Equivalence Relations
for Net and Algebraic Models of Concurrency

Igor V. Tarasyuk

A.P. Ershov Institute of Informatics Systems
Siberian Division of the Russian Academy of Sciences
6, Acad. Lavrentiev pr., Novosibirsk 630090, Russia

itar@iis.nsk.su
itar.iis.nsk.su

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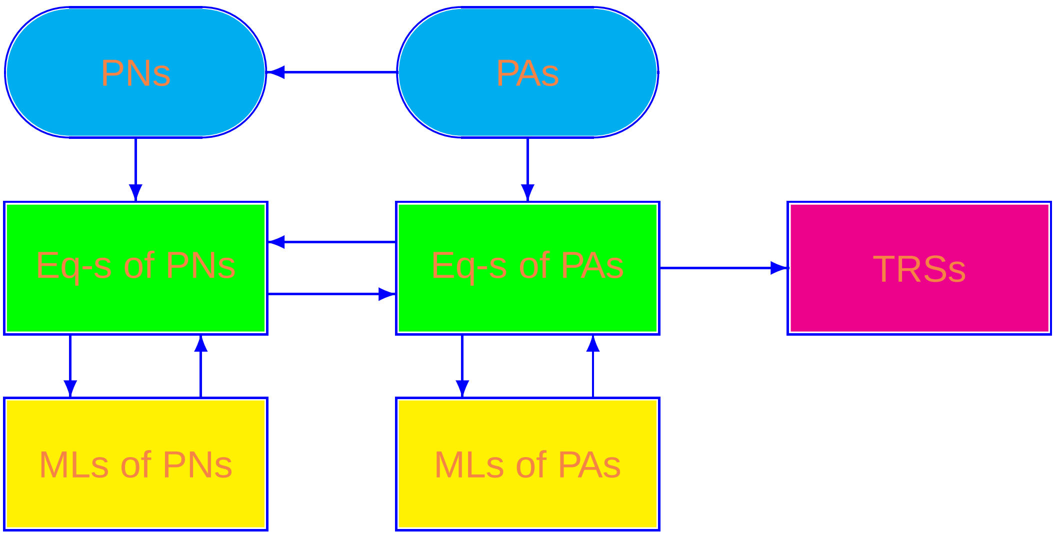
The results



Interrelations of formalisms and equivalences

Equivalences for Petri Nets

Abstract: Behavioural equivalences of concurrent systems modeled by Petri nets are considered.

Known basic, back-forth and place bisimulation equivalences are supplemented by new ones.

The equivalence interrelations are examined for the general Petri nets as well as for their subclasses of sequential nets (no concurrent transitions), strictly labeled nets (unlabeled) and T-nets (no place branching).

A logical characterization of back-forth bisimulation equivalences in terms of logics with past modalities is proposed.

An effective net reduction method based on place bisimulation relations is presented.

A preservation of all the equivalences by refinements is investigated to find out their appropriateness for top-down design.

Keywords: Petri nets, sequential nets, strictly labeled nets, T-nets, basic equivalences, back-forth bisimulations, place bisimulations, logical characterization, net reduction, refinement.

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Introduction

Previous work

The following basic equivalences are known:

- *Trace equivalences* (respect protocols of behaviour):

  interleaving ($\equiv_i$) [Hoa80], step ($\equiv_s$) [Pom86], partial word ($\equiv_{pw}$) [Gra81] and pomset ($\equiv_{pom}$) [Pra86].

- *Usual bisimulation equivalences* (respect branching structure of behaviour):

  interleaving ($\underline{\leftrightarrow}_i$) [Par81], step ($\underline{\leftrightarrow}_s$) [NT84], partial word ($\underline{\leftrightarrow}_{pw}$)[Vog91a], pomset ($\underline{\leftrightarrow}_{pom}$) [BCa87] and process ($\underline{\leftrightarrow}_{pr}$) [AS92].

- *ST-bisimulation equivalences* (respect the duration or maximality of events in behaviour):

  interleaving ($\underline{\leftrightarrow}_{iST}$) [GV87], partial word ($\underline{\leftrightarrow}_{pwST}$) [Vog91a] and pomset ($\underline{\leftrightarrow}_{pomST}$) [Vog91a].

- *History preserving bisimulation equivalences* (respect the “history” of behaviour):

  pomset ($\underline{\leftrightarrow}_{pomh}$) [RT88].

- *Conflict preserving equivalences* (respect conflicts of events):

  occurrence ($\equiv_{occ}$) [NPW81].

- *Isomorphism* (coincidence up to renaming of components):

  ($\simeq$).

Back-forth bisimulation equivalences: bisimulation relation do not only require systems to simulate each other behavior in the forward direction but also when going back in history, backward.

They are connected with equivalences of logics with past modalities.

Interleaving *back* interleaving *forth bisimulation equivalence* ($\underline{\leftrightarrow}_{ibif} = \underline{\leftrightarrow}_i$) [NMV90].

Step *back* step *forth* ($\underline{\leftrightarrow}_{sbsf}$), partial word *back* partial word *forth* ($\underline{\leftrightarrow}_{pwbpwf}$) and pomset *back* pomset *forth* ($\underline{\leftrightarrow}_{pombpomf}$) *bisimulation equivalences* [Che92a,Che92b,Che92c].

All possible *back-forth equivalences* in interleaving, step, partial word and pomset semantics s.t. types of backward and forward simulations may differ. New relations: step *back* partial word *forth* ($\underline{\leftrightarrow}_{sbpwf}$) and step *back* pomset *forth* ($\underline{\leftrightarrow}_{sbpomf}$) *bisimulation equivalences* [Pin93].

Place bisimulation equivalences [ABS91] are based on definition from [Old89,Old91]. They are relations over places instead of markings or processes. The relation on markings is obtained via *"lifting"* that on places.

The main application of the place equivalences is effective behaviour preserving reduction of Petri nets.

Interleaving *place bisimulation equivalence* ($\sim_i$) and interleaving *strict place bisimulation equivalence* ($\approx_i$) [ABS91].

Step ($\sim_s$), partial word ($\sim_{pw}$), pomset ($\sim_{pom}$) and process ($\sim_{pr}$) *place bisimulation equivalences*. Their *strict analogues*: ($\approx_s, \approx_{pw}, \approx_{pom}, \approx_{pr}$).

Merging: $\sim_i = \sim_s = \sim_{pw}$ and $\approx_s = \approx_{pw} = \approx_{pom} = \approx_{pr} = \sim_{pr}$. Three different relations remain: $\sim_i, \sim_{pom}$ and $\sim_{pr}$ [AS92].

New equivalences

- Basic equivalences:

  process *trace* ($\equiv_{pr}$),

  process *ST-bisimulation* ($\underline{\leftrightarrow}_{prST}$),

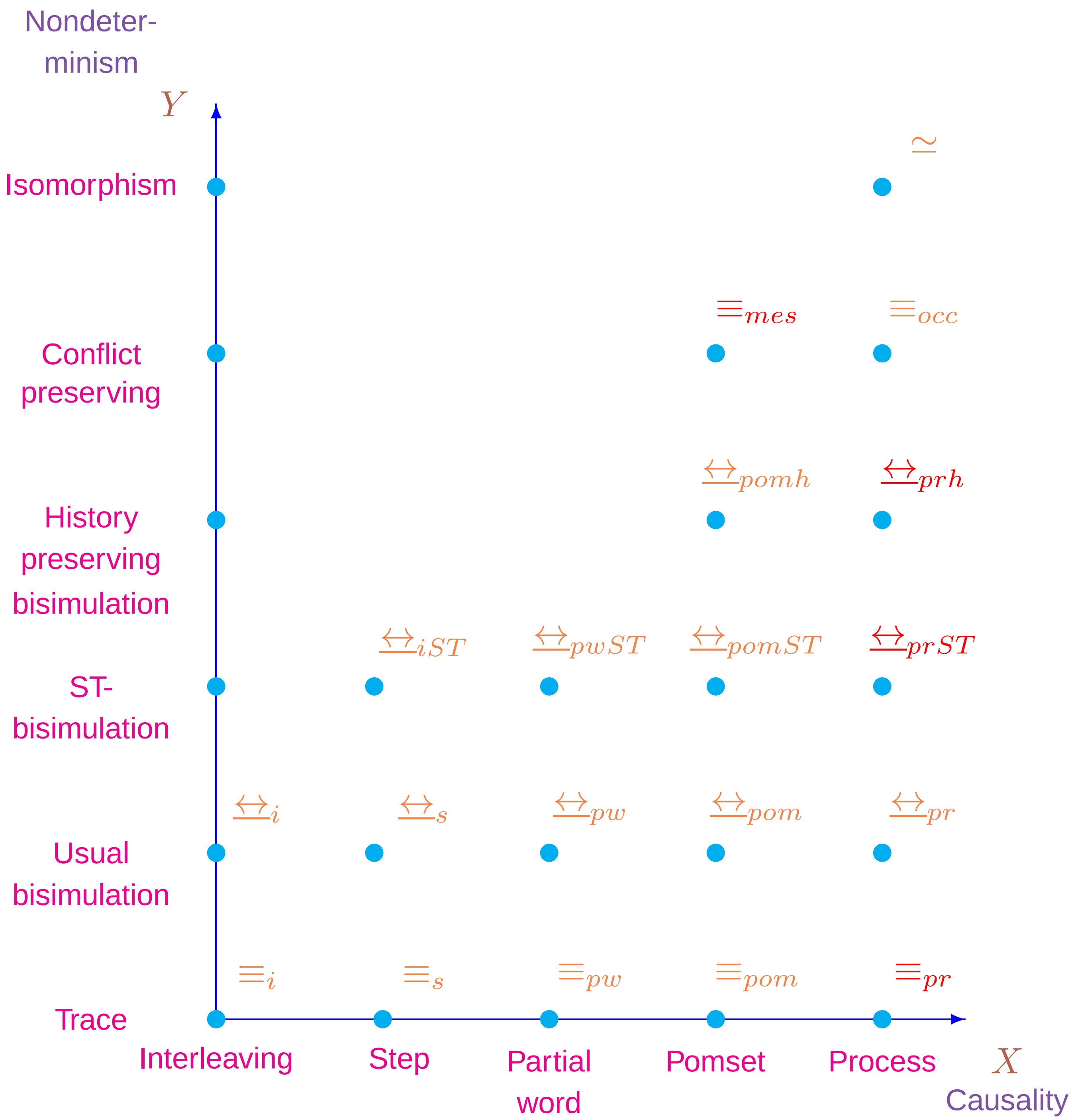
  process *history preserving bisimulation* ($\underline{\leftrightarrow}_{prh}$) and

  multi event structure ($\equiv_{mes}$).

- Back-forth bisimulation equivalences:

  step *back* process *forth* ($\underline{\leftrightarrow}_{sbprf}$) and

  pomset *back* process *forth* ($\underline{\leftrightarrow}_{pombprf}$).



Classification of basic equivalences

Basic equivalences are positioned on coordinate plane.

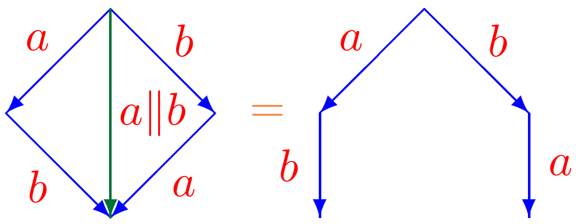
New relations are depicted in red colour.

Moving along X axis: a degree of causality grows.

Moving along Y axis: a degree of non-determinism grows.

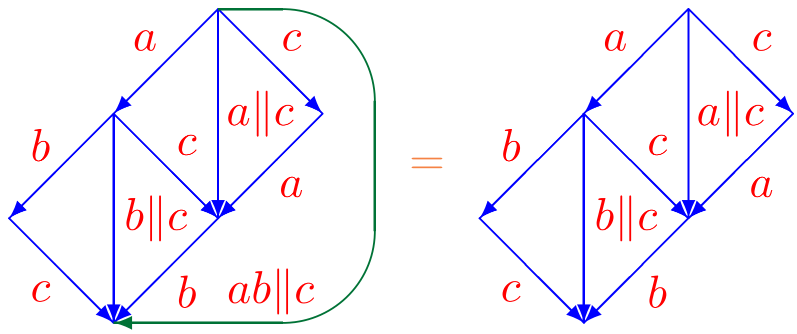
Interleaving

$a \| b = ab + ba$



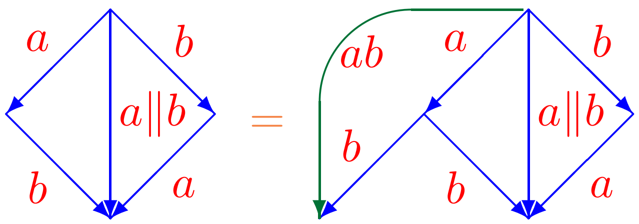
Step

$ab \| c = a(b \| c) + (a \| c)b$



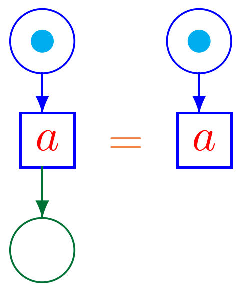
Partial word

$a \| b = a \| b + ab$



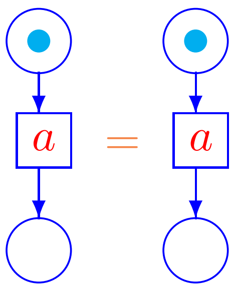
Pomset

$a = a$



Process

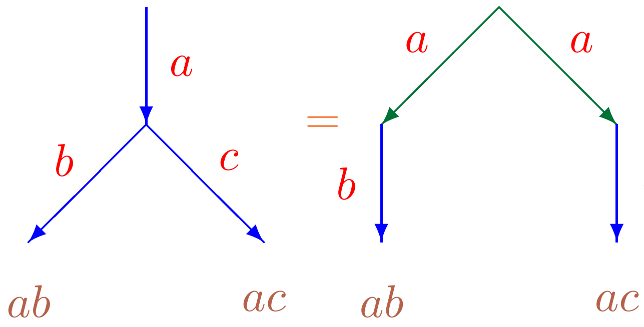
$a = a$



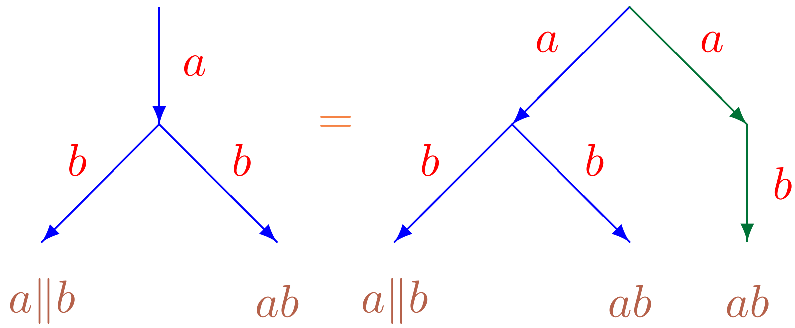
Causality degrees

Trace

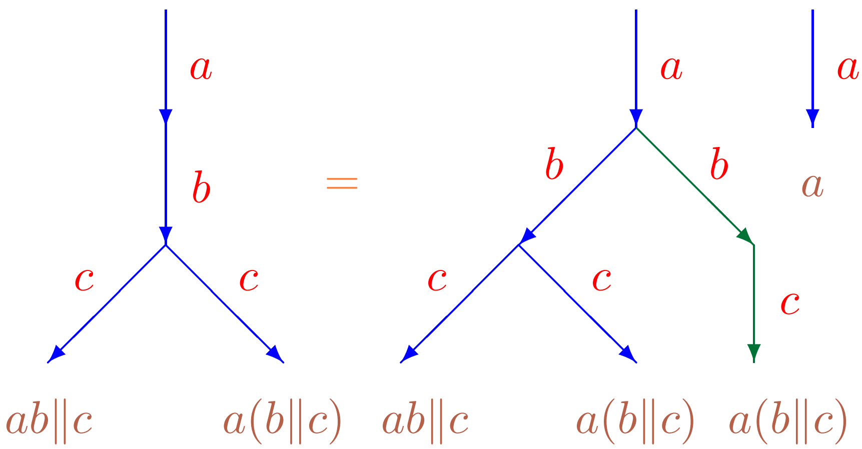
$a(b+c)=ab+ac$



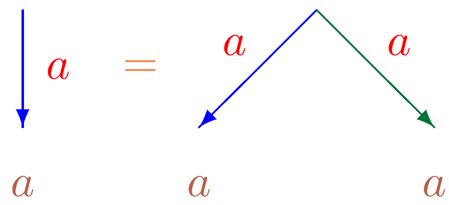
Usual bisimulation



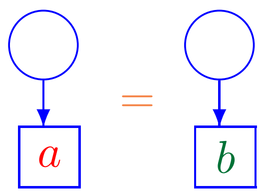
ST-bisimulation



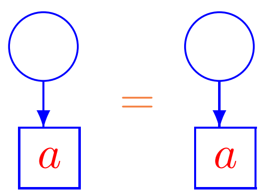
History preserving bisimulation



Conflict preserving



Isomorphism



Nondeterminism degrees

Basic definitions

Multisets

Definition 1 *A finite multiset (bag) M over a set X is a mapping $M : X \rightarrow I\!N$ s.t. $|\{x \in X \mid M(x) > 0\}| < \infty$.*

The *set of all finite multisets* over X is $I\!N_{fin}^X$.

The *set of all subsets (powerset)* of X is 2^X.

For $x \in X,\ M(x)$ is a number of elements x in M.

When $\forall x \in X\ M(x) \leq 1,\ M$ is a proper set s.t. $M \subseteq X$.

The *cardinality* of a multiset M: $|M| = \sum_{x \in X} M(x)$.

If $M_1, M_2 \in I\!N_{fin}^X$ and $x \in X$ then

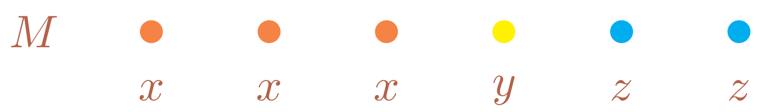
$$\begin{array}{rcl} (M_1 + M_2)(x) & = & M_1(x) + M_2(x); \\ (M_1 - M_2)(x) & = & \max\{M_1(x) - M_2(x), 0\}; \\ (M_1 \cup M_2)(x) & = & \max\{M_1(x), M_2(x)\}; \\ (M_1 \cap M_2)(x) & = & \min\{M_1(x), M_2(x)\}; \\ M_1 \subseteq M_2 & \Leftrightarrow & \forall x \in X\ M_1(x) \leq M_2(x); \\ x \in M & \Leftrightarrow & M(x) > 0. \end{array}$$

We write $M + x - y$ for $M + \{x\} - \{y\}$.

The *empty multiset*: $\emptyset$.

Multisets: sets with identical elements.

$M = \{x, x, x, y, z, z\}$ denotes the multiset M s.t. $M(x) = 3,\ M(y) = 1,\ M(z) = 2$, and for other elements M is equal to 0.



Example of multiset

Labeled nets

Let $Act = \{a, b, \ldots\}$ be a set of *action names* or *labels*.

$\tau \notin Act$ denotes *silent* action that represents an internal activity. Let $Act_\tau = Act \cup \{\tau\}$.

Definition **2** *A labeled net is a quadruple* $N = (P_N, T_N, W_N, L_N)$*:*

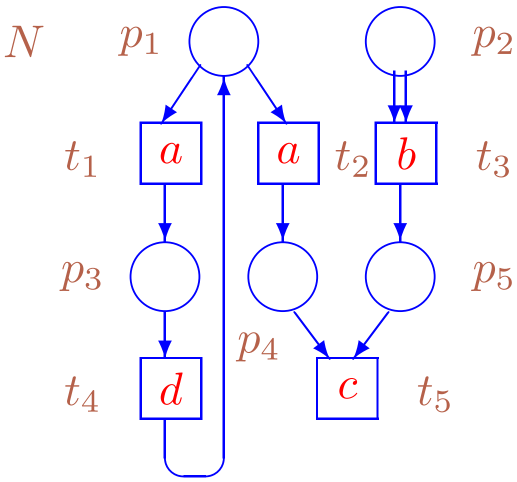
- $P_N = \{p, q, \ldots\}$ *is a set of places;*
- $T_N = \{t, u, \ldots\}$ *is a set of transitions;*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \rightarrow \mathbb{N}$ *is the flow relation with weights;*
- $L_N : T_N \rightarrow Act_\tau$ *is a labeling of transitions with action names.*

Given labeled nets $N = (P_N, T_N, W_N, L_N)$ and $N' = (P_{N'}, T_{N'}, W_{N'}, L_{N'})$.

A mapping $\beta : P_N \cup T_N \rightarrow P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', $\beta : N \simeq N'$, if:

1. β is a bijection s.t. $\beta(P_N) = P_{N'}$ and $\beta(T_N) = T_{N'}$;
2. $\forall p \in P_N\ \forall t \in T_N\ W_N(p, t) = W_{N'}(\beta(p), \beta(t))$ and $W_N(t, p) = W_{N'}(\beta(t), \beta(p))$;
3. $\forall t \in T_N\ L_N(t) = L_{N'}(\beta(t))$.

N and N' are *isomorphic*, $N \simeq N'$, if $\exists \beta : N \simeq N'$.



Example of labeled net

Let N be a labeled net and $t \in T_N,\ p \in P_N,\ U \in I\!N_{fin}^{T_N},\ R \in I\!N_{fin}^{P_N}$.

The *precondition* ${}^\bullet t$ and the *postcondition* $t^\bullet$ of t are the multisets $({}^\bullet t)(p) = W_N(p,t)$ and $(t^\bullet)(p) = W_N(t,p)$.

The *precondition* ${}^\bullet p$ and the *postcondition* $p^\bullet$ of p are the multisets $({}^\bullet p)(t) = W_N(t,p)$ and $(p^\bullet)(t) = W_N(p,t)$.

The *precondition* ${}^\bullet U$ and the *postcondition* $U^\bullet$ of U are the multisets ${}^\bullet U = \sum_{t\in U} {}^\bullet t$ and $U^\bullet = \sum_{t\in U} t^\bullet$.

The *precondition* ${}^\bullet R$ and the *postcondition* $R^\bullet$ of R are the multisets ${}^\bullet R = \sum_{p\in R} {}^\bullet p$ and $R^\bullet = \sum_{p\in R} p^\bullet$.

${}^\bullet N = \{p \in P_N \mid {}^\bullet p = \emptyset\}$ is the set of *initial (input)* places of N.

$N^\bullet = \{p \in P_N \mid p^\bullet = \emptyset\}$ is the set of *final (output)* places of N.

A labeled net N is *acyclic*, if there exist no transitions $t_0, \ldots, t_n \in T_N$ s.t. $t_{i-1}^\bullet \cap {}^\bullet t_i \neq \emptyset\ (1 \leq i \leq n)$ and $t_0 = t_n$.

A labeled net N is *ordinary* if $\forall p \in P_N\ {}^\bullet p$ and $p^\bullet$ are proper sets (not multisets).

Let $N = (P_N, T_N, W_N, L_N)$ be acyclic ordinary labeled net and $x, y \in P_N \cup T_N$. Then

- $x \prec_N y \Leftrightarrow W_N^*(x, y) = 1$, where W_N^* is a transitive closure of W_N (*strict causal dependence* relation);
- $x \preceq_N y \Leftrightarrow (x \prec_N y) \vee (x = y)$ (a relation of *causal dependence*);
- $x \#_N y \Leftrightarrow \exists t, u \in T_N\ (t \neq u,\ {}^\bullet t \cap {}^\bullet u \neq \emptyset,\ t \preceq_N x,\ u \preceq_N y)$ (a relation of *conflict*);
- $\downarrow_N x = \{y \in P_N \cup T_N \mid y \prec_N x\}$ (the set of *strict predecessors* of x).

