

[62] M.K. Molloy, *Performance analysis using stochastic Petri nets*, IEEE Transactions on Computing, **31**:9 (1982), 913–917.

[63] M.K. Molloy, *Discrete time stochastic Petri nets*, IEEE Transactions on Software Engineering, **11**:4 (1985), 417–423. Zbl 0558.68053

[64] S.O. Natkin, *Les reseaux de Petri stochastiques et leur application a l'evalution des systemes informatiques*, Ph. D. thesis, Conseratoire National des Arts et Metiers, France, 1980 (in French).

[65] M.R. Pedersen, G. Bacci, K.G. Larsen, R. Mardare, *A hemimetric extension of simulation for semi-Markov decision processes*, Lecture Notes in Computer Science, **11024** (2018), 339–355.

[66] A. Sharma, *Trace relations and logical preservation for continuous-time Markov decision processes*, Lecture Notes in Computer Science, **10580** (2017), 192–209. Zbl 06802664

[67] A. Sharma, *Trace relations and logical preservation for Markov automata*, Lecture Notes in Computer Science, **11022** (2018), 162–178. Zbl 06989605

[68] L. Song, L. Zhang, J.C. Godskesen, *Bisimulations and logical characterizations on continuous-time Markov decision processes*, Lecture Notes in Computer Science, **8318** (2014), 98–117. Zbl 06385877

[69] J. Sproston, S. Donatelli, *Backward stochastic bisimulation in CSL model checking*, Proc. 1^{st} Int. Conf. on the Quantitative Evaluation of Systems (QEST) 2004, 220–229, IEEE Computer Society Press, 2004. http://www.di.unito.it/~sproston/Research/qest04.ps.gz

[70] A. Syropoulos, *Mathematics of multisets*, Lecture Notes in Computer Science, **2235** (2001), 347–358. MR2054267

[71] I.V. Tarasyuk, *Place bisimulation equivalences for design of concurrent and sequential systems*, Electronic Notes in Theoretical Computer Science, **18** (1998), 191–206. MR1672359

[72] I.V. Tarasyuk, *τ-equivalences and refinement for Petri nets based design*, Technische Berichte, **TUD-FI00-11**, 41 p., Fakultät Informatik, Technische Universität Dresden, Germany, 2000. http://itar.iis.nsk.su/files/itar/pages/eqtbrtud_cov.pdf, http://www.qucosa.de/fileadmin/data/qucosa/documents/10037/tud_TB_2000-11.pdf

[73] I.V. Tarasyuk, *Equivalences for behavioural analysis of concurrent and distributed computing systems*, Academic Publisher "Geo", Novosibirsk, 2007 (ISBN 978-5-9747-0098-9, in Russian).

[74] I.V. Tarasyuk, P. Buchholz, *Bisimulation for fluid stochastic Petri nets*, Bulletin of the Novosibirsk Computing Center, Series Computer Science, IIS Special Issue, **38** (2015), 121–149. Zbl 1374.68344

[75] I.V. Tarasyuk, P. Buchholz, *Equivalences for fluid stochastic Petri nets*, Siberian Electronic Mathematical Reports, **14** (2017), 317–366. Zbl 1373.68299

[76] K.S. Trivedi, V.G. Kulkarni, *FSPNs: fluid stochastic Petri nets*, Lecture Notes in Computer Science, **691** (1993), 24–31.

[77] M. Tribastone, A. Vandin, *Speeding up stochastic and deterministic simulation by aggregation: an advanced tutorial*, Proc. the 2018 Winter Simulation Conf. (WSC) 2018 (M. Rabe, A. Skoogh, N. Mustafee, A.A. Juan, eds.), Gothenburg, Sweeden, December 2018, 15 p., IEEE Computer Society Press, 2018. http://cse.lab.imtlucca.it/~mirco.tribastone/papers/wsc18.pdf

[78] M. Tschaikowski, M. Tribastone, *Exact fluid lumpability for Markovian process algebra*, Lecture Notes in Computer Science, **7454** (2012), 380–394. Zbl 1364.68297

[79] M. Tschaikowski, M. Tribastone, *Exact fluid lumpability in Markovian process algebra*, Theoretical Computer Science, **538** (2014), 140–166. Zbl 1359.68228

[80] M. Tschaikowski, M. Tribastone, *A unified framework for differential aggregations in Markovian process algebra*, Journal of Logical and Algebraic Methods in Programming, **84** (2015), 238–258. Zbl 1319.68151

[81] M. Tschaikowski, M. Tribastone, *Approximate reduction of heterogenous nonlinear models with differential hulls*, IEEE Transactions on Automatic Control, **61**:4 (2016), 1099–1104. Zbl 1359.93084

[82] V. Wolf, *Testing theory for probabilistic processes*, Lecture Notes in Computer Science, **347** (2005), 233–275.

[83] V. Wolf, C. Baier, M. Majster-Cederbaum, *Trace machines for observing continuous-time Markov chains*, Electronic Notes in Theoretical Computer Science, **153**:2 (2006), 259–277.

[84] V. Wolf, C. Baier, M. Majster-Cederbaum, *Trace semantics for stochastic systems with nondeterminism*, Electronic Notes in Theoretical Computer Science, **164**:3 (2006), 187–204.

[85] K. Wolter, *Second order fluid stochastic Petri nets: an extension of GSPNs for approximate and continuous modelling*, Proc. Workshop on Analytical and Numerical Modelling Techniques, 1^{st} World Congress on Systems Simulation (WCSS) 1997, Singapore, September 1997, 328–332, IEEE Singapore Section, Singapore, 1997. http://page.mi.fu-berlin.de/katinkaw/publications/anmt.ps

Igor Valerievich Tarasyuk
A.P. Ershov Institute of Informatics Systems,
Siberian Branch of the Russian Academy of Sciences,
Acad. Lavrentiev pr. 6,
630090 Novosibirsk, Russian Federation
Email address: itar@iis.nsk.su

Peter Buchholz
Faculty of Computer Science,
Technical University of Dortmund,
Otto-Hahn-Str. 16,
44227 Dortmund, Germany
Email address: peter.buchholz@cs.uni-dortmund.de