A set $T \subseteq T_N$ is *left-closed* in N, if $\forall t \in T\ (\downarrow_N t) \cap T_N \subseteq T$.

Marked nets

A *marking* of a labeled net N is $M \in \mathbb{N}_{fin}^{P_N}$.

Definition 3 *A* marked net (net) *is a tuple* $N = (P_N, T_N, W_N, L_N, M_N)$*:*

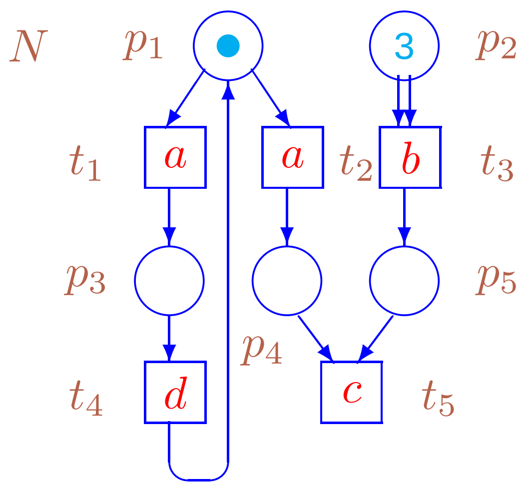
- (P_N, T_N, W_N, L_N) *is a labeled net;*
- $M_N \in \mathbb{N}_{fin}^{P_N}$ *is the* initial *marking.*

Given nets $N = (P_N, T_N, W_N, L_N, M_N)$ and $N' = (P_{N'}, T_{N'}, W_{N'}, L_{N'}, M_{N'})$.

A mapping $\beta : P_N \cup T_N \to P_{N'} \cup T_{N'}$ is an *isomorphism* between N and N', $\beta : N \simeq N'$, if:

1. $\beta : (P_N, T_N, W_N, L_N) \simeq (P_{N'}, T_{N'}, W_{N'}, L_{N'})$;
2. $\forall p \in P_N \; M_N(p) = M_{N'}(\beta(p))$.

N and N' are *isomorphic*, $N \simeq N'$, if $\exists \beta : N \simeq N'$.



Example of marked net

Let $M \in \mathbb{N}_{fin}^{P_N}$ be a marking of a net N.

A transition $t \in T_N$ is *enabled (fireable)* in M, if ${}^\bullet t \subseteq M$.

$Ena(M)$ is the set of *all transitions enabled in marking* M.

If $t \in Ena(M)$, its firing yields a new marking $\widetilde{M} = M - {}^\bullet t + t^\bullet$, $M \stackrel{t}{\rightarrow} \widetilde{M}$ or $M \stackrel{a}{\rightarrow} \widetilde{M}$, if $L_N(t) = a$.

We write $M \rightarrow \widetilde{M}$, if $\exists t \in T_N\ M \stackrel{t}{\rightarrow} \widetilde{M}$.

A marking $\widetilde{M}$ of a net N is *reachable from marking* M, if $\widetilde{M} = M$ or there exists a reachable marking $\widehat{M}$ of N s.t. $\widehat{M} \rightarrow \widetilde{M}$.

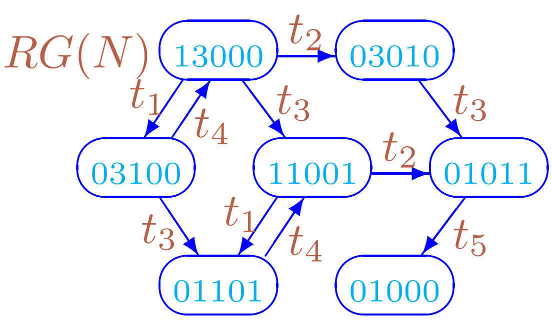
A marking M of a net N is *reachable*, if it is reachable from M_N.

$RS(N, M)$ is the *set of all reachable from* M markings of a net N.

$RS(N)$ is the *set of all reachable* markings of a net N.

$RG(N)$ is the *reachabiliy graph* of a net N, an oriented graph with vertex set $RS(N)$ and arcs from M to $\widetilde{M}$ iff $M \rightarrow \widetilde{M}$.

The arcs could be labeled by transition names or labels.



Reachability graph of the marked net

Let $\sigma = t_1 \cdots t_n \in T_N^*$ be a sequence of transitions and $M \stackrel{t_1}{\to} M_1 \stackrel{t_2}{\to} \ldots \stackrel{t_n}{\to} M_n = \widetilde{M}$.

Then firing of σ in M yields a new marking $\widetilde{M}$, $M \stackrel{\sigma}{\to} \widetilde{M}$ or $M \stackrel{\omega}{\to} \widetilde{M}$, if $L_N(\sigma) = L_N(t_1) \cdots L_N(t_n) = \omega$.

A multiset of transitions $U \in I\!N_{fin}^{T_N}$ is *fireable* in M, if ${}^\bullet U \subseteq M$.

If U is fireable in M, its firing yields a new marking $\widetilde{M} = M - {}^\bullet U + U^\bullet$, $M \stackrel{U}{\to} \widetilde{M}$ or $M \stackrel{A}{\to} \widetilde{M}$, if $L_N(U) = \sum_{t \in U} L_N(t) = A$.

A net N is n*-bounded* ($n \in I\!N$), if $\forall M \in RS(N)\ \forall p \in P_N\ M(p) \leq n$.

A net N is *bounded*, if $\exists n \in I\!N$ s.t. N is n-bounded.

A net N is *safe*, if it is 1-bounded.

An action $a \in Act$ is *auto-concurrent* in N, if $\exists M \in RS(N)\ \exists t, u \in T_N$ s.t. $L_N(t) = a = L_N(u)$ and ${}^\bullet t + {}^\bullet u \subseteq M$.

A net N is *auto-concurrency free*, if no action is auto-concurrent in N.

An action $a \in Act$ is *self-concurrent* in N, if $\exists M \in RS(N)\ \exists t \in T_N$ s.t. $L_N(t) = a$ and ${}^\bullet t + {}^\bullet t \subseteq M$.

A net N is *self-concurrency free*, if no action is self-concurrent in N.

A net N is *live*, if $\forall t \in T_N\ \exists M \in RS(N)\ t \in Ena(M)$.

A net N is *reversible*, if $\forall M \in RS(N)\ M_N \in RS(N, M)$.

Partially ordered sets [Pra86]

Definition 4 *A partially ordered set (poset) is a pair $\rho = (X, \prec)$:*

- $X = \{x, y, \ldots\}$ *is an underlying set;*
- $\prec\, \subseteq X \times X$ *is a strict partial order (irreflexive transitive relation) over X.*

Let $\rho = (X, \prec)$ be a poset. A *restriction* of ρ to the set $Y \subseteq X$ is $\rho|_Y = (Y, \prec \cap (Y \times Y))$. A set of *strict predecessors* of $x \in X$ is $\downarrow x = \{y \in X \mid y \prec x\}$. A set $Y \subseteq X$ is *left-closed*, if $\forall y \in Y\ \downarrow y \subseteq Y$.

Let $\rho_1 = (X_1, \prec_1)$ and $\rho_2 = (X_2, \prec_2)$ be posets. ρ_1 is a *strict prefix* of ρ_2, $\rho_1 \triangleleft \rho_2$, if $\rho_1 = \rho_2|_Y$ s.t. $Y \subset X$ is a finite left-closed set. ρ_1 is a *prefix* of ρ_2, notation $\rho_1 \trianglelefteq \rho_2$, if $\rho_1 \triangleleft \rho_2$ or $\rho_1 = \rho_2$.

Definition 5 *A labeled partially ordered set (lposet, causal structure) is a triple $\rho = (X, \prec, l)$:*

- $(X, \prec)$ *is a poset;*
- $l : X \to Act_\tau$ *is a labeling function.*

The notions defined for posets are transferred to lposets.

Let $\rho = (X, \prec, l)$ and $\rho' = (X', \prec', l')$ be lposets.

A mapping $\beta : X \to X'$ is a *label-preserving bijection* between ρ and ρ', $\beta : \rho \asymp \rho'$, if:

1. β is a bijection;
2. $\forall x \in X\ l(x) = l'(\beta(x))$.

We write $\rho \asymp \rho'$, if $\exists \beta : \rho \asymp \rho'$.

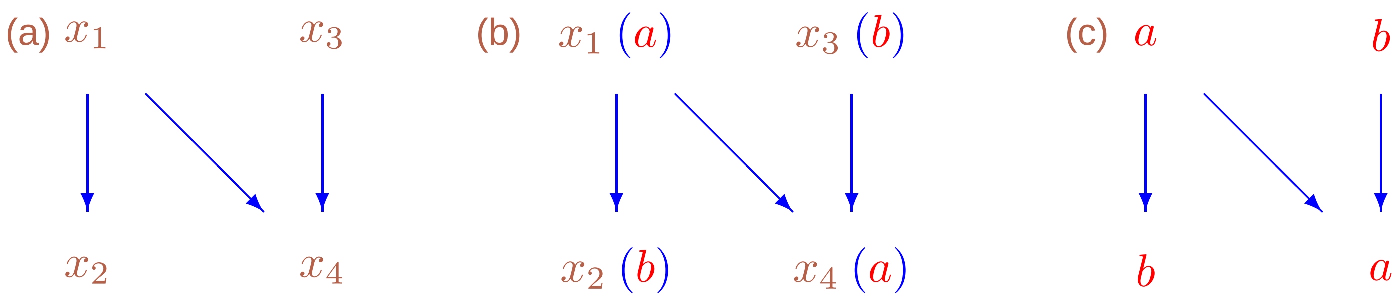
A mapping $\beta : X \to X'$ is a *homomorphism* between ρ and ρ', $\beta : \rho \sqsubseteq \rho'$, if:

1. $\beta : \rho \asymp \rho'$;
2. $\forall x, y \in X \; x \prec y \; \Rightarrow \; \beta(x) \prec' \beta(y)$.

We write $\rho \sqsubseteq \rho'$, if $\exists \beta : \rho \sqsubseteq \rho'$.

A mapping $\beta : X \to X'$ is an *isomorphism* between ρ and ρ', $\beta : \rho \simeq \rho'$, if $\beta : \rho \sqsubseteq \rho'$ and $\beta^{-1} : \rho' \sqsubseteq \rho$. Lposets ρ and ρ' are *isomorphic*, $\rho \simeq \rho'$, if $\exists \beta : \rho \simeq \rho'$.

Definition 6 Partially ordered multiset (pomset) *is the equivalence class of lposets w.r.t. isomorphism (the isomorphism class).*



Examples of poset, lposet and pomset

Event structures [NPW81]

Definition 7 *An event structure (ES) is a triple* $\xi = (X, \prec, \#)$*:*

- $X = \{x, y, \ldots\}$ *is a set of events;*
- $\prec \subseteq X \times X$ *is a strict partial order, a causal dependence relation, which satisfies to the principle of finite causes:* $\forall x \in X \; |\downarrow x| < \infty$*;*
- $\# \subseteq X \times X$ *is an irreflexive symmetrical conflict relation, which satisfies to the principle of conflict heredity:* $\forall x, y, z \in X \; x\#y \prec z \; \Rightarrow \; x\#z$*.*

Let $\xi = (X, \prec, \#)$ be LES and $Y \subseteq X$. A *restriction* of ξ to the set Y is: $\xi|_Y = (Y, \prec \cap (Y \times Y), \# \cap (Y \times Y))$.

Definition 8 *A labeled event structure (LES) is a quadruple* $\xi = (X, \prec, \#, l)$*:*

- $(X, \prec, \#)$ *is an event structure;*
- $l : X \to Act_\tau$ *is a labeling function.*

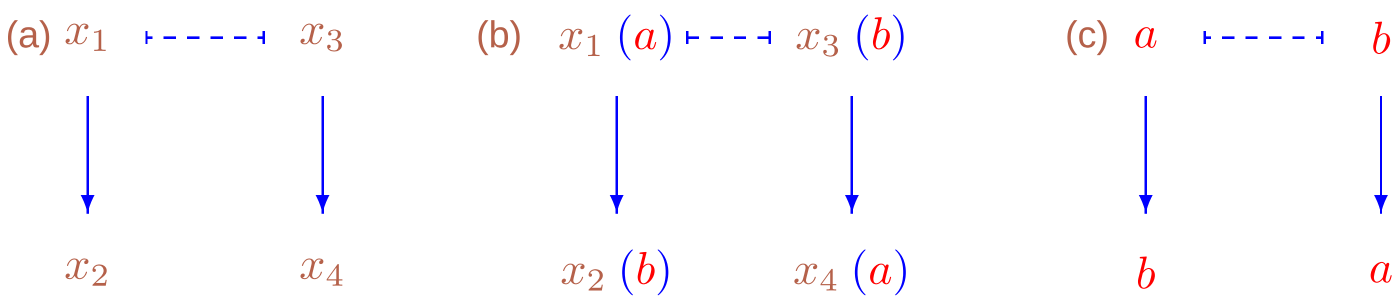
The notions defined for ES's are transferred to LES's.

Let $\xi = (X, \prec, \#, l)$ and $\xi' = (X', \prec', \#', l')$ be LES's. A mapping $\beta : X \to X'$ is an *isomorphism* between ξ and ξ', $\beta : \xi \simeq \xi'$, if:

1. β is a bijection;
2. $\forall x \in X \; l(x) = l'(\beta(x))$;
3. $\forall x, y \in X \; x \prec y \;\Leftrightarrow\; \beta(x) \prec' \beta(y)$;
4. $\forall x, y \in X \; x \# y \;\Leftrightarrow\; \beta(x) \#' \beta(y)$.

ξ and ξ' are *isomorphic*, $\xi \simeq \xi'$, if $\exists \beta : \xi \simeq \xi'$.

Definition 9 *A* multi-event structure (MES) *is an isomorphism class of LES's.*



Examples of ES, LES and MES

Processes [BD87]

Definition 10 *A causal net is an acyclic ordinary labeled net* $C = (P_C, T_C, W_C, L_C)$, *s.t.:*

1. $\forall r \in P_C \; |^\bullet r| \leq 1$ *and* $|r^\bullet| \leq 1$, *places are unbranched;*
2. $\forall x \in P_C \cap T_C \; |\downarrow_C x| < \infty$, *a set of causes is finite.*

Based on causal net $C = (P_C, T_C, W_C, L_C)$, one can define lposet $\rho_C = (T_C, \prec_N \cap (T_C \times T_C), L_C)$.

For any causal net C there is a sequence of transition firings: $^\bullet C = L_0 \stackrel{v_1}{\to} \cdots \stackrel{v_n}{\to} L_n = C^\bullet$ s.t. $L_i \subseteq P_C \; (0 \leq i \leq n), \; P_C = \cup_{i=0}^n L_i$ and $T_C = \{v_1, \ldots, v_n\}$. It is called a *full execution* of C.

Definition 11 *Given a net* N *and a causal net* C. *A mapping* $\varphi : P_C \cup T_C \to P_N \cup T_N$ *is an homomorphism of* C *into* N, $\varphi : C \rightarrow N$, *if:*

1. $\varphi(P_C) \in I\!N_{fin}^{P_N}$ *and* $\varphi(T_C) \in I\!N_{fin}^{T_N}$, *sorts are preserved;*
2. $\forall v \in T_C \; ^\bullet\varphi(v) = \varphi(^\bullet v)$ *and* $\varphi(v)^\bullet = \varphi(v^\bullet)$, *flow relation is respected;*
3. $\forall v \in T_C \; L_C(v) = L_N(\varphi(v))$, *labeling is preserved.*

Since homomorphisms respect the flow relation, if $^\bullet C \stackrel{v_1}{\to} \cdots \stackrel{v_n}{\to} C^\bullet$ is a full execution of C, then $M = \varphi(^\bullet C) \stackrel{\varphi(v_1)}{\to} \cdots \stackrel{\varphi(v_n)}{\to} \varphi(C^\bullet) = \widetilde{M}$ is a sequence of transition firings in N.

Definition 12 *An enabled (fireable) in marking M process of a net N is a pair $\pi = (C, \varphi)$, where C is a causal net and $\varphi : C \to N$ is an homomorphism s.t. $M = \varphi({}^\bullet C)$. An enabled in M_N process is a process of N.*

$\Pi(N, M)$ is a *set of all enabled* in marking M, and $\Pi(N)$ is the *set of all* processes of a net N.

The *initial* process of a net N is $\pi_N = (C_N, \varphi_N) \in \Pi(N)$, s.t. $T_{C_N} = \emptyset$.

If $\pi \in \Pi(N, M)$, then firing of this process transforms a marking M into $\widetilde{M} = M - \varphi({}^\bullet C) + \varphi(C^\bullet) = \varphi(C^\bullet),\ M \xrightarrow{\pi} \widetilde{M}$.

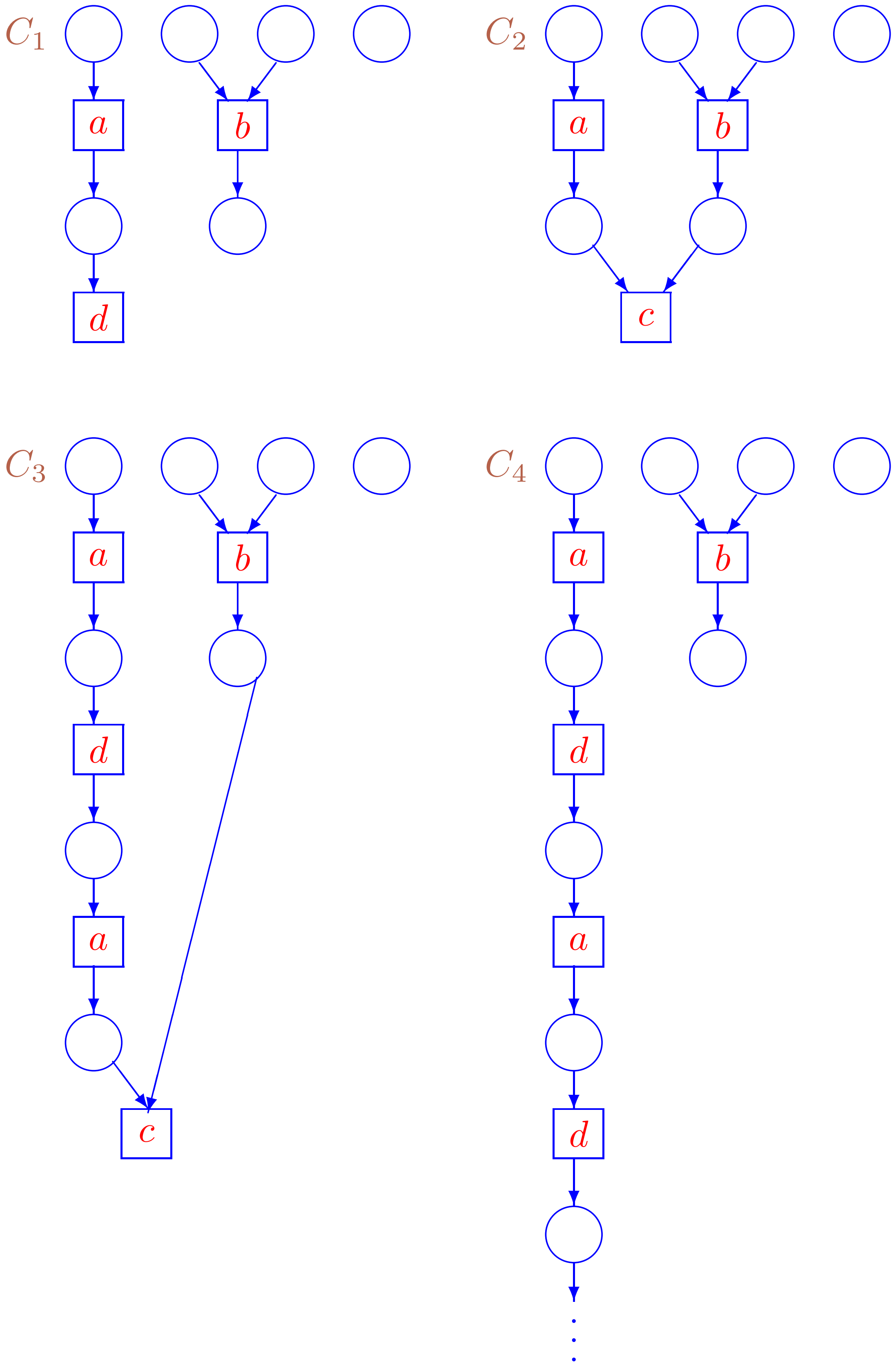
Let $\pi = (C, \varphi),\ \tilde{\pi} = (\widetilde{C}, \tilde{\varphi}) \in \Pi(N),\ \hat{\pi} = (\widehat{C}, \hat{\varphi}) \in \Pi(N, \varphi(C^\bullet))$. A process π is a *prefix* of a process $\tilde{\pi}$, if $T_C \subseteq T_{\widetilde{C}}$ is a left-closed set in $\widetilde{C}$. A process $\hat{\pi}$ is a *suffix* of a process $\tilde{\pi}$, if $T_{\widehat{C}} = T_{\widetilde{C}} \setminus T_C$.

In such a case a process $\tilde{\pi}$ is an *extension* of π *by process* $\hat{\pi}$, and $\hat{\pi}$ is an *extending* process for π, $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$. We write $\pi \to \tilde{\pi}$, if $\exists \hat{\pi}\ \pi \xrightarrow{\hat{\pi}} \tilde{\pi}$.

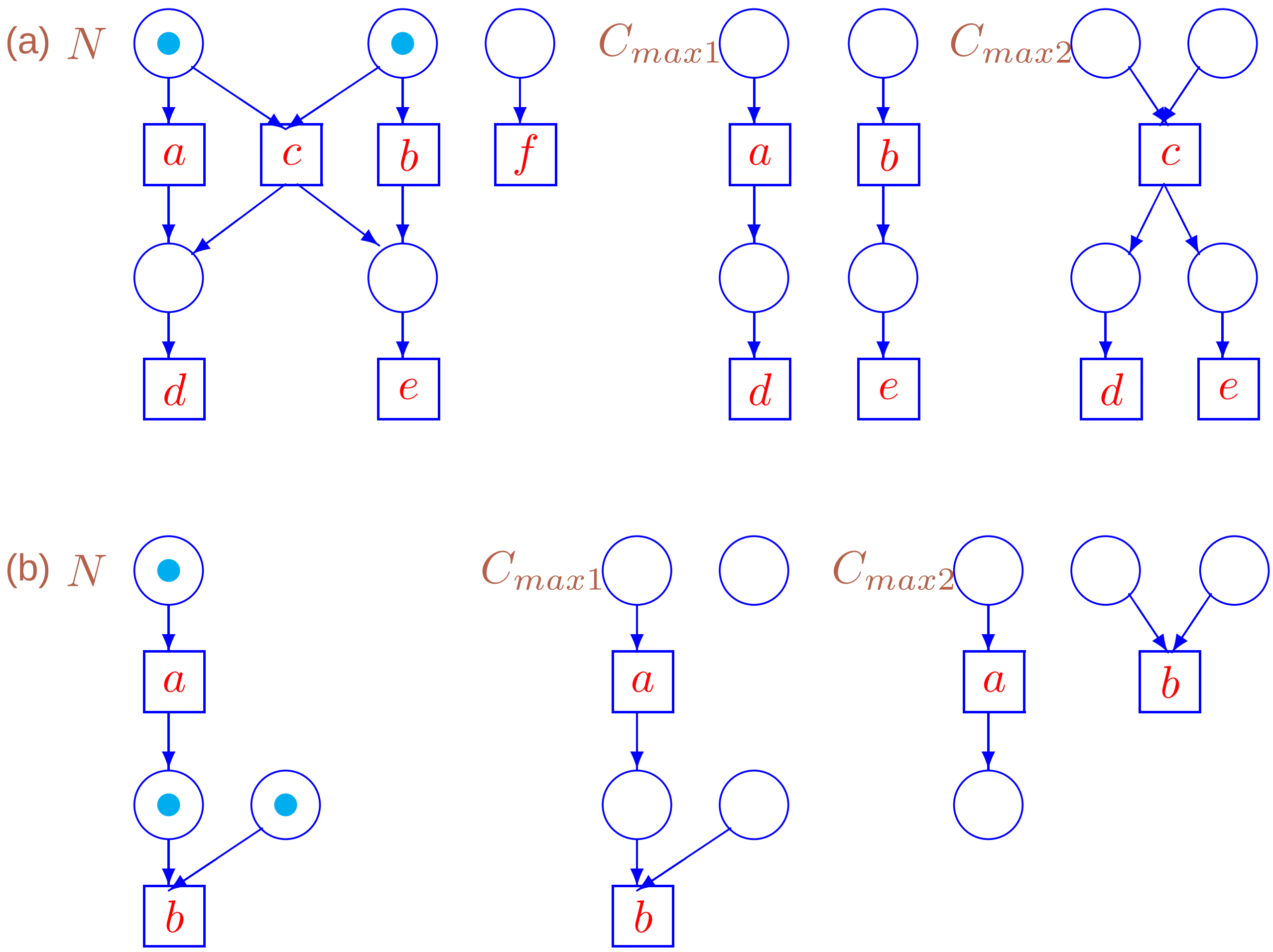
A process $\tilde{\pi}$ is an extension of a process π *by one transition*, $\pi \xrightarrow{v} \tilde{\pi}$ or $\pi \xrightarrow{a} \tilde{\pi}$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi},\ T_{\widehat{C}} = \{v\}$ and $L_{\widehat{C}}(v) = a$.

A process $\tilde{\pi}$ is an extension of a process π *by sequence of transitions*, $\pi \xrightarrow{\sigma} \tilde{\pi}$ or $\pi \xrightarrow{\omega} \tilde{\pi}$, if
$\exists \pi_i \in \Pi(N)\ (1 \le i \le n)\ \pi \xrightarrow{v_1} \pi_1 \xrightarrow{v_2} \ldots \xrightarrow{v_n} \pi_n = \tilde{\pi},\ \sigma = v_1 \cdots v_n$ and $L_{\widehat{C}}(\sigma) = \omega$.

A process $\tilde{\pi}$ is an extension of a process π *by multiset of transitions*, $\pi \xrightarrow{V} \tilde{\pi}$ or $\pi \xrightarrow{A} \tilde{\pi}$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi},\ \prec_{\widehat{C}} = \emptyset,\ T_{\widehat{C}} = V$ and $L_{\widehat{C}}(V) = A$.



Causal nets of processes



Causal nets of maximal processes

Branching processes [Eng91]

Definition 13 *An occurrence net is an acyclic ordinary labeled net $O = (P_O, T_O, W_O, L_O)$, s.t.:*

1. $\forall r \in P_O\ |{}^\bullet r| \leq 1$, *there are no backwards conflicts;*
2. $\forall x \in P_O \cup T_O\ \neg(x \#_O x)$, *conflict relation is irreflexive;*
3. $\forall x \in P_O \cup T_O\ |\downarrow_O x| < \infty$, *set of causes is finite.*

Let $O = (P_O, T_O, W_O, L_O)$ be occurrence net and $N = (P_N, T_N, W_N, L_N, M_N)$ be some net. A mapping $\psi : P_O \cup T_O \to P_N \cup T_N$ is an *homomorphism* O into N, $\psi : O \rightarrow N$, if:

1. $\psi(P_O) \in I\!N_{fin}^{P_N}$ and $\psi(T_O) \in I\!N_{fin}^{T_N}$, sorts are preserved;
2. $\forall v \in T_O\ L_O(v) = L_N(\psi(v))$, labeling is preserved;
3. $\forall v \in T_O\ {}^\bullet\psi(v) = \psi({}^\bullet v)$ and $\psi(v)^\bullet = \psi(v^\bullet)$, flow relation is respected;
4. $\forall v, w \in T_O\ ({}^\bullet v = {}^\bullet w) \wedge (\psi(v) = \psi(w)) \Rightarrow\ v = w$, there are no "superfluous" conflicts.

Based on occurrence net $O = (P_O, T_O, W_O, L_O)$, one can define LES $\xi_O = (T_O, \prec_O \cap (T_O \times T_O), \#_O \cap (T_O \times T_O), L_O)$.

Definition 14 *A branching process of a net N is a pair $\varpi = (O, \psi)$, where O is an occurrence net and $\psi : O \to N$ is an homomorphism s.t. $M_N = \psi({}^\bullet O)$.*

$\wp(N)$ is the set of *all branching processes* of a net N. The *initial* branching process of a net N coincides with its initial process, $\varpi_N = \pi_N$.

Let $\varpi = (O, \psi),\ \tilde{\varpi} = (\widetilde{O}, \tilde{\psi}) \in \wp(N),\ O = (P_O, T_O, W_O, L_O)$, $\widetilde{O} = (P_{\widetilde{O}}, T_{\widetilde{O}}, W_{\widetilde{O}}, L_{\widetilde{O}})$. ϖ is a *prefix* of $\tilde{\varpi}$, if $T_O \subseteq T_{\widetilde{O}}$ is a left-closed set in $\widetilde{O}$.

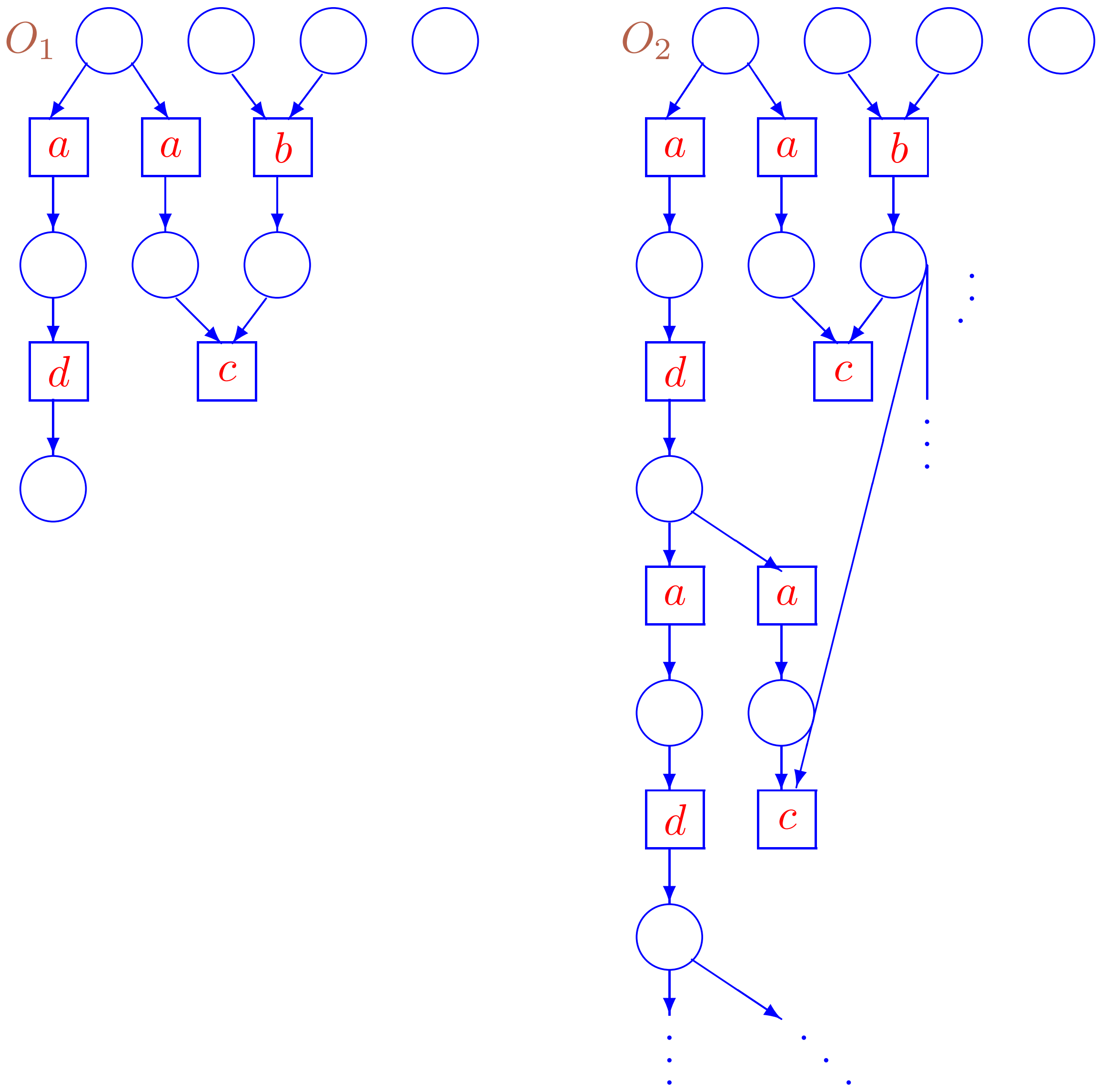
Then $\tilde{\varpi}$ is an *extension* of ϖ, and $\hat{\varpi}$ is an *extending* branching process for $\varpi,\ \varpi \rightarrow \tilde{\varpi}$.

A branching process $\varpi = (O, \psi)$ of a net N is *maximal*, if it cannot be extended, $\forall \tilde{\varpi} = (\widetilde{O}, \tilde{\psi})$ s.t. $\varpi \rightarrow \tilde{\varpi} :\ T_{\widetilde{O}} \setminus T_O = \emptyset$.

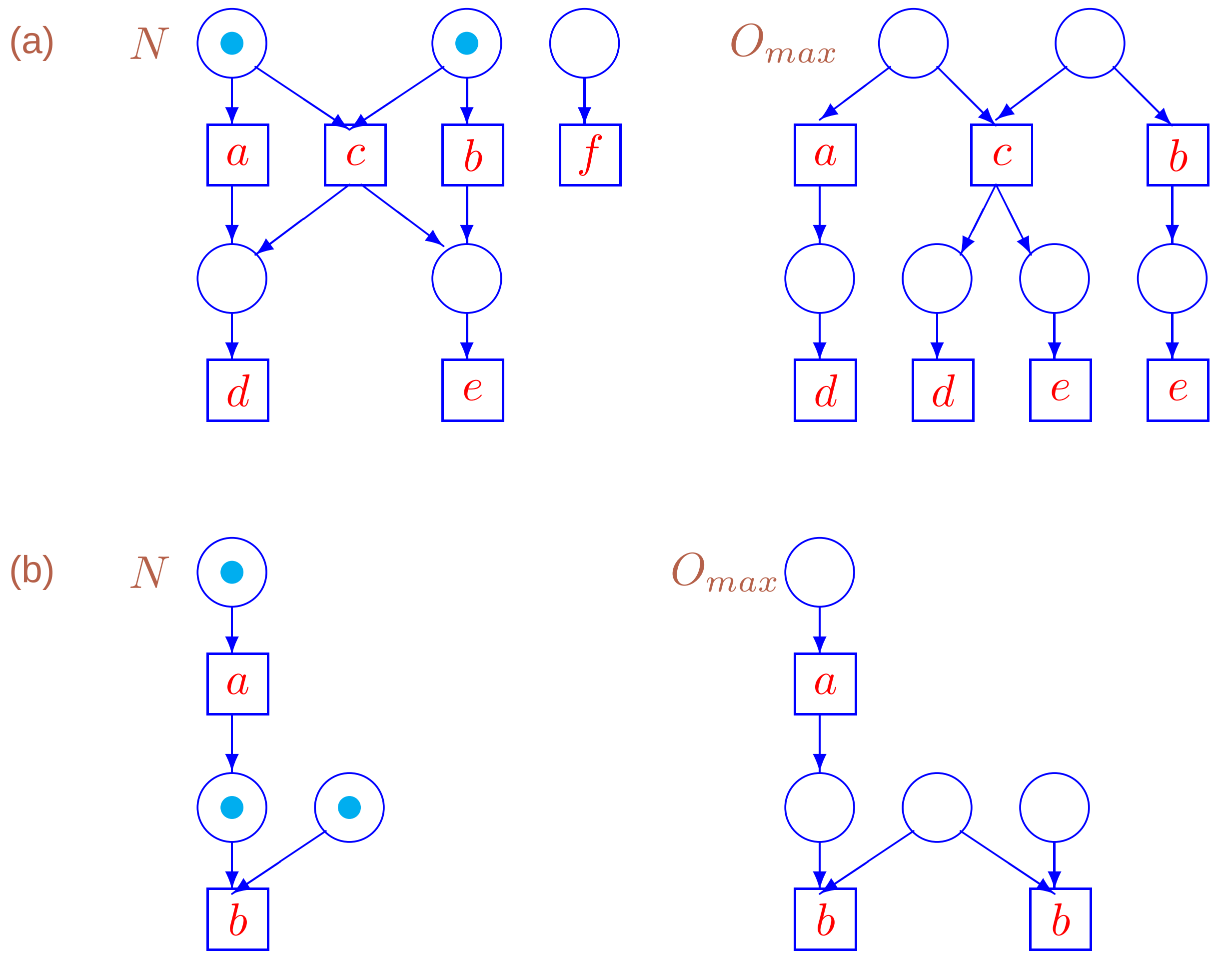
The set of all maximal branching processes of a net N consists of the unique (up to isomorphism) branching process $\varpi_{max} = (O_{max}, \psi_{max})$.

An isomorphism class of occurrence net O_{max} is an *unfolding* of a net N, notation $\mathcal{U}(N)$.

On the basis of unfolding $\mathcal{U}(N)$ of a net N, one can define MES $\mathcal{E}(N) = \xi_{\mathcal{U}(N)}$ which is an isomorphism class of LES ξ_O for $O \in \mathcal{U}(N)$.



Occurrence nets of branching processes



Occurrence nets of maximal branching processes

Basic simulation

Trace equivalences

Definition 15 *An* interleaving trace *of a net* N *is a sequence* $a_1 \cdots a_n \in Act^*$ s.t. $\pi_N \stackrel{a_1}{\to} \pi_1 \stackrel{a_2}{\to} \ldots \stackrel{a_n}{\to} \pi_n,\ \pi_i \in \Pi(N)\ (1 \le i \le n)$.

The set of all interleaving traces *of* N is $IntTraces(N)$.

N *and* N' *are* interleaving trace equivalent, $N \equiv_i N'$, *if*

$$IntTraces(N) = IntTraces(N').$$

Definition 16 *A* step trace *of a net* N *is a sequence* $A_1 \cdots A_n \in (I\!N_{fin}^{Act})^*$ s.t. $\pi_N \stackrel{A_1}{\to} \pi_1 \stackrel{A_2}{\to} \ldots \stackrel{A_n}{\to} \pi_n,\ \pi_i \in \Pi(N)\ (1 \le i \le n)$.

The set of all step traces *of* N is $StepTraces(N)$.

N *and* N' *are* step trace equivalent, $N \equiv_s N'$, *if*

$$StepTraces(N) = StepTraces(N').$$

Definition 17 *A* pomset trace *of a net* N *is a pomset* ρ, *an isomorphism class of lposet* ρ_C *for* $\pi = (C, \varphi) \in \Pi(N)$.

The set of all pomset traces *of* N is $Pomsets(N)$.

N *and* N' *are* partial word trace equivalent, $N \equiv_{pw} N'$, *if*

$$Pomsets(N) \sqsubseteq Pomsets(N') \text{ and } Pomsets(N') \sqsubseteq Pomsets(N).$$

Definition 18 N *and* N' *are* pomset trace equivalent, $N \equiv_{pom} N'$, *if*

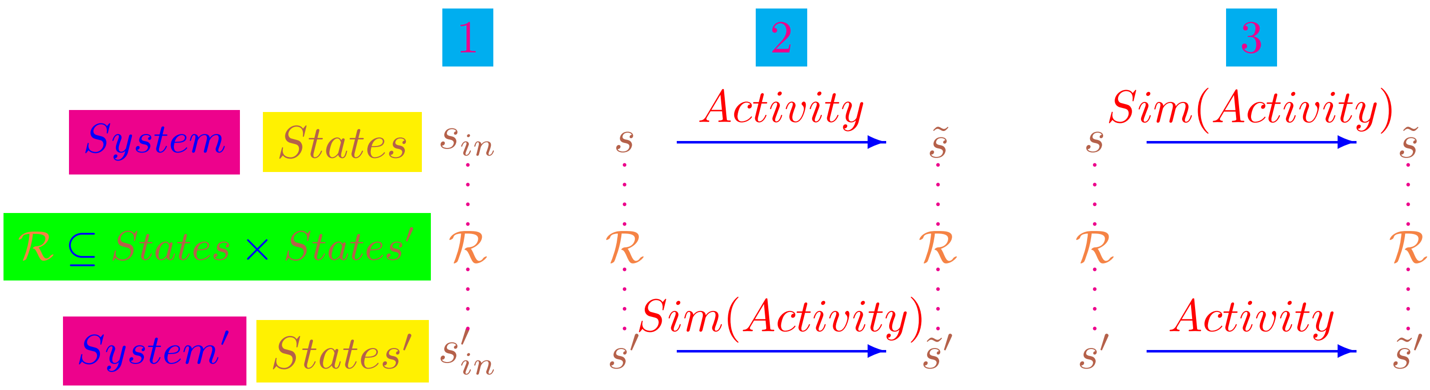
$$Pomsets(N) = Pomsets(N').$$

Definition 19 *A* process trace *of a net* N *is an isomorphism class of causal net* C *for* $\pi = (C, \varphi) \in \Pi(N)$.

The set of all process traces *of* N *is* $ProcessNets(N)$.

N *and* N' *are* process trace equivalent, $N \equiv_{pr} N'$, *if*

$$ProcessNets(N) = ProcessNets(N').$$



Bisimulation equivalence

Usual bisimulation equivalences

Definition 20 $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ *is a* $\star$-bisimulation *between nets* N *and* N', $\star \in$ {interleaving, step, partial word, pomset, process},
$\mathcal{R} : N \underline{\leftrightarrow}_\star N'$, $\star \in \{i, s, pw, pom, pr\}$, *if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.
2. $(\pi, \pi') \in \mathcal{R}$, $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$,
   (a) $|T_{\widehat{C}}| = 1$, *if* $\star = i$;
   (b) $\prec_{\widehat{C}} = \emptyset$, *if* $\star = s$;

   $\Rightarrow$ $\exists \tilde{\pi}' : \pi' \xrightarrow{\hat{\pi}'} \tilde{\pi}'$, $(\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$ *and*

   (a) $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$, *if* $\star = pw$;
   (b) $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, *if* $\star \in \{i, s, pom\}$;
   (c) $\widehat{C} \simeq \widehat{C}'$, *if* $\star = pr$.
3. *As item 2, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* $\star$-bisimulation equivalent, $\star \in$ {interleaving, step, partial word, pomset, process}, $N \underline{\leftrightarrow}_\star N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_\star N'$, $\star \in \{i, s, pw, pom, pr\}$.

ST-bisimulation equivalences

Definition 21 *[Vog92] An ST-marking of a net N is a pair (M, U):*

- *$M \in \mathbb{N}_{fin}^{P_N}$ is the current marking;*
- *$U \in \mathbb{N}_{fin}^{T_N}$ are the working transitions.*

$(M_N, \emptyset)$ is the *initial ST-marking* of a net N.

$T_N^{\pm} = \{t^+, t^- \mid t \in T_N\}$ is a set of *transition parts*.

t^+ is the *beginning*, and t^- is the *end* of t.

A transition part $q \in T_N^{\pm}$ is *enabled* in ST-marking $Q = (M, U)$, $Q \xrightarrow{q}$, if:

1. $M \xrightarrow{t}$, if $q = t^+$ or
2. $t \in U$, if $q = t^-$.

If q is enabled in M, its occurrence transforms ST-marking Q into $\widetilde{Q}$, $Q \xrightarrow{q} \widetilde{Q}$, as:

1. $\widetilde{M} = M - {}^\bullet t$ and $\widetilde{U} = U + t$, if $q = t^+$ or
2. $\widetilde{M} = M + t^\bullet$ and $\widetilde{U} = U - t$, if $q = t^-$.

We write $Q \rightarrow \widetilde{Q}$, if $\exists q \; Q \xrightarrow{q} \widetilde{Q}$.

$Act^{\pm} = \{a^+, a^- \mid a \in Act\}$ is the set of *action parts*.

a^+ is the *beginning*, and a^- is the *end* of a.

For $t \in T_N$, we define $L_N(t^+) = L_N(t)^+$ and $L_N(t^-) = L_N(t)^-$.

For $z \in Act^{\pm}$, we write $Q \xrightarrow{z} \widetilde{Q}$, if $\exists q \; Q \xrightarrow{q} \widetilde{Q}$ and $L_N(q) = z$.

An ST-marking $\widetilde{Q}$ of N is *reachable from* Q, if:

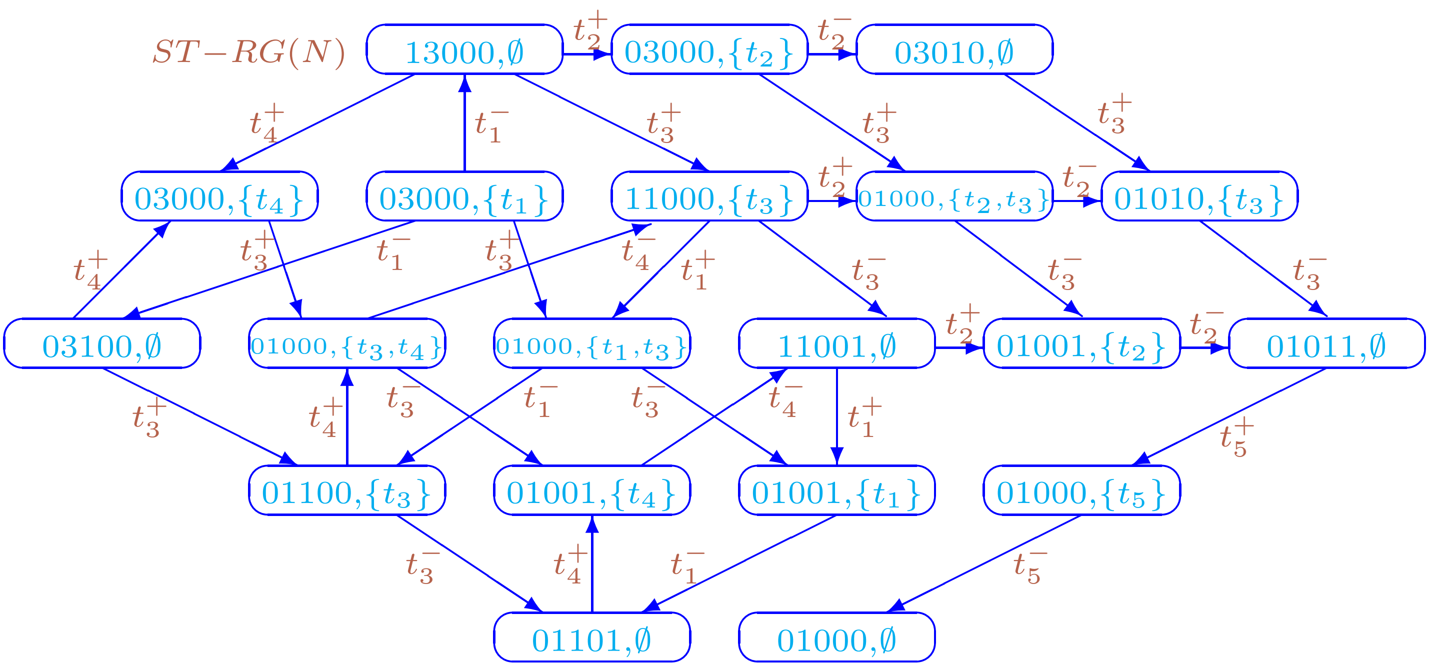
1. $\widetilde{Q} = Q$ or
2. there is a reachable from Q ST-marking $\widehat{Q}$ s.t. $\widehat{Q} \to \widetilde{Q}$.

An ST-marking Q of N is *reachable*, if it is reachable from M_N.

$ST-RS(N)$ is the set of *all reachable* ST-markings of N.

$ST-RG(N)$ is the *ST-reachabiliy graph* of a net N, an oriented graph with vertex set $ST-RS(N)$ and arcs from Q to $\widetilde{Q}$ iff $Q \to \widetilde{Q}$.

The arcs could be labeled by transition part names or labels.



ST-reachability graph of the marked net

Definition 22 *An ST-process of a net N is a pair (π_E, π_P):*

1. $\pi_E, \pi_P \in \Pi(N),\ \pi_P \stackrel{\pi_W}{\to} \pi_E$;
2. $\forall v, w \in T_{C_E}\ v \prec_{C_E} w\ \Rightarrow\ v \in T_{C_P}$.

- π_E is the current process;
- π_P is the completed part;
- π_W is the still working part.

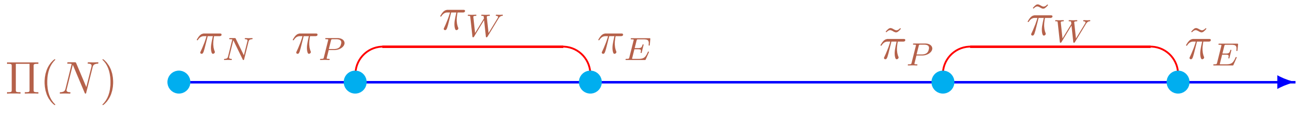
Obviously, $\prec_{C_W} = \emptyset$.

$ST-\Pi(N)$ is the set of *all ST-processes* of a net N.

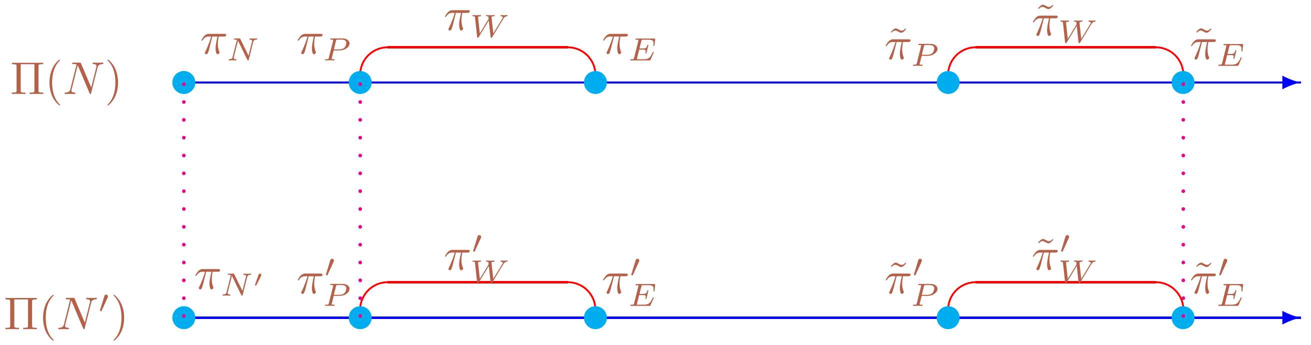
(π_N, π_N) is the *initial ST-process* of a net N.

Let $(\pi_E, \pi_P),\ (\tilde{\pi}_E, \tilde{\pi}_P) \in ST-\Pi(N)$.

We write $(\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \to \tilde{\pi}_E$ and $\pi_P \to \tilde{\pi}_P$.



ST-processes



ST-bisimulation equivalence

Definition 23 $\mathcal{R} \subseteq ST - \Pi(N) \times ST - \Pi(N') \times \mathcal{B}$, *where* $\mathcal{B} = \{\beta \mid \beta : T_C \to T_{C'},\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$, *is a* $\star$-ST-bisimulation *between nets* N *and* N', $\star \in \{$interleaving, partial word, pomset, process$\}$, $\mathcal{R} : N \underline{\leftrightarrow}_{\star ST} N'$, $\star \in \{i, pw, pom, pr\}$, *if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \Rightarrow \beta : \rho_{C_E} \asymp \rho_{C'_E}$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \Rightarrow \exists \tilde{\beta},\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{T_{C_E}} = \beta,$ $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$, *and if* $\pi_P \xrightarrow{\pi} \tilde{\pi}_E,\ \pi'_P \xrightarrow{\pi'} \tilde{\pi}'_E,\ \gamma = \tilde{\beta}|_{T_C}$, *then:*
   (a) $\gamma^{-1} : \rho_{C'} \sqsubseteq \rho_C$, *if* $\star = pw$;
   (b) $\gamma : \rho_C \simeq \rho_{C'}$, *if* $\star = pom$;
   (c) $C \simeq C'$, *if* $\star = pr$.
4. *As item 3, but the roles of* N *and* N' *are reversed.*

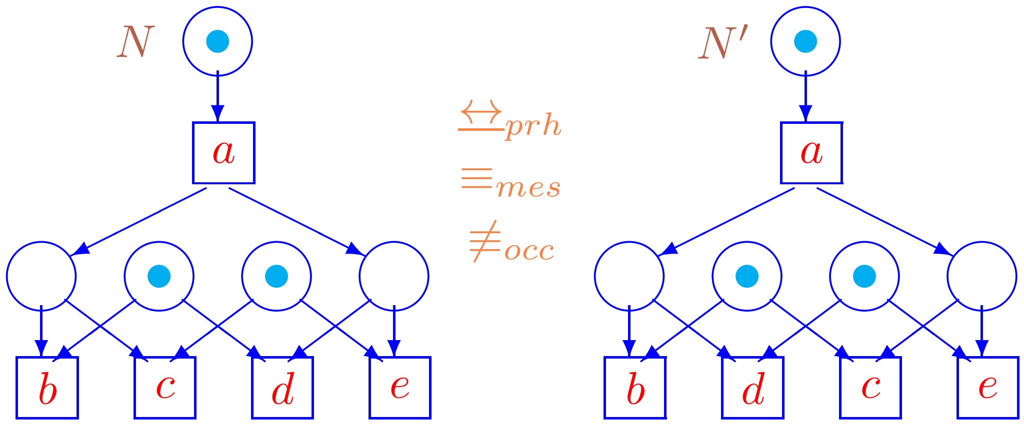
N *and* N' *are* $\star$-ST-bisimulation equivalent, $\star \in \{$interleaving, partial word, pomset, process$\}$, $N \underline{\leftrightarrow}_{\star ST} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_{\star ST} N'$, $\star \in \{i, pw, pom, pr\}$.

History preserving bisimulation equivalences

Definition 24 $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, *where*
$\mathcal{B} = \{\beta \mid \beta : T_C \to T_{C'},\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$,
is a $\star$-history preserving bisimulation *between nets* N *and* N',
$\star \in$ {pomset,process}, $N \underline{\leftrightarrow}_{\star h} N',\ \star \in \{pom, pr\}$, *if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.
2. $(\pi, \pi', \beta) \in \mathcal{R} \Rightarrow$
   (a) $\tilde{\beta} : \rho_{\tilde{C}} \simeq \rho_{\tilde{C}'}$, *if* $\star \in \{pom, pr\}$;
   (b) $\tilde{C} \simeq \tilde{C}'$, *if* $\star = pr$.
3. $(\pi, \pi', \beta) \in \mathcal{R},\ \pi \to \tilde{\pi} \Rightarrow \exists \tilde{\beta},\ \tilde{\pi}' : \pi' \to \tilde{\pi}',\ \tilde{\beta}|_{T_C} = \beta,$
   $(\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of* N *and* N' *are reversed.*

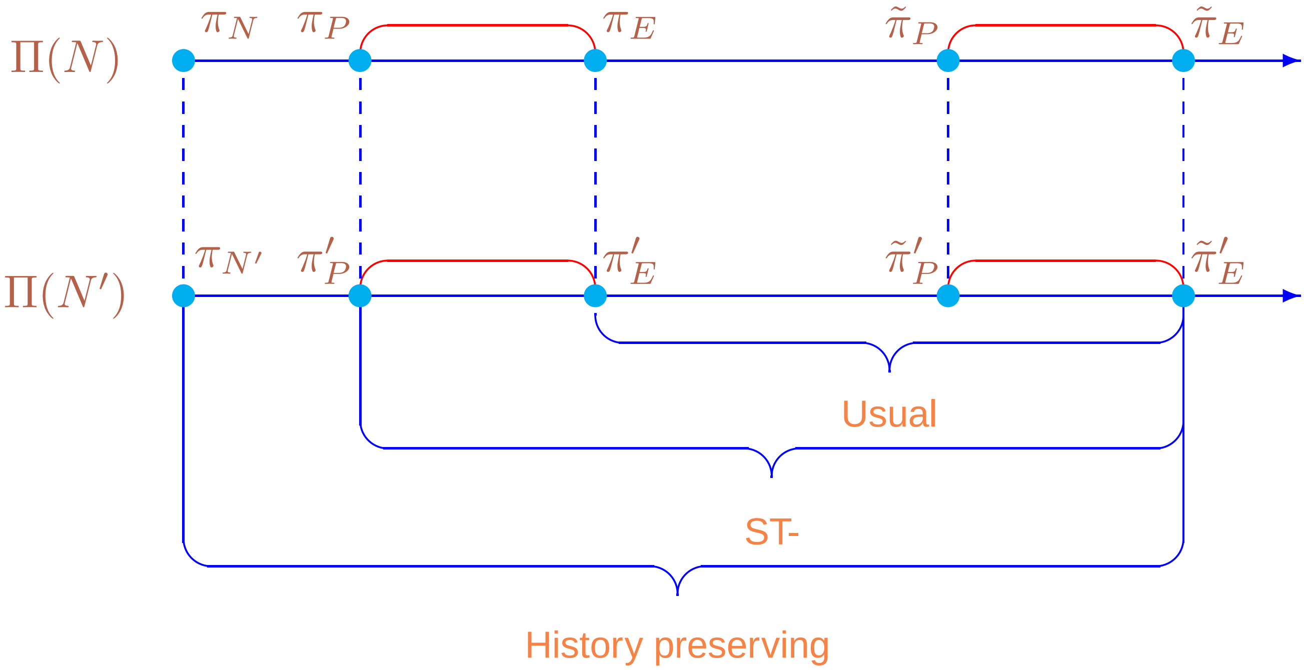
N *and* N' *are* $\star$-history preserving bisimulation equivalent, $\star \in$ {pomset, process}, $N \underline{\leftrightarrow}_{\star h} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_{\star h} N',\ \star \in \{pom, pr\}$.



Nets that are not equivalent w.r.t. strict version of $\underline{\leftrightarrow}_{prh}$

Strict version of $\underline{\leftrightarrow}_{prh}$: suppose $\beta : C \simeq C'$ in the definition.

N and N' are not equivalent since any isomorphism "reverts" output places of their transitions labeled by a. For any correspondence between the left and right places in N and the ones in N' there is an extension (by a process with action b or c) in N that cannot be imitated in N'. The two places of N' should be "revered" in any case to allow the correct extension of isomorphism to C-nets of the resulted processes.



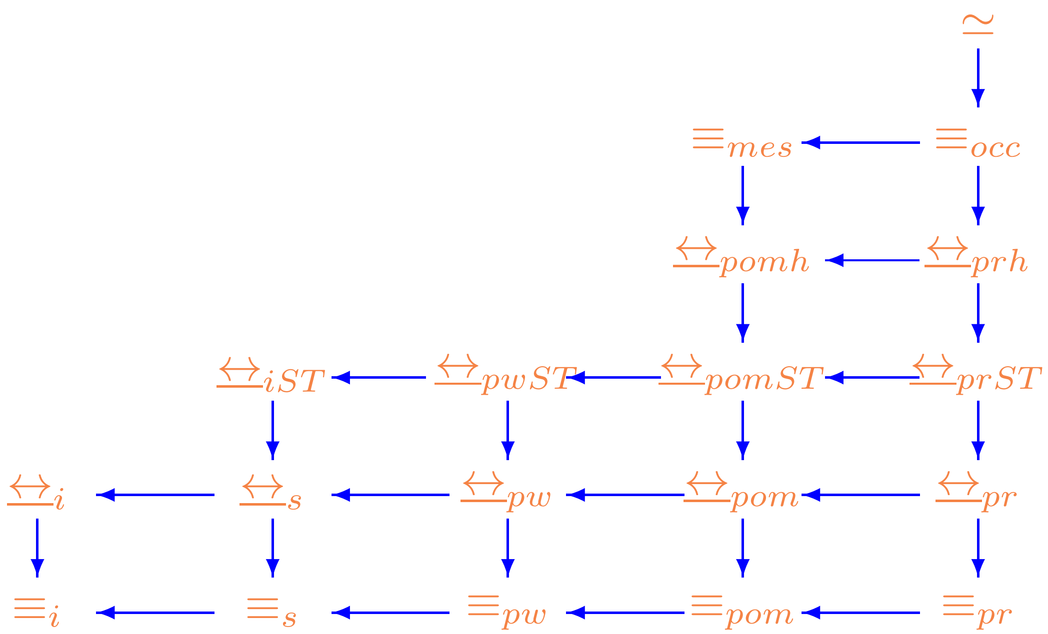
A distinguish ability of the bisimulation equivalences

Conflict preserving equivalences

Definition 25 N *and* N' *are* MES conflict preserving equivalent, $N \equiv_{mes} N'$, *if* $\mathcal{E}(N) = \mathcal{E}(N')$.

Definition 26 N *and* N' *are* occurrence conflict preserving equivalent, $N \equiv_{occ} N'$, *if* $\mathcal{U}(N) = \mathcal{U}(N')$.

Comparing basic equivalences

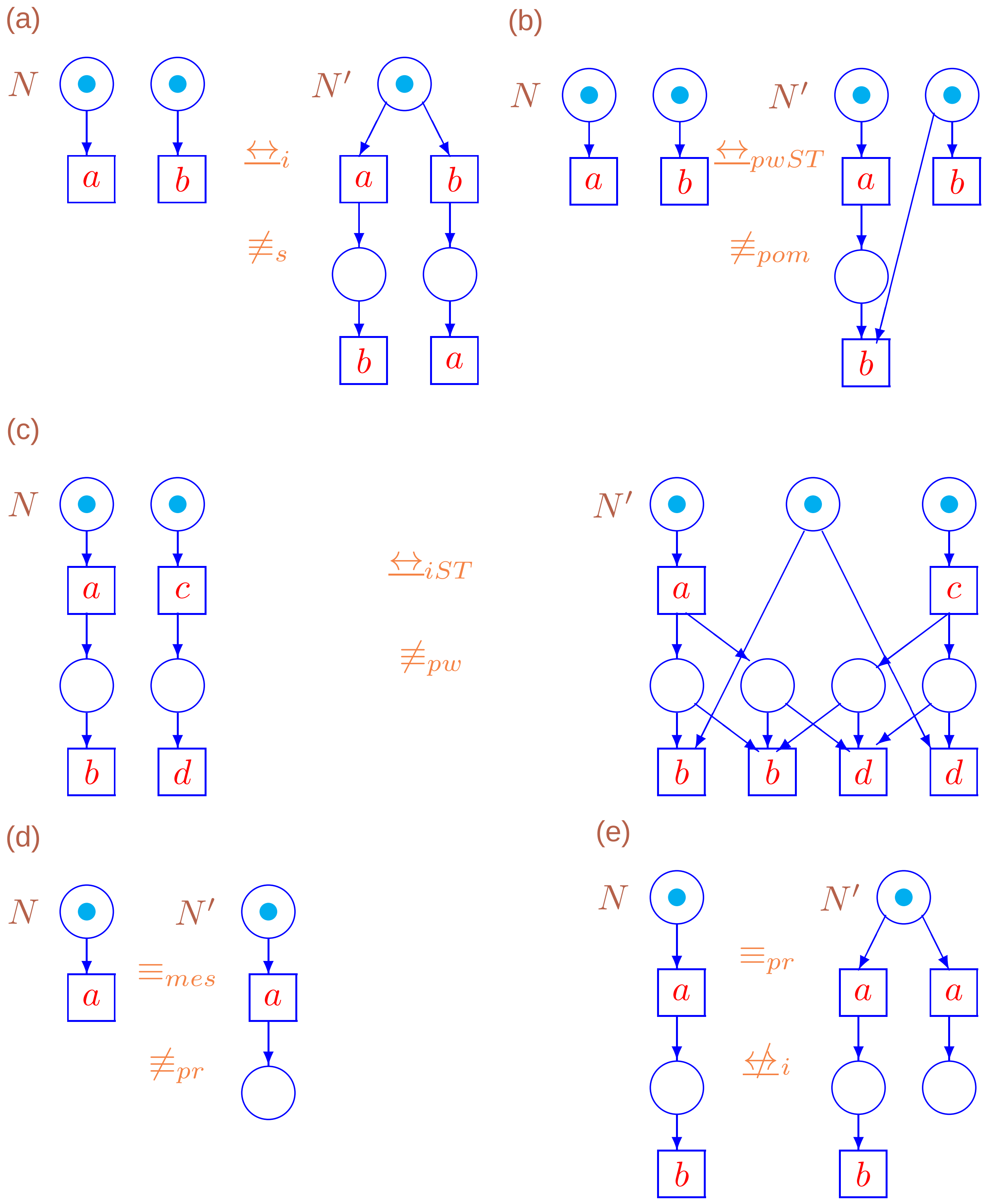


Interrelations of basic equivalences

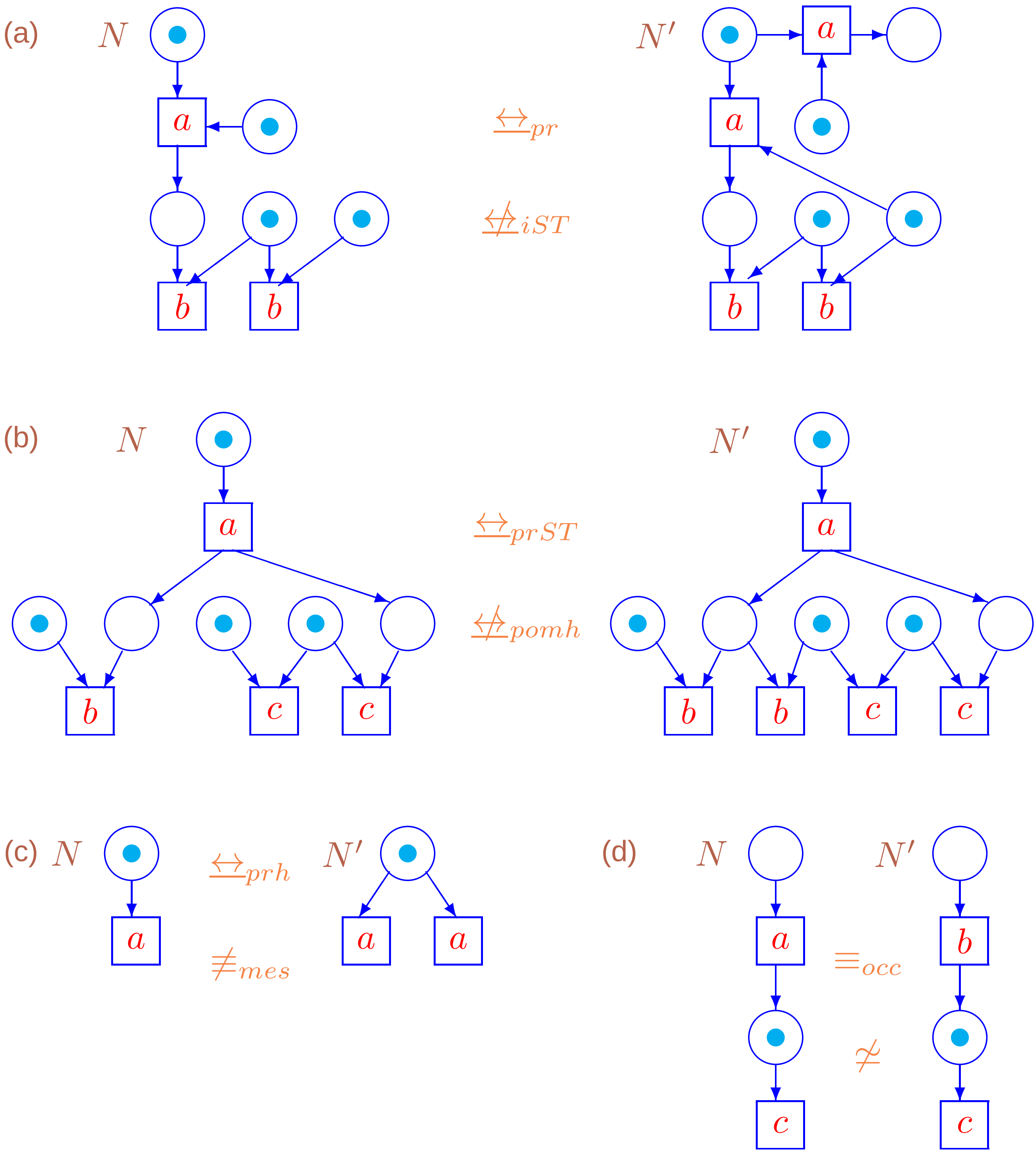
Theorem 1 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\!\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \simeq\}$ *and* $\star, \star\star \in \{\_, i, s, pw, pom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ\}$. *For nets* N *and* N'

$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \leftrightarrow\!\!\!\!\leftrightarrow_{\star\star} N'$$

iff there exists a directed path from $\leftrightarrow_{\star}$ *to* $\leftrightarrow\!\!\!\!\leftrightarrow_{\star\star}$ *in the graph above.*



B: Examples of basic equivalences



B1: Examples of basic equivalences (continued)

- In Figure B(a), $N \underline{\leftrightarrow}_i N'$, but $N \not\equiv_s N'$, since only in the net N' actions a and b can occur concurrently.
- In Figure B(c), $N \underline{\leftrightarrow}_{iST} N'$, but $N \not\equiv_{pw} N'$, since for the pomset corresponding to the net N there is no even less sequential pomset in N'.
- In Figure B(b), $N \underline{\leftrightarrow}_{pwST} N'$, but $N \not\equiv_{pom} N'$, since only in the net N' an action b can depend on action a.
- In Figure B(d), $N \equiv_{mes} N'$, but $N \not\equiv_{pr} N'$, since N' is a causal net which is not isomorphic to N (because of additional output place).
- In Figure B(e), $N \equiv_{pr} N'$, but $N \underline{\not\leftrightarrow}_i N'$, since only in net N' action a can occur so that action b cannot occur afterwards.
- In Figure B1(a), $N \underline{\leftrightarrow}_{pr} N'$, but $N \underline{\not\leftrightarrow}_{iST} N'$, since only in net N' action a can start so that no action b can begin working until a finishes.
- In Figure B1(b), $N \underline{\leftrightarrow}_{prST} N'$, but $N \underline{\not\leftrightarrow}_{pomh} N'$, since only in net N' actions a and b can occur so that action c must depend on a.
- In Figure B1(c), $N \underline{\leftrightarrow}_{prh} N'$, but $N \not\equiv_{mes} N'$, since only net N' has corresponding MES with two conflict actions a.
- In Figure B1(d), $N \equiv_{occ} N'$, but $N \not\simeq N'$, since upper transitions of nets N and N' are labeled by different actions (a and b).

Back-forth simulation and logics

Sequential runs [Che92a,Tar97]

Definition 27 *A sequential run of a net N is a pair (π, σ):*

- *a process $\pi \in \Pi(N)$:*

  causal dependencies of transitions;

- *a sequence $\sigma \in T_C^*$ s.t. $\pi_N \xrightarrow{\sigma} \pi$:*

  occurrence order of transitions.

The *set of all sequential runs* of a net N is $Runs(N)$.

The *initial* sequential run of a net N is a pair (π_N, ε) (ε is the empty sequence).

Let $(\pi, \sigma), \ (\tilde{\pi}, \tilde{\sigma}) \in Runs(N)$.

We write $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$, if $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}, \ \exists \hat{\sigma} \in T_{\tilde{C}}^* \ \pi \xrightarrow{\hat{\sigma}} \tilde{\pi}$ and $\tilde{\sigma} = \sigma\hat{\sigma}$.

We write $(\pi, \sigma) \rightarrow (\tilde{\pi}, \tilde{\sigma})$, if $\exists \hat{\pi} \ (\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$.

$|\sigma|$ is the *length* of a sequence σ.

Let $(\pi, \sigma) \in Runs(N),\ (\pi', \sigma') \in Runs(N')$ and $\sigma = v_1 \cdots v_n,\ \sigma' = v_1' \cdots v_n'$.

We define a mapping $\beta_\sigma^{\sigma'} : T_C \to T_{C'}$:

- $\beta_\varepsilon^\varepsilon = \emptyset$;
- $\beta_\sigma^{\sigma'} = \{(v_i, v_i') \mid 1 \leq i \leq n\}$.

Let $(\pi, \sigma) \in Runs(N)$ and $\sigma = v_1 \cdots v_n,\ \pi_N \stackrel{v_1}{\to} \ldots \stackrel{v_i}{\to} \pi_i\ (1 \leq i \leq n)$. Then:

- $\pi(0) = \pi_N$,
  $\pi(i) = \pi_i\ (1 \leq i \leq n)$;
- $\sigma(0) = \varepsilon$,
  $\sigma(i) = v_1 \cdots v_i\ (1 \leq i \leq n)$.

Back-forth bisimulation equivalences

Definition 28 $\mathcal{R} \subseteq Runs(N) \times Runs(N')$ *is a* $\star$-back $\star\star$-forth bisimulation *between nets* N *and* N', $\star, \star\star \in\{$interleaving, step, partial word, pomset, process$\}$, $\mathcal{R} : N \underline{\leftrightarrow}_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom, pr\}$, *if:*

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$
   - *(back)* $(\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$,

     (a) $|T_{\widehat{C}}| = 1$, *if* $\star = i$;

     (b) $\prec_{\widehat{C}} = \emptyset$, *if* $\star = s$;

     $\Rightarrow \exists (\tilde{\pi}', \tilde{\sigma}') : (\tilde{\pi}', \tilde{\sigma}') \xrightarrow{\hat{\pi}'} (\pi', \sigma'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*

     (a) $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$, *if* $\star = pw$;

     (b) $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, *if* $\star \in \{i, s, pom\}$;

     (c) $\widehat{C} \simeq \widehat{C}'$, *if* $\star = pr$;
   - *(forth)* $(\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$,

     (a) $|T_{\widehat{C}}| = 1$, *if* $\star\star = i$;

     (b) $\prec_{\widehat{C}} = \emptyset$, *if* $\star\star = s$;

     $\Rightarrow \exists (\tilde{\pi}', \tilde{\sigma}') : (\pi', \sigma') \xrightarrow{\hat{\pi}'} (\tilde{\pi}', \tilde{\sigma}'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*

     (a) $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$, *if* $\star\star = pw$;

     (b) $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, *if* $\star\star \in \{i, s, pom\}$;

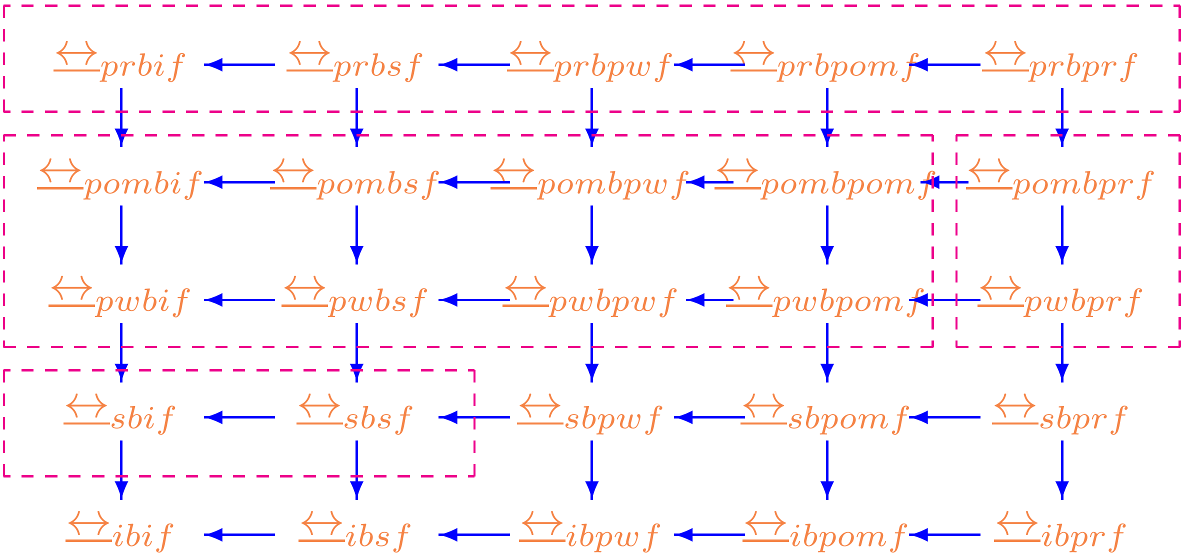
     (c) $\widehat{C} \simeq \widehat{C}'$, *if* $\star\star = pr$.
3. *As item 2, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* $\star$-back $\star\star$-forth bisimulation equivalent, $\star, \star\star \in\{$interleaving, step, partial word, pomset, process$\}$, $N \underline{\leftrightarrow}_{\star b \star\star f} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom, pr\}$.

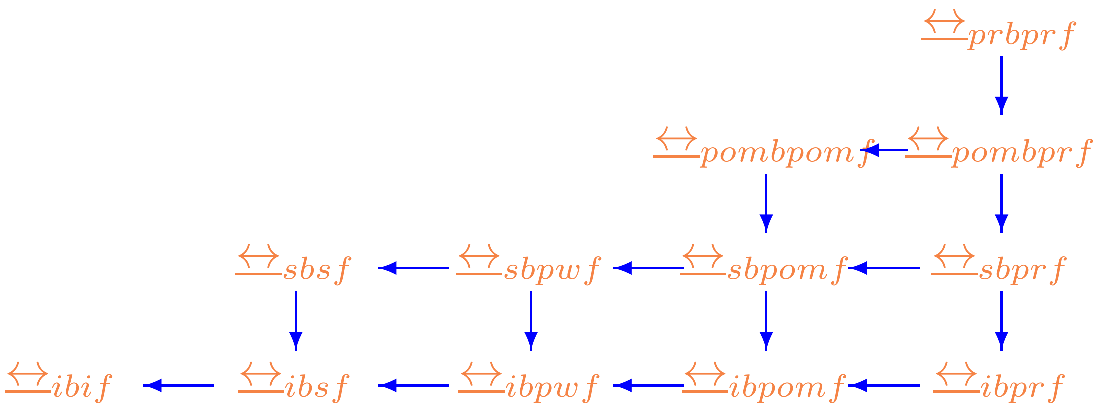
Comparing back-forth bisimulation equivalences

Proposition 1 [Pin93,Tar97] Let $\star \in \{i, s, pw, pom, pr\}$. For nets N and N'

1. $N \underline{\leftrightarrow}_{pwb\star f} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}_{pomb\star f} N'$;
2. $N \underline{\leftrightarrow}_{\star bif} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}_{\star b\star f} N'$.



Merging of back-forth bisimulation equivalences

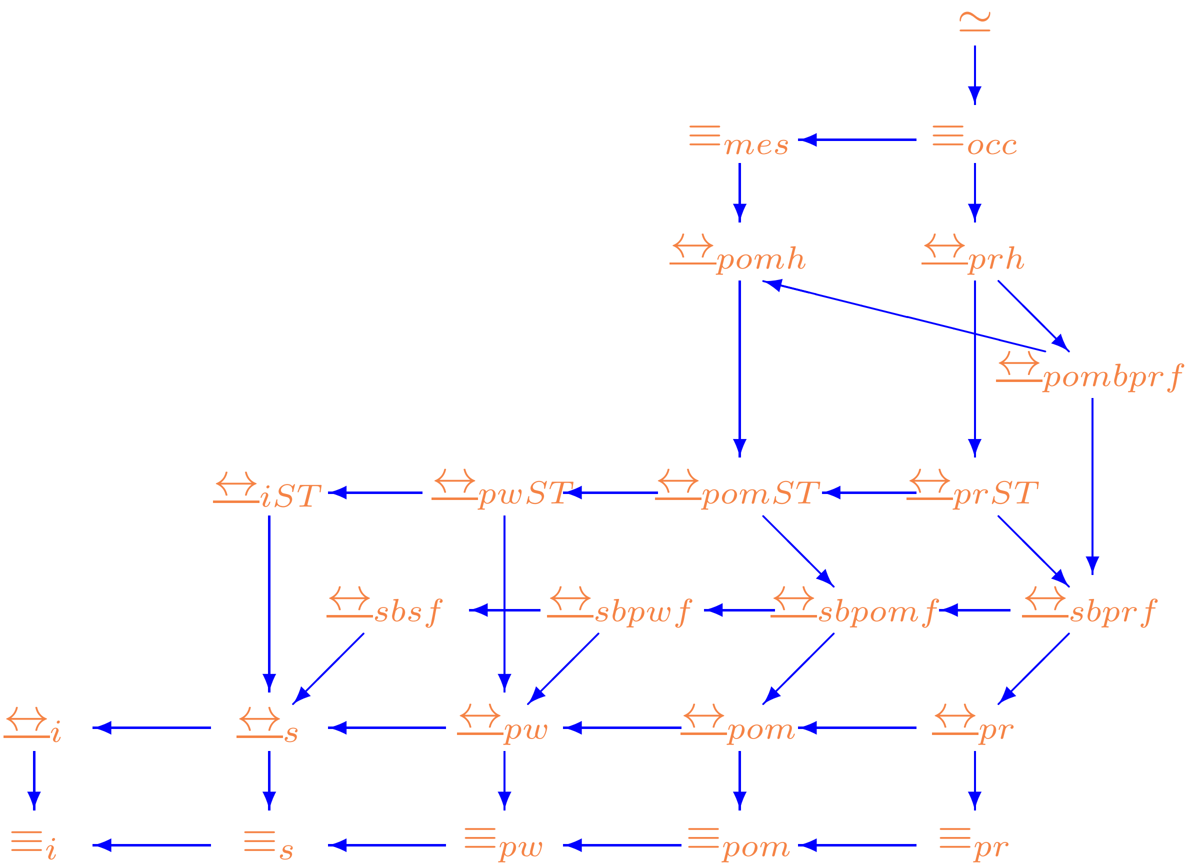


Interrelations of back-forth bisimulation equivalences

Comparing back-forth bisimulation equivalences with basic ones

Proposition 2 *[Pin93,Tar97] Let $\star \in \{i, s, pw, pom, pr\}$ and $\star\star \in \{pom, pr\}$. For nets N and N'*

1. $N \underline{\leftrightarrow}_{ib\star f} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}_{\star} N'$;
2. $N \underline{\leftrightarrow}_{\star\star ST} N' \;\Rightarrow\; N \underline{\leftrightarrow}_{sb\star\star f} N'$.

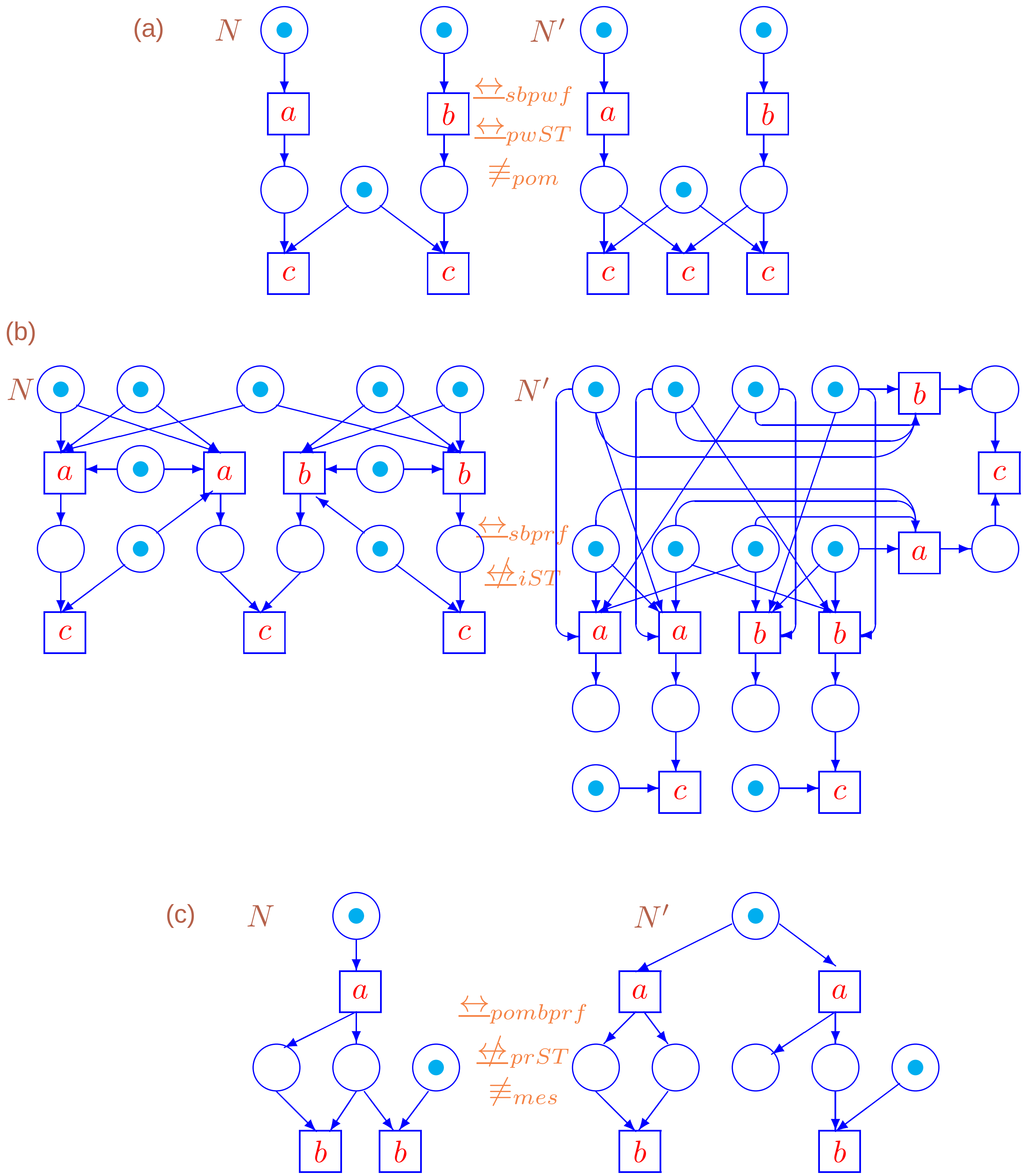


Interrelations of back-forth bisimulation equivalences with basic ones

Theorem 2 *Let $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, \simeq\}$ and $\star, \star\star \in \{\_, i, s, pw, pom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$. For nets N and N'*

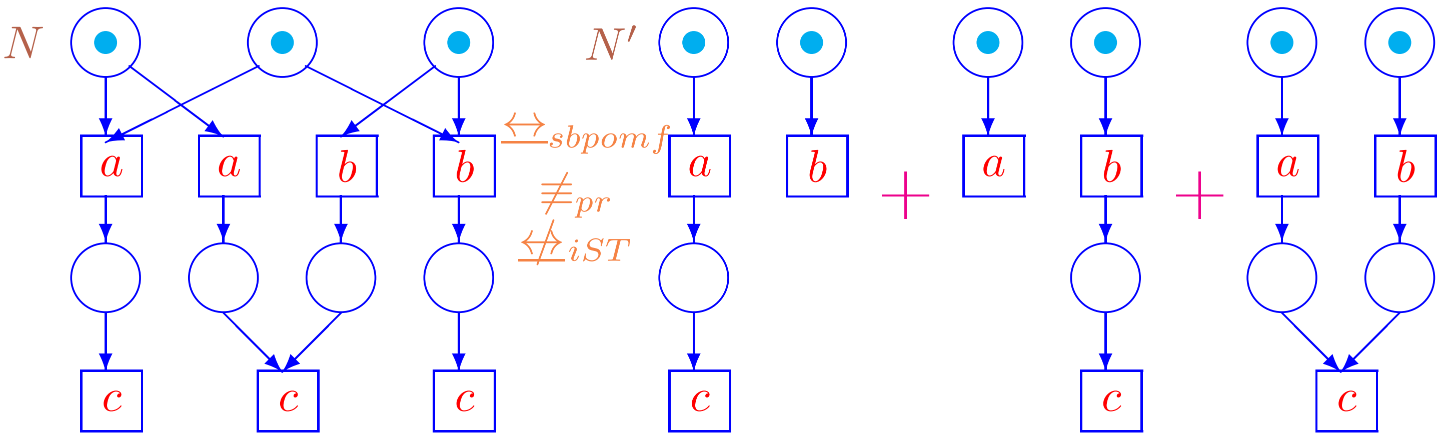
$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \twoheadleftarrow\!\!\twoheadrightarrow_{\star\star} N'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ to $\twoheadleftarrow\!\!\twoheadrightarrow_{\star\star}$.



BF: Examples of back-forth bisimulation equivalences

- In Figure B(c), $N \underline{\leftrightarrow}_{sbsf} N'$, but $N \not\equiv_{pw} N'$.
- In Figure BF(a), $N \underline{\leftrightarrow}_{sbpwf} N'$, but $N \not\equiv_{pom} N'$, since only in the net N' action c can depend on actions a and b.
- In Figure BF(b), $N \underline{\leftrightarrow}_{sbprf} N'$, but $N \underline{\not\leftrightarrow}_{iST} N'$, since only in the net N' action a can start so that:
  1. until finishing of a the sequence of actions bc cannot occur, and
  2. immediately after finishing of a action c cannot occur.
- In Figure BF(c), $N \underline{\leftrightarrow}_{pombprf} N'$, but $N \underline{\not\leftrightarrow}_{prST} N'$, since only in the net N' the process with action a can start so that it can be extended by process with action b in the only way (so that extended process be unique).
- In Figure B(b), $N \underline{\leftrightarrow}_{pwST} N'$, but $N \underline{\not\leftrightarrow}_{sbsf} N'$, since only in the net N' the sequence of actions ab can occur so that b must depend on a.
- In Figure B1(a), $N \underline{\leftrightarrow}_{pr} N'$, but $N \underline{\not\leftrightarrow}_{sbsf} N'$, since only in the net N' action a can occur so that action b must depend on a.



More clear, but weaker example of back-forth bisimulation equivalences

Logic HML [HM85]

Definition 29 *$\top$ denotes the truth, $a \in Act$.*

A formula of HML:

$$\Phi ::= \ \top \mid \neg\Phi \mid \Phi \wedge \Phi \mid \langle a \rangle \Phi$$

HML is the set of *all formulas* of HML.

Definition 30 *Let N be a net and $\pi \in \Pi(N)$. The satisfaction relation $\models_N \in \Pi(N) \times \mathbf{HML}$:*

1. *$\pi \models_N \top$ — always;*
2. *$\pi \models_N \neg\Phi$, if $\pi \not\models_N \Phi$;*
3. *$\pi \models_N \Phi \wedge \Psi$, if $\pi \models_N \Phi$ and $\pi \models_N \Psi$;*
4. *$\pi \models_N \langle a \rangle \Phi$, if $\exists \tilde{\pi} \in \Pi(N)\ \pi \xrightarrow{a} \tilde{\pi}$ and $\tilde{\pi} \models_N \Phi$.*

$[a]\Phi = \neg\langle a \rangle \neg\Phi$. $N \models_N \Phi$, if $\pi_N \models_N \Phi$.

Definition 31 *N and N' are are logical equivalent in HML, $N =_{HML} N'$, if $\forall \Phi \in \mathbf{HML}\ N \models_N \Phi \Leftrightarrow N' \models_{N'} \Phi$.*

Let for a net N $\pi \in \Pi(N)$, $a \in Act$.

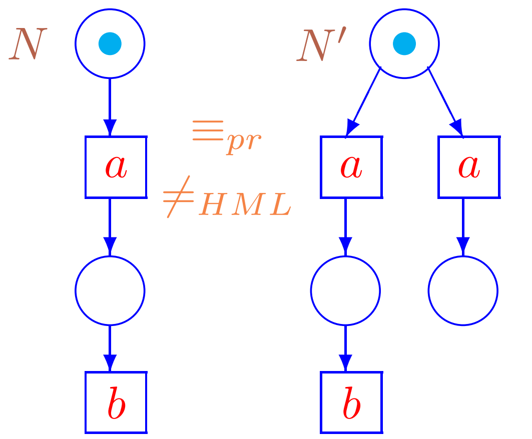
The set of *extensions* of a process π by action a (*image set*)is $Image(\pi, a) = \{\tilde{\pi} \mid \pi \xrightarrow{a} \tilde{\pi}\}$.

A net N is a *image-finite* one, if $\forall \pi \in \Pi(N)\ \forall a \in Act\ |Image(\pi, a)| < \infty$.

Theorem 3 *For image-finite nets N and N'*

$$N \underline{\leftrightarrow}_i N' \ \Leftrightarrow \ N \underline{\leftrightarrow}_{ibif} N' \ \Leftrightarrow \ N =_{HML} N'.$$

Example on logical equivalence of HML



Differentiating power of $=_{HML}$

$N \equiv_{pr} N'$, but $N \not=_{HML} N'$, because for $\Phi = [a]\langle b\rangle\top$, $N \models_N \Phi$, but $N' \not\models_{N'} \Phi$ since only in N' an action a can occur so that no b is possible afterwards.

Logic $PBFL$ [CLP92]

Definition 32 *$\top$ denotes the truth, $a \in Act$ and ρ is a pomset with labeling into Act.*

A formula of $PBFL$:

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \langle\leftarrow \rho\rangle\Phi \mid \langle a\rangle\Phi$$

PBFL is the set of *all formulas* of $PBFL$.

Definition 33 *Let $(\pi, \sigma) \in Runs(N)$ for a net N. The satisfaction relation $\models_N \in Runs(N) \times \mathbf{PBFL}$:*

1. $(\pi, \sigma) \models_N \top$ *— always;*
2. $(\pi, \sigma) \models_N \neg\Phi$*, if* $(\pi, \sigma) \not\models_N \Phi$*;*
3. $(\pi, \sigma) \models_N \Phi\wedge\Psi$*, if* $(\pi, \sigma) \models_N \Phi$ *and* $(\pi, \sigma) \models_N \Psi$*;*
4. $(\pi, \sigma) \models_N \langle\leftarrow \rho\rangle\Phi$*, if* $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$*, where* $\hat{\pi} = (\widehat{C}, \hat{\varphi}),\ \rho_{\widehat{C}} \in \rho$ *and* $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$*;*
5. $(\pi, \sigma) \models_N \langle a\rangle\Phi$*, if* $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$*, where* $\hat{\pi} = (\widehat{C}, \hat{\varphi}),\ L_{\widehat{C}}(T_{\widehat{C}}) = a$ *and* $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$*.*

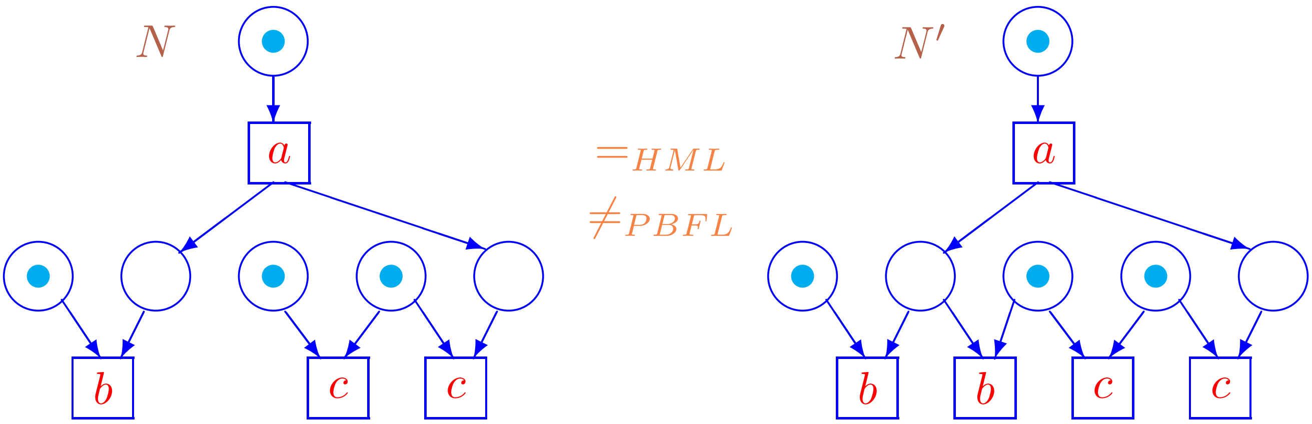
$[a]\Phi = \neg\langle a\rangle\neg\Phi$, $[\leftarrow \rho]\Phi = \neg\langle\leftarrow \rho\rangle\neg\Phi$. $N \models_N \Phi$, if $(\pi_N, \varepsilon) \models_N \Phi$.

Definition 34 *N and N' are logical equivalent in $PBFL$, $N =_{PBFL} N'$, if $\forall\Phi \in \mathbf{PBFL}\ N \models_N \Phi \ \Leftrightarrow\ N' \models_{N'} \Phi$.*

Theorem 4 *For image-finite nets N and N'*

$$N \underline{\leftrightarrow}_{pomh} N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{pombpomf} N' \ \Leftrightarrow\ N =_{PBFL} N'.$$

Example on logical equivalence of $PBFL$



Differentiating power of $=_{PBFL}$

$N =_{HML} N'$, but $N \neq_{PBFL} N'$, because for $\Phi = [a][b]\langle c\rangle\langle\leftarrow (a;b)\|c\rangle\top$, $N \models_N \Phi$, but $N' \not\models_{N'} \Phi$ since only in N' after action a an action b can occur so that c must depend on a.

Here $(a;b)\|c$ denotes the pomset where b depends on a, and $a,\ b$ are independent with c.

Logic $PrBFL$ [Tar97]

Definition 35 *$\top$ denotes the truth, $a \in Act$ and $\mathbf{C}$ is the isomorphism class of a causal net C.*

A formula of $PrBFL$:

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \langle\leftarrow \mathbf{C}\rangle\Phi \mid \langle a\rangle\Phi$$

PrBFL is the set of *all formulas* of $PrBFL$.

Definition 36 *Let $(\pi, \sigma) \in Runs(N)$ for a net N. The satisfaction relation $\models_N \in Runs(N) \times \mathbf{PrBFL}$:*

1. $(\pi, \sigma) \models_N \top$ *— always;*
2. $(\pi, \sigma) \models_N \neg\Phi$*, if* $(\pi, \sigma) \not\models_N \Phi$*;*
3. $(\pi, \sigma) \models_N \Phi\wedge\Psi$*, if* $(\pi, \sigma) \models_N \Phi$ *and* $(\pi, \sigma) \models_N \Psi$*;*
4. $(\pi, \sigma) \models_N \langle\leftarrow \mathbf{C}\rangle\Phi$*, if* $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$*, where* $\hat{\pi} = (\widehat{C}, \hat{\varphi})$, $\widehat{C} \in \mathbf{C}$ *and* $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$*;*
5. $(\pi, \sigma) \models_N \langle a\rangle\Phi$*, if* $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$*, where* $\hat{\pi} = (\widehat{C}, \hat{\varphi})$, $L_{\widehat{C}}(T_{\widehat{C}}) = a$ *and* $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$*.*

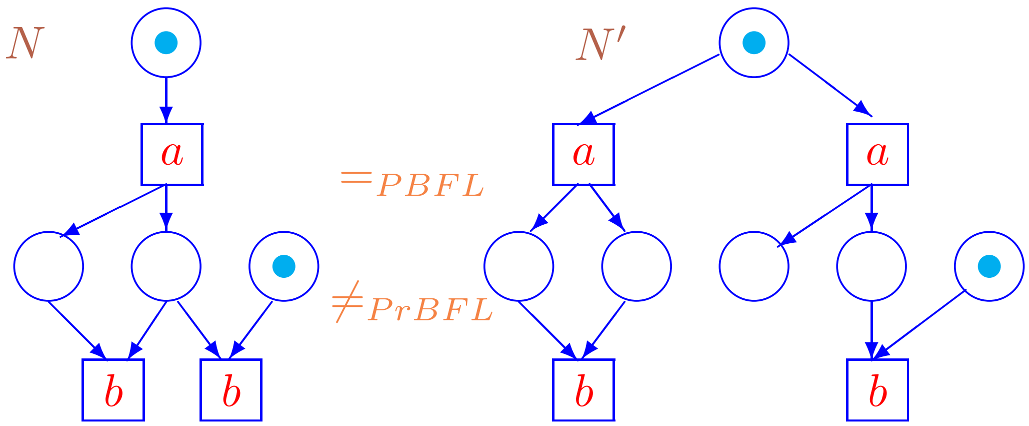
$[a]\Phi = \neg\langle a\rangle\neg\Phi$, $[\leftarrow \mathbf{C}]\Phi = \neg\langle\leftarrow \mathbf{C}\rangle\neg\Phi$. $N \models_N \Phi$, if $(\pi_N, \varepsilon) \models_N \Phi$.

Definition 37 *N and N' are logical equivalent in $PrBFL$, $N =_{PrBFL} N'$, if $\forall\Phi \in \mathbf{PrBFL}\ N \models_N \Phi \Leftrightarrow N' \models_{N'} \Phi$.*

Theorem 5 *For image-finite nets N and N'*

$$N \underline{\leftrightarrow}_{prh} N' \Leftrightarrow N \underline{\leftrightarrow}_{prbprf} N' \Leftrightarrow N =_{PrBFL} N'.$$

Example on logical equivalence of $PrBFL$



Differentiating power of $=_{PrBFL}$

$N =_{PBFL} N'$, but $N \neq_{PrBFL} N'$, because for $\Phi = [a]\langle b\rangle\langle\leftarrow \mathbf{C}\rangle\top$, $N \models_N \Phi$, but $N' \not\models_{N'} \Phi$, since only in the net N' a process with action a can start so that it can be extended by b in the only way (connecting pairwise output and input places).

Here $\mathbf{C}$ is an isomorphism class of causal net where two output places of an a-labeled transition are both the input places of b-labeled one.

Place simulation and net reduction

Place bisimulation equivalences

Definition 38 $\mathcal{R} \subseteq \mathbb{N}_{fin}^{P_N} \times \mathbb{N}_{fin}^{P_{N'}}$ *is a* $\star$-bisimulation *between nets* N *and* N', $\star \in \{$interleaving, step, partial word, pomset, process$\}$,
$\mathcal{R} : N \underline{\leftrightarrow}_\star N',\ \star \in \{i, s, pw, pom, pr\}$, *if:*

1. $(M_N, M_{N'}) \in \mathcal{R}$.
2. $(M, M') \in \mathcal{R},\ M \xrightarrow{\hat{\pi}} \widetilde{M}$,
   (a) $|T_{\widehat{C}}| = 1$, *if* $\star = i$;
   (b) $\prec_{\widehat{C}} = \emptyset$, *if* $\star = s$;
   $\Rightarrow\ \exists \widetilde{M}' :\ M' \xrightarrow{\hat{\pi}'} \widetilde{M}',\ (\widetilde{M}, \widetilde{M}') \in \mathcal{R}$ *and*
   (a) $\rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$, *if* $\star = pw$;
   (b) $\rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, *if* $\star \in \{i, s, pom\}$;
   (c) $\widehat{C} \simeq \widehat{C}'$, *if* $\star = pr$.
3. *As item 2, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* $\star$-bisimulation equivalent, $\star \in \{$interleaving, step, partial word, pomset, process$\}$, $N \underline{\leftrightarrow}_\star N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_\star N',\ \star \in \{i, s, pw, pom, pr\}$.

Definition 39 *Let for nets N and N' $\mathcal{R} \subseteq P_N \times P_{N'}$.*

A lifting of $\mathcal{R}$ is $\overline{\mathcal{R}} \subseteq \mathbb{N}_{fin}^{P_N} \times \mathbb{N}_{fin}^{P_{N'}}$, defined as:

$$(M, M') \in \overline{\mathcal{R}} \Leftrightarrow \begin{cases} \exists\{(p_1, p_1'), \ldots, (p_n, p_n')\} \in \mathbb{N}_{fin}^{\mathcal{R}}) : \\ M = \{p_1, \ldots p_n\},\ M' = \{p_1', \ldots p_n'\} \end{cases}$$

Definition 40 $\mathcal{R} \subseteq P_N \times P_{N'}$ *is a $\star$-place bisimulation between nets N and N', $\star \in${interleaving, step, partial word, pomset, process}, $\mathcal{R} : N \sim_\star N'$, if $\overline{\mathcal{R}} : N \underline{\leftrightarrow}_\star N'$, $\star \in \{i, s, pw, pom, pr\}$.*

N and N' are $\star$-place bisimulation equivalent, $\star \in${interleaving, step, partial word, pomset, process}, $N \sim_\star N'$, if $\exists \mathcal{R} : N \sim_\star N'$, $\star \in \{i, s, pw, pom, pr\}$.

Strict place bisimulations require additionally the corresponding transitions to be related by $\overline{\mathcal{R}}$.

Definition 41 *Let for nets N and N' $t \in T_N$, $t' \in T_{N'}$. Then:*

$$(t, t') \in \overline{\mathcal{R}} \Leftrightarrow \begin{cases} ({}^\bullet t, {}^\bullet t') \in \overline{\mathcal{R}} \wedge \\ (t^\bullet, t'^\bullet) \in \overline{\mathcal{R}} \wedge \\ L_N(t) = L_{N'}(t') \end{cases}$$

Definition 42 $\mathcal{R} \subseteq P_N \times P_{N'}$ *is a* strict $\star$-place bisimulation *between nets* N *and* N', $\star \in${interleaving, step, partial word, pomset, process}, $\mathcal{R} : N \approx_\star N'$, $\star \in \{i, s, pw, pom, pr\}$, *if:*

1. $\overline{\mathcal{R}} : N \underline{\leftrightarrow}_\star N'$.
2. *The new requirement is added to item 2 (and to 3) of the definition of* $\star$*-bisimulation:*
   $\forall v \in T_{\widehat{C}}\ (\hat{\varphi}(v), \hat{\varphi}'(\beta(v))) \in \overline{\mathcal{R}}$, *where:*
   (a) $\beta : \rho_{\widehat{C}'} \sqsubseteq \rho_{\widehat{C}}$, *if* $\star = pw$;
   (b) $\beta : \rho_{\widehat{C}} \simeq \rho_{\widehat{C}'}$, *if* $\star \in \{i, s, pom\}$;
   (c) $\beta : \widehat{C} \simeq \widehat{C}'$, *if* $\star = pr$.

N *and* N' *are* strict $\star$-place bisimulation equivalent, $\star \in${interleaving, step, partial word, pomset, process}, $N \approx_\star N'$, *if* $\exists \mathcal{R} : N \approx_\star N'$, $\star \in \{i, s, pw, pom, pr\}$.

An important property of place bisimulations: *additivity*.

Let for nets N and N' $\mathcal{R} : N \sim_{\star} N'$, $\star \in \{i, s, pw, pom, pr\}$.

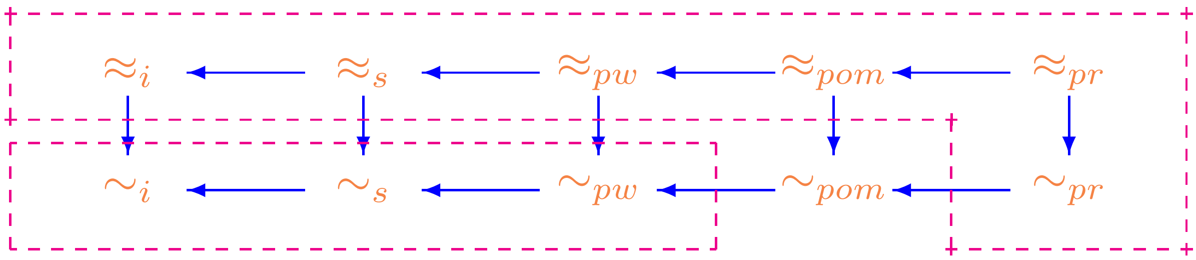
Then $(M_1, M_1') \in \overline{\mathcal{R}}$ and $(M_2, M_2') \in \overline{\mathcal{R}}$ implies $((M_1 + M_2), (M_1' + M_2')) \in \overline{\mathcal{R}}$.

If we add n tokens in each of the places $p \in P_N$ and $p' \in P_{N'}$ s.t. $(p, p') \in \mathcal{R}$, then the resulting nets must also be place bisimulation equivalent.

Comparing place bisimulation equivalences

Proposition 3 [AS92] *For nets* N *and* N'

1. $N \sim_i N' \Leftrightarrow N \sim_{pw} N'$;
2. $N \sim_{pr} N' \Leftrightarrow N \approx_i N' \Leftrightarrow N \approx_{pr} N'$.



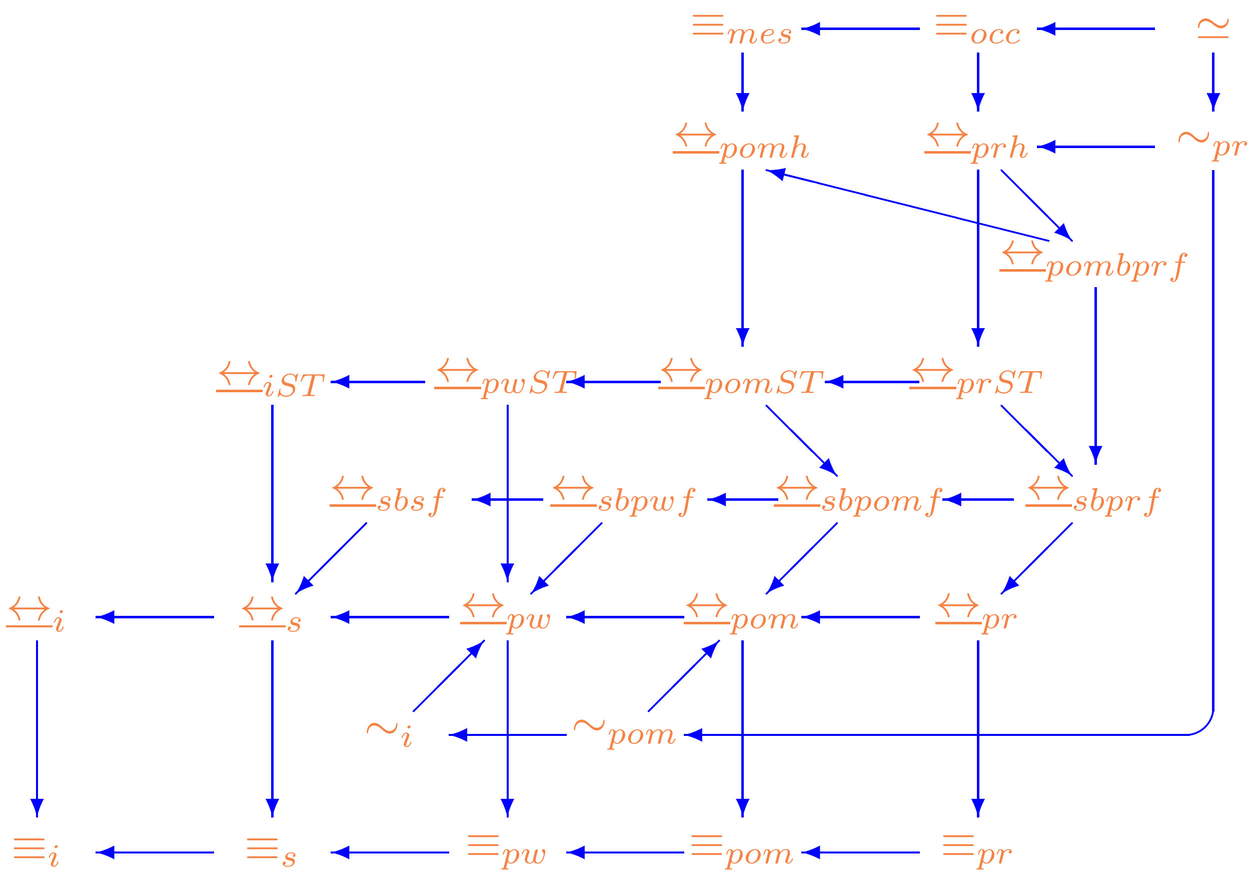
Merging of place bisimulation equivalences

Interrelations of place bisimulation equivalences

Comparing place bisimulation equivalences with basic and back-forth ones

Proposition 4 *[Tar97,Tar98b] For nets N and N'*

$$N \sim_{pr} N' \Rightarrow N \underline{\leftrightarrow}_{prh} N'.$$

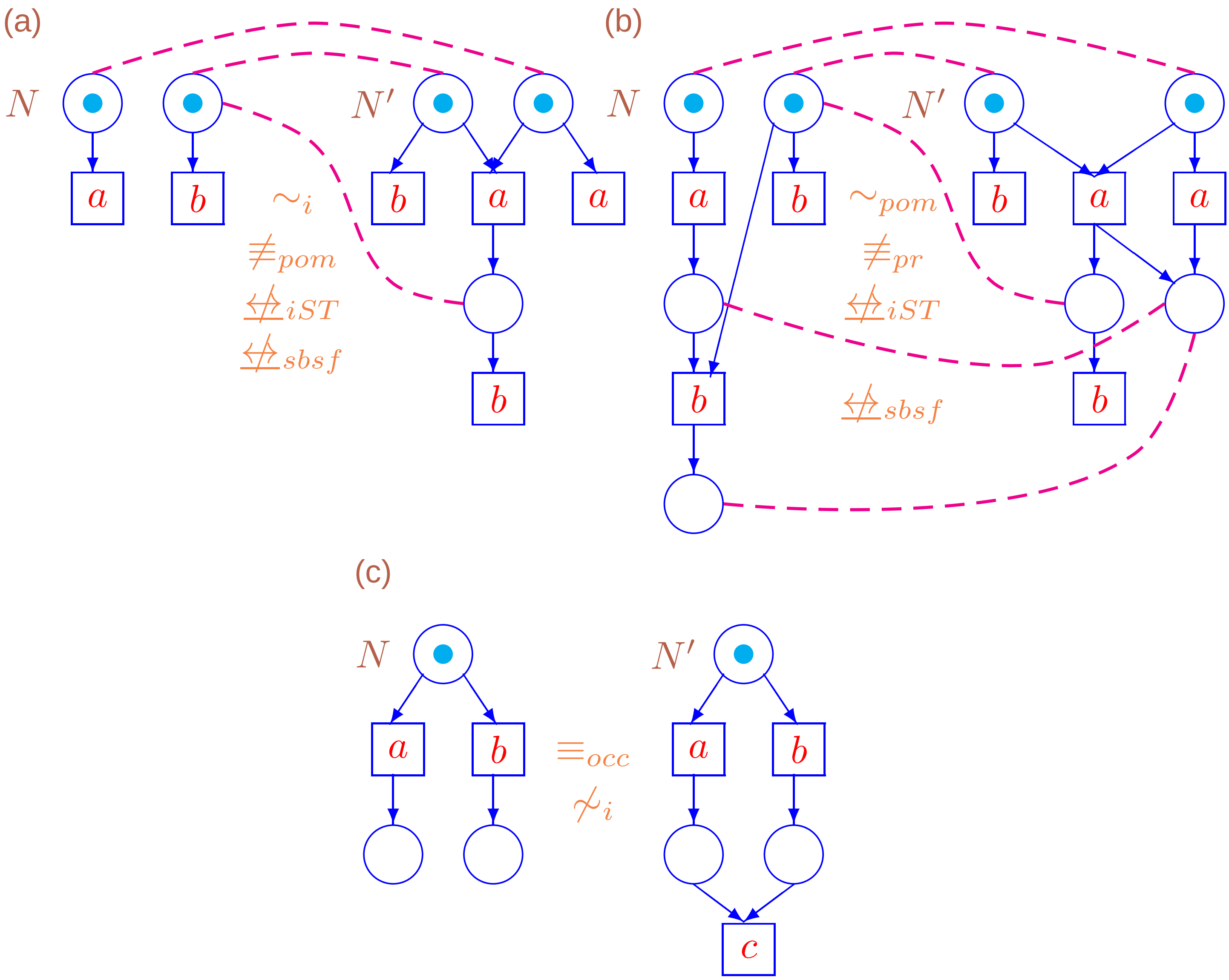


Interrelations of place bisimulation equivalences with basic and back-forth ones

Theorem 6 *Let* $\leftrightarrow, \twoheadleftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \sim, \simeq\}$, $\star, \star\star \in \{\_, i, s, pw, pom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$. *For nets N and N'*

$$N \leftrightarrow_{\star} N' \Rightarrow N \twoheadleftrightarrow_{\star\star} N'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ to $\twoheadleftrightarrow_{\star\star}$.



P: Examples of place bisimulation equivalences

- In Figure P(a), $N \sim_i N'$, but $N \not\equiv_{pom} N'$, since only in the net N' action b can depend on a.
- In Figure P(b), $N \sim_{pom} N'$, but $N \not\equiv_{pr} N'$, since only in the net N' the transition with label a has two input (and two output) places.
- In Figure P(c), $N \equiv_{occ} N'$, but $N \not\sim_i N'$, since any place bisimulation must relate input places of the nets N and N'. But if we add one additional token in each of these places, then only in N' the action c can occur.
- In Figure P(b), $N \sim_{pom} N'$, but $N \not\underline{\leftrightarrow}_{iST} N'$, since only in the net N' action a can start so that no b can begin working until ending a.
- In Figure B1(c), $N \sim_{pr} N'$, but $N \not\equiv_{mes} N'$, since only the MES corresponding to the net N' has two conflict actions a.
- In Figure P(b), $N \sim_{pom} N'$, but $N \not\underline{\leftrightarrow}_{sbsf} N'$, since only in the net N' action a can occur so that b must depend on a.

Net reduction based on place bisimulation equivalences

An *autobisimulation* is a bisimulation between a net and itself.

An *equibisimulation* is an autobisimulation that is an equivalence.

Proposition **5** *[AS92] Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be reflexive interleaving place autobisimulations of a net N. Then $(\mathcal{R}_1 \cup \mathcal{R}_2)^*$ (transitive closure of $(\mathcal{R}_1 \cup \mathcal{R}_2)$) is an interleaving place autobisimulation.*

Definition **43** *For a net N, $\mathcal{R}_i(N) = \bigcup\{\mathcal{R} \mid \mathcal{R} : N \sim_i N,\ \mathcal{R}$ is reflexive$\}$ is a canonical interleaving place bisimulation.*

Definition **44** *Let for a net N $\mathcal{E} \subseteq P_N \times P_N$ be an equivalence.*

For $p \in P_N,\ [p]_{\mathcal{E}} = \{q \mid (p, q) \in \mathcal{E}\}$ is an equivalence class of p w.r.t. $\mathcal{E}$.

For $M \in I\!N_{fin}^{P_N},\ M/_{\mathcal{E}} = \sum_{p \in P_N} [p]_{\mathcal{E}}$ is a categorization (partitioning) of M w.r.t. $\mathcal{E}$.

$N/_{\mathcal{E}} = (P_N/_{\mathcal{E}}, T_N, W_N/_{\mathcal{E}}, L_N, M_N/_{\mathcal{E}})$, where $W_N/_{\mathcal{E}}$ is constructed as:

1. *${}^\bullet t = M$ in $N \ \Rightarrow\ {}^\bullet t = M/_{\mathcal{E}}$ in $N/_{\mathcal{E}}$;*
2. *$t^\bullet = M$ in $N \ \Rightarrow\ t^\bullet = M/_{\mathcal{E}}$ in $N/_{\mathcal{E}}$.*

$M \xrightarrow{t} \widetilde{M}$ in N implies $M/_{\mathcal{E}} \xrightarrow{t} \widetilde{M}/_{\mathcal{E}}$ in $N/_{\mathcal{E}}$.

Proposition **6** *[AS92] If $\mathcal{R} : N \sim_i N$ is an equivalence then*

$$[\cdot]_{\mathcal{R}} : N \sim_i N/_{\mathcal{E}}.$$

Definition **45** *A canonical interleaving categorization of a net N is a net*

$$N/_{\sim_i} = N/_{\mathcal{R}_i(N)}.$$

Definition 46 *For a net N, $\mathcal{R} \subseteq P_N \times P_N$ has a transfer property, if*
$\forall t \in T_N\ \forall p \in {}^\bullet t\ \forall q :\ (p, q) \in \mathcal{R}$ *holds:*
$\exists u \in T_N :\ L_N(t) = L_N(u),\ {}^\bullet t - p + q \xrightarrow{u} \widetilde{M}$ *and* $(t^\bullet, \widetilde{M}) \in \mathcal{R}$.

Theorem 7 *[AS92] If for a net N, $\mathcal{R} \subseteq P_N \times P_N$ is a reflexive and symmetrical relation having transfer property then $\mathcal{R}^*$ (transitive closure of $\mathcal{R}$) is an interleaving place bisimulation in N.*

Theorem 8 *[AS92] For a net N, the maximal relation $\mathcal{R} \subseteq P_N \times P_N$ having transfer property is $\mathcal{R}_i(N)$.*

An effective algorithm of computing $\mathcal{R}_i(N)$ [AS92]:

1. The initial relation: $\mathcal{R} = P_N \times P_N$.
2. Check all pairs $(p, q) \in \mathcal{R}$ for transfer property.
   (a) If the property is valid for all that pairs then $\mathcal{R} = \mathcal{R}_i(N)$.
   (b) Otherwise, there exists a pair (p, q), for which transfer property is not valid. Then we remove the pairs (p, q) and (q, p) from $\mathcal{R}$ and go to item 2.

If a net is finite then a number of the pairs is finite too.

A complexity: $\mathcal{O}(|P_N|^2 \cdot |T_N|^2)$, if $\forall t \in T_N\ |{}^\bullet t| + |t^\bullet| \leq d$ (the constant depends on d) [Pfi92].

An implementation: a system CAESAR on LOTOS programming language [Pfi92].

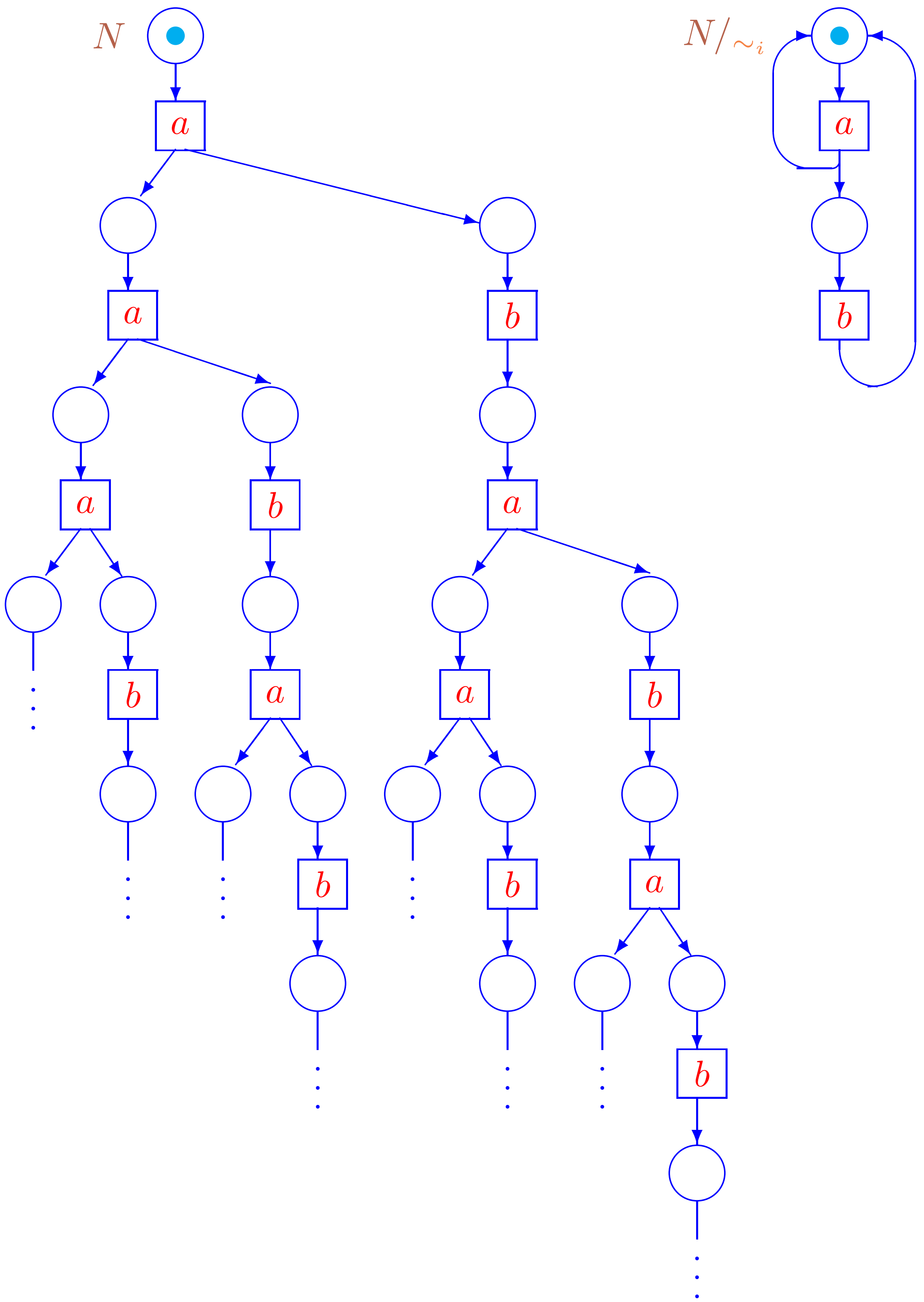
The results on using $\sim_{pom}$ and $\sim_{pr}$ for net reduction

- We cannot use $\sim_{pom}$ for net simplification, since there is an example s.t. for a net $N: N \not\sim_{pom} N/_{\sim_{pom}}$ [AS92].
- Since $\sim_{pr} = \approx_i$, we can modify the algorithm for $\mathcal{R}_i$ to obtain $\mathcal{R}_{pr}$: we shall look for bisimulation between transitions in the pairs appearing during check of the transfer property.

  A complexity of the algorithm will be the same. Thus, it is possible to reduce net effectively modulo $\sim_{pr}$.

Important results (due to interrelations of $\sim_{pr}$ with the other equivalences).

1. Since $\sim_{pr}$ implies $\underline{\leftrightarrow}_{prh}$ and $\underline{\leftrightarrow}_{prST}$, a reduced net has the same histories of behavior and *timed traces* [GV87] as the initial one.
2. Since $\underline{\leftrightarrow}_{prh}$ coincide with $=_{PrBFL}$, all the properties that can be specified in logic $PrBFL$ are preserved in the reduced net.



Reduction of the net corresponding to a PBC formula $\mu X.(a;(X\|(b;X)))$ modulo $\sim_i$

Refinements

SM-refinements [BDKP91]

Definition 47 *An* SM-net *is a net* $D = (P_D, T_D, W_D, L_D, M_D)$ *s.t.:*

1. $\forall t \in T_D \ |{}^\bullet t| = |t^\bullet| = 1$, *each transition has exactly one input and one output place;*

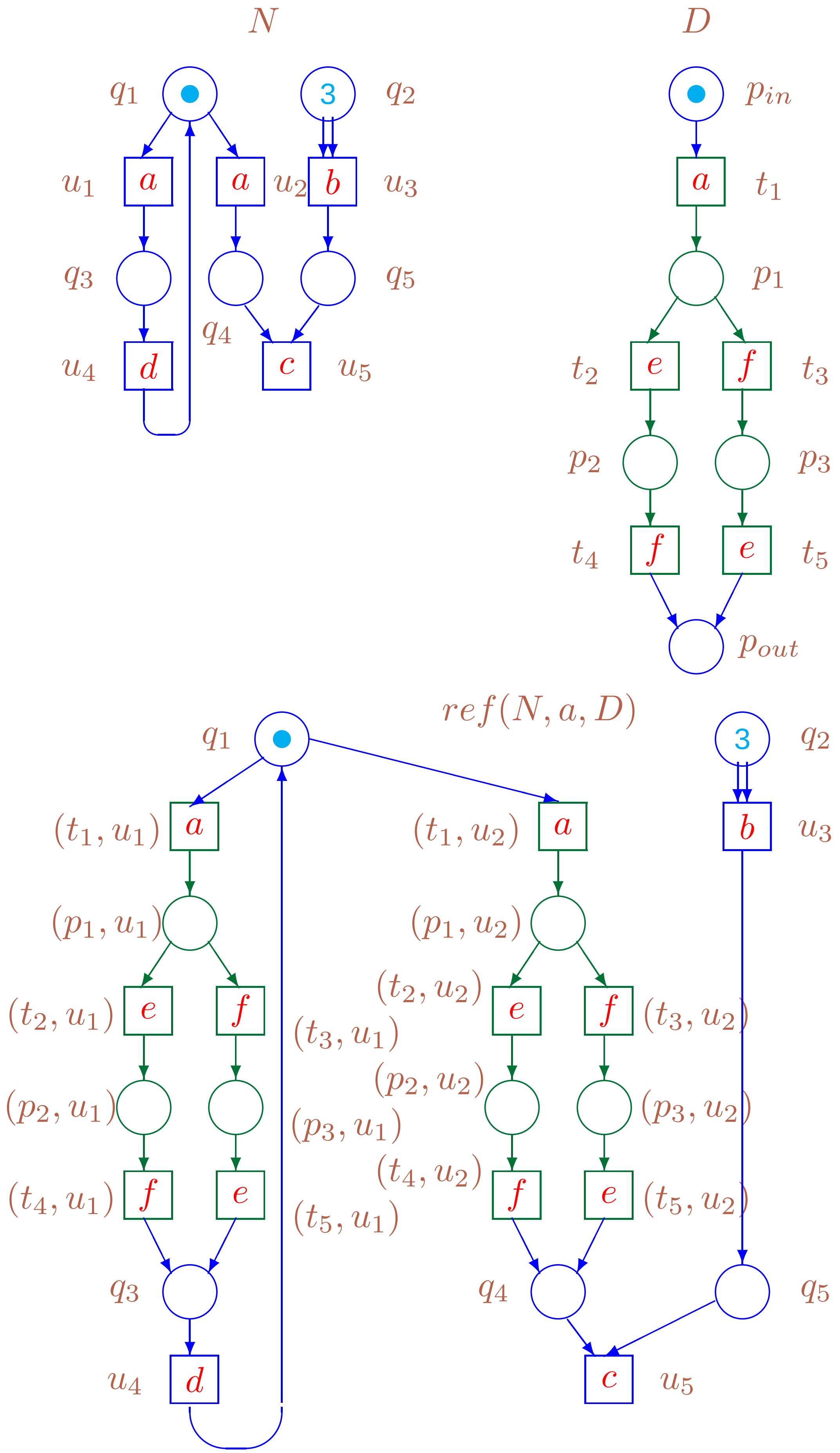
2. $\exists p_{in}, \ p_{out} \in P_D$ *s.t.* $p_{in} \neq p_{out}$ *and* ${}^\bullet D = \{p_{in}\}, \ D^\bullet = \{p_{out}\}$, *net* D *has a unique input and a unique output place.*

3. $M_D = \{p_{in}\}$, *at the beginning there is a unique token in* p_{in}.

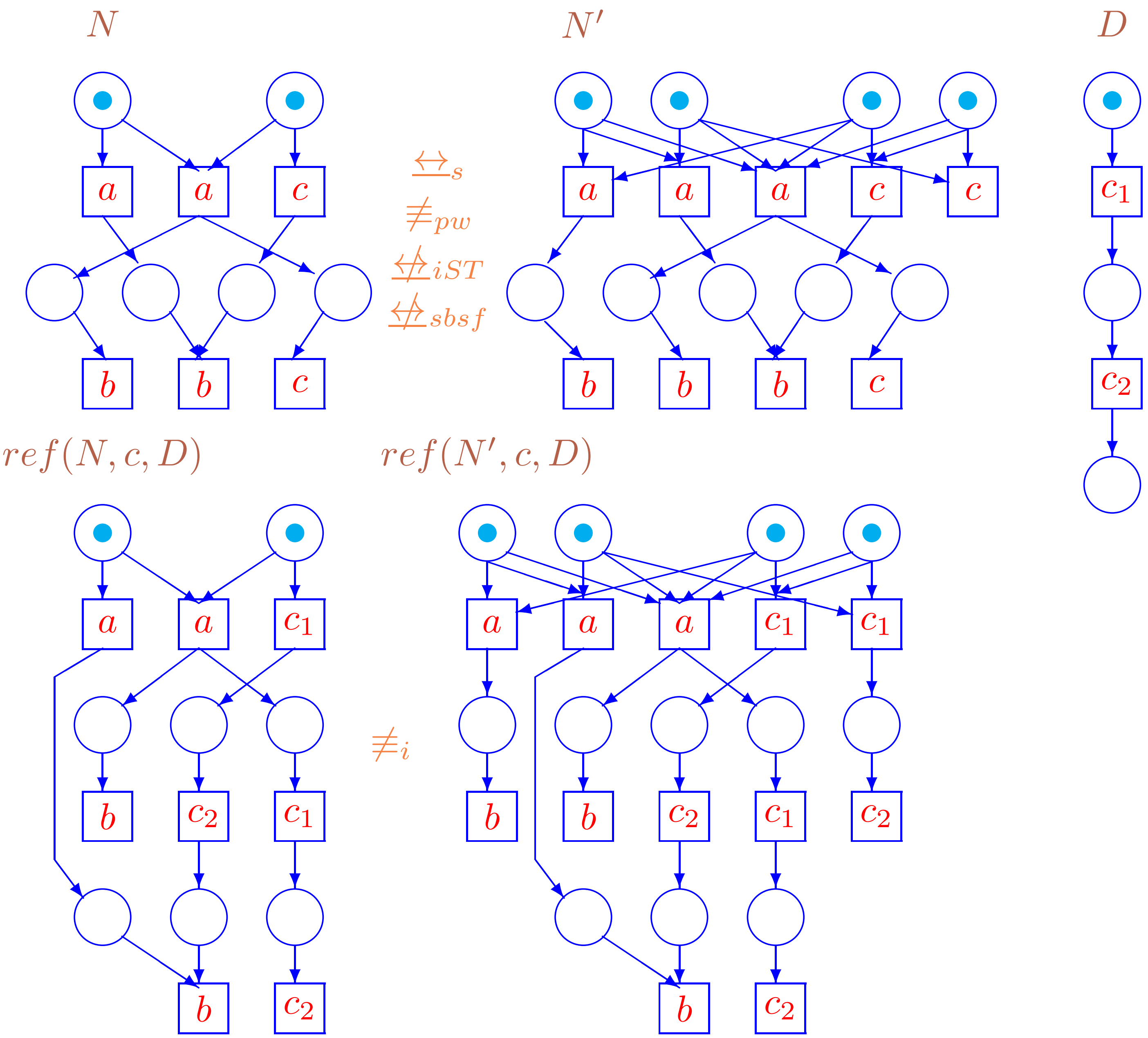
Definition 48 *Let* $N = (P_N, T_N, W_N, L_N, M_N)$ *be a net,* $a \in L_N(T_N)$ *and* $D = (P_D, T_D, W_D, L_D, M_D)$ *be SM-net. An* SM-refinement, $ref(N, a, D)$, *is a net* $\overline{N} = (P_{\overline{N}}, T_{\overline{N}}, W_{\overline{N}}, L_{\overline{N}}, M_{\overline{N}})$*:*

- $P_{\overline{N}} = P_N \cup \{(p, u) \mid p \in P_D \setminus \{p_{in}, p_{out}\}, \ u \in L_N^{-1}(a)\}$;
- $T_{\overline{N}} = (T_N \setminus L_N^{-1}(a)) \cup \{(t, u) \mid t \in T_D, \ u \in L_N^{-1}(a)\}$;
- $W_{\overline{N}}(\bar{x}, \bar{y}) =$
$$\begin{cases} W_N(\bar{x}, \bar{y}), & \bar{x}, \bar{y} \in P_N \cup (T_N \setminus L_N^{-1}(a)); \\ W_D(x, y), & \bar{x} = (x, u), \ \bar{y} = (y, u), \ u \in L_N^{-1}(a); \\ W_N(\bar{x}, u), & \bar{y} = (y, u), \ \bar{x} \in {}^\bullet u, \ u \in L_N^{-1}(a), \ y \in p_{in}^\bullet; \\ W_N(u, \bar{y}), & \bar{x} = (x, u), \ \bar{y} \in {}^\bullet u, \ u \in L_N^{-1}(a), \ x \in {}^\bullet p_{out}; \\ 0, & \text{otherwise}; \end{cases}$$
- $L_{\overline{N}}(\bar{u}) = \begin{cases} L_N(\bar{u}), & \bar{u} \in T_N \setminus L_N^{-1}(a); \\ L_D(t), & \bar{u} = (t, u), \ t \in T_D, \ u \in L_N^{-1}(a); \end{cases}$
- $M_{\overline{N}}(p) = \begin{cases} M_N(p), & p \in P_N; \\ 0, & \text{otherwise}. \end{cases}$

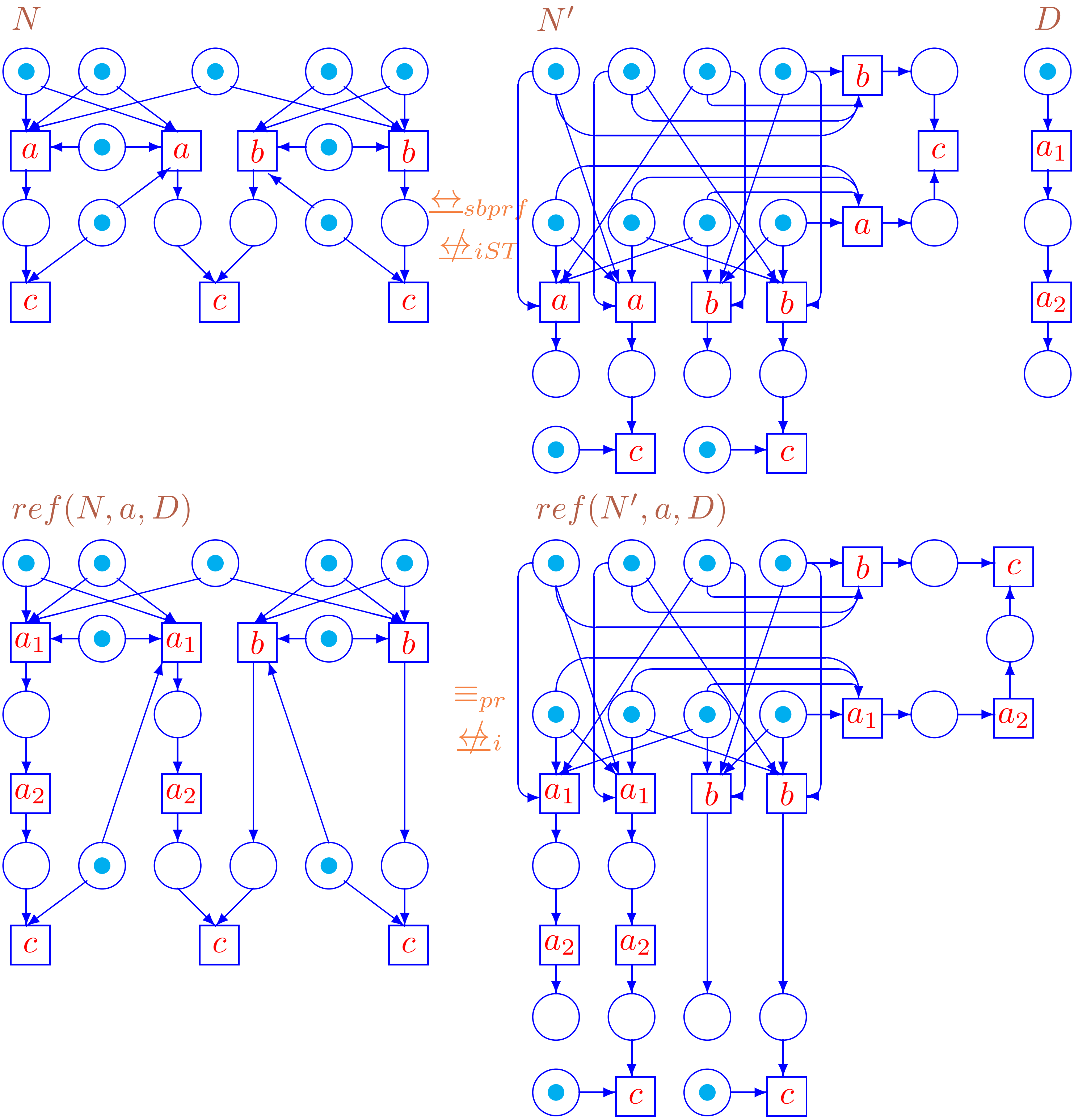
An equivalence is *preserved by refinements*, if equivalent nets remain equivalent after applying any refinement operator to them accordingly.



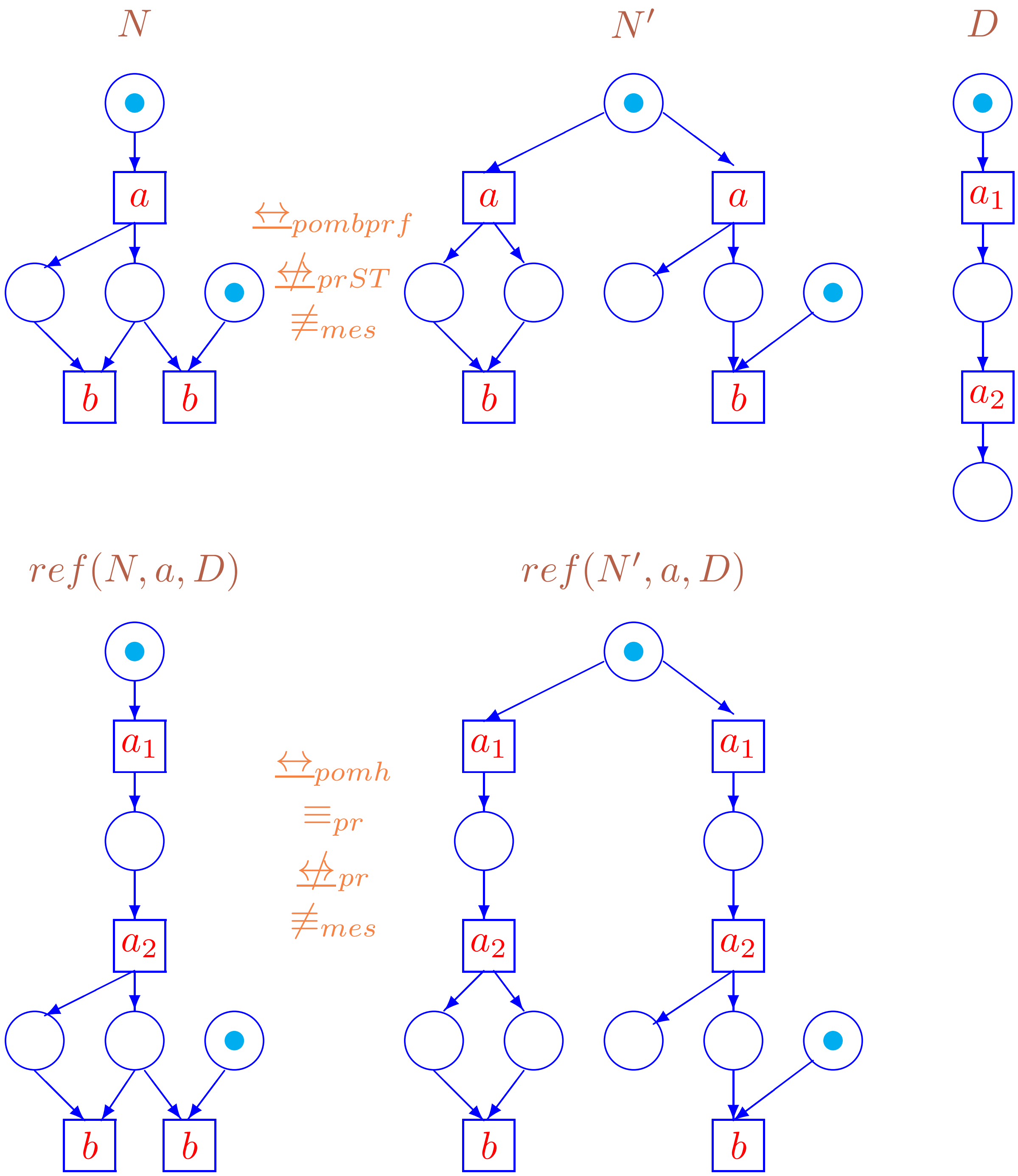
Example of SM-refinement



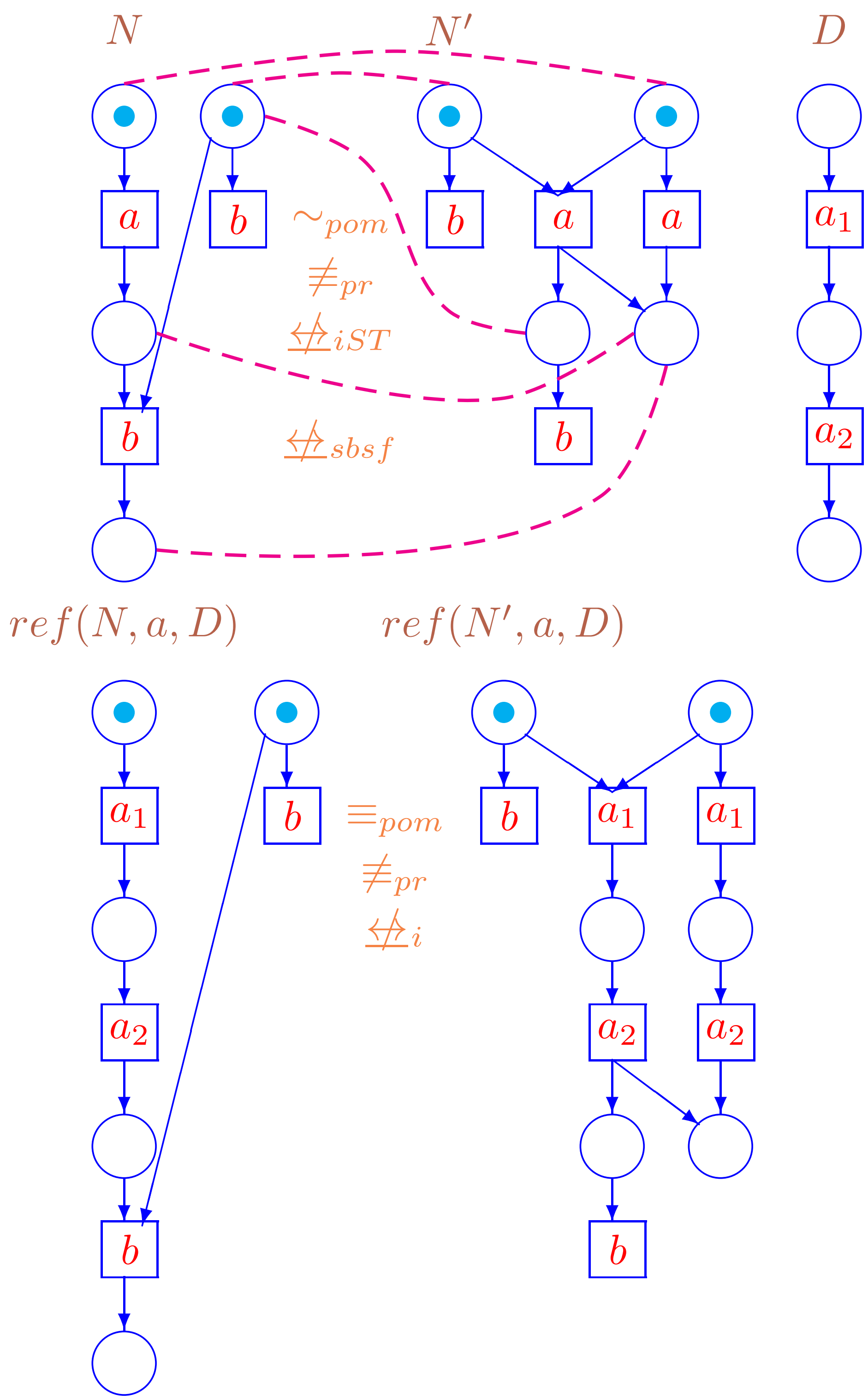
RB: The equivalences between $\equiv_i$ and $\underline{\leftrightarrow}_s$ are not preserved by SM-refinements



RBF: The equivalences between $\underline{\leftrightarrow}_i$ and $\underline{\leftrightarrow}_{sbprf}$ are not preserved by SM-refinements



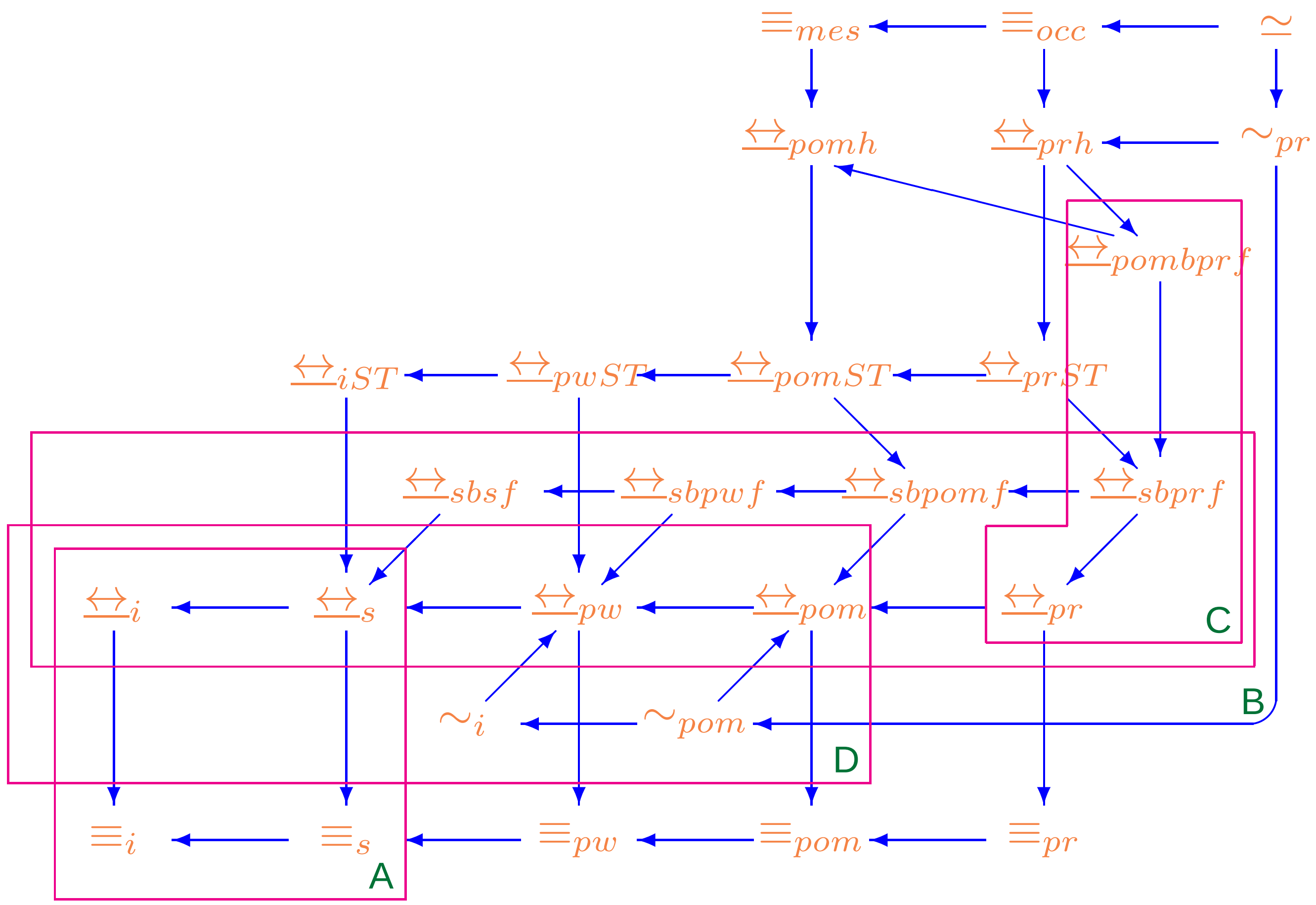
RBF1: The equivalences between $\underline{\leftrightarrow}_{pr}$ and $\underline{\leftrightarrow}_{pombprf}$ are not preserved by SM-refinements



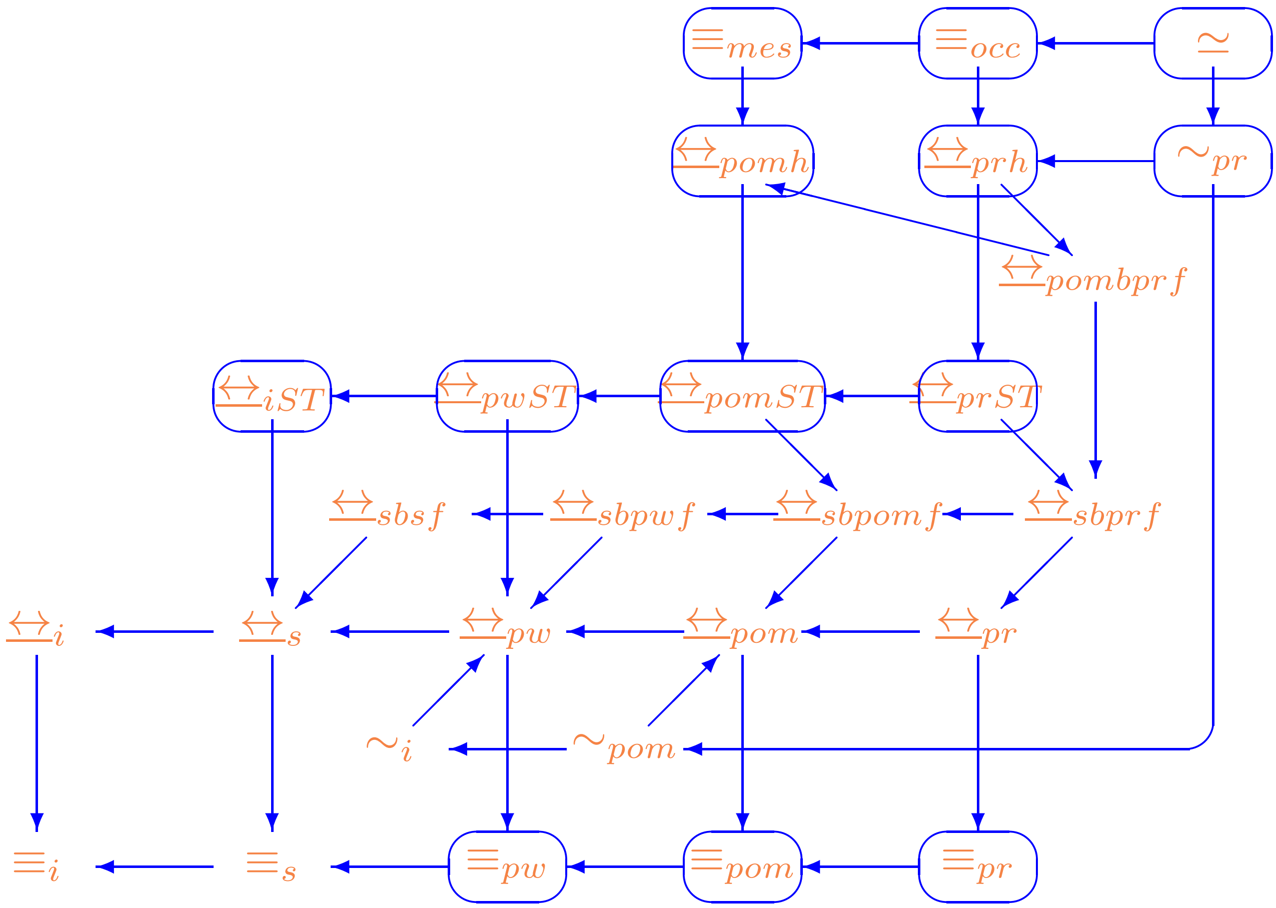
RP: The equivalences between $\underline{\leftrightarrow}_i$ and $\sim_{pom}$ are not preserved by SM-refinements

- In Figure RB, $N\underline{\leftrightarrow}_s N'$, but $ref(N, c, D) \not\equiv_i ref(N', c, D)$, since only in $ref(N', c, D)$ the sequence of actions c_1abc_2 can occur.
- In Figure RBF, $N\underline{\leftrightarrow}_{sbprf} N'$, but $ref(N, a, D) \underline{\not\leftrightarrow}_i ref(N', a, D)$, since only in the net $ref(N', a, D)$ action a_1 can occur so that immediately after it:
  1. the sequence of actions bc cannot occur, and
  2. the sequence of actions a_2c cannot occur.
- In Figure RBF1, $N\underline{\leftrightarrow}_{pombprf} N'$, but $ref(N, a, D) \underline{\not\leftrightarrow}_{pr} ref(N', a, D)$, since only in the net $ref(N', a, D)$ action a_1 can occur so that after it the sequence of actions a_2b can occur which has only one corresponding process (the transition labeled by b connects with transition with label a_2 in the only way).
- In Figure RP, $N \sim_{pom} N'$, but $ref(N, a, D) \underline{\not\leftrightarrow}_i ref(N', a, D)$, since only in the net $ref(N', a, D)$ after action a_1 action b cannot occur.

Proposition 7 *[BDKP91,Tar97] Let* $\star \in \{i, s\}$, $\star\star \in \{i, s, pw, pom, pr, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$, $\star\star\star \in \{i, pom\}$*. Then the equivalences* $\equiv_\star$, $\underline{\leftrightarrow}_{\star\star}$, $\sim_{\star\star\star}$ *are not preserved by SM-refinements.*



The equivalences which are not preserved by SM-refinements



Preservation of the equivalences by SM-refinements

Theorem 9 *Let* $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \sim, \simeq\}$ *and* $\star \in \{\_, i, s, pw, pom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ, sbsf, sbpwf, sbpomf, sbprf, pombprf\}$. *For nets* $N,\ N'$ *s.t.* $a \in L_N(T_N) \cap L_{N'}(T_{N'})$ *and SM-net* D

$$N \leftrightarrow_\star N' \ \Rightarrow\ ref(N, a, D) \leftrightarrow_\star ref(N', a, D)$$

iff the equivalence $\leftrightarrow_\star$ *is in oval in the figure above.*

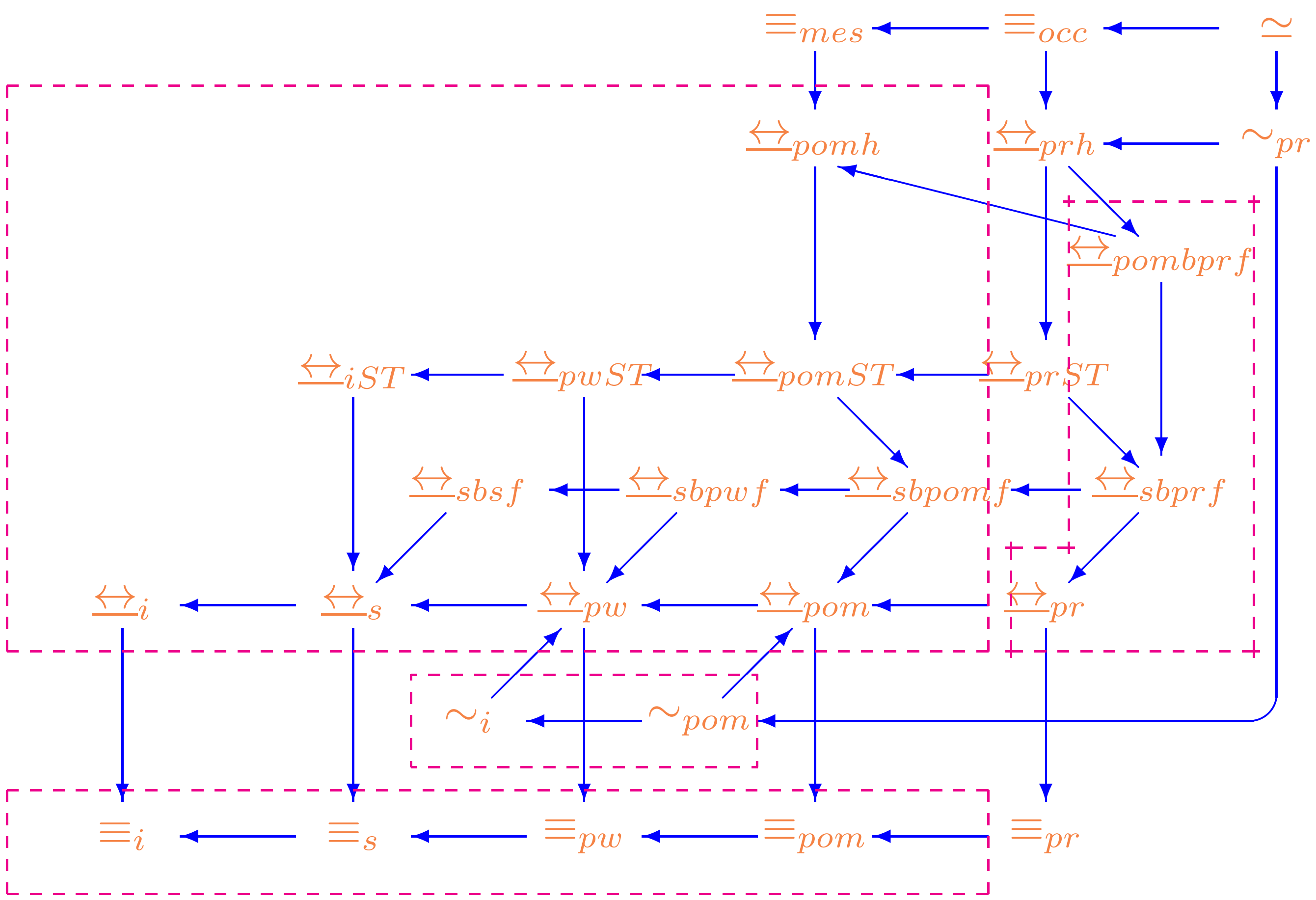
Net subclasses

The equivalences on sequential nets

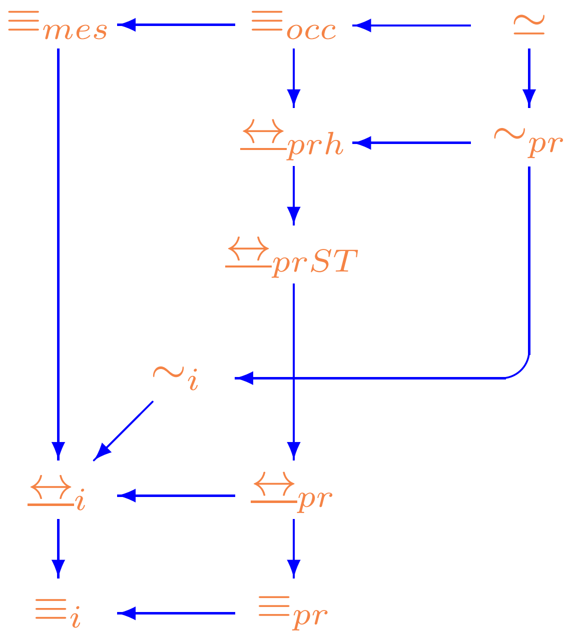
Definition **49** *A net* $N = (P_N, T_N, W_N, L_N, M_N)$ *is sequential, if* $\forall M \in RS(N)\ \neg\exists t, u \in T_N : {}^\bullet t + {}^\bullet u \subseteq M$.

Proposition **8** *For sequential nets* N *and* N'

1. $N \equiv_i N' \Leftrightarrow N \equiv_{pom} N'$ [Eng85];
2. $N \underline{\leftrightarrow}_i N' \Leftrightarrow N \underline{\leftrightarrow}_{pomh} N'$ [BDKP91];
3. $N \underline{\leftrightarrow}_{pr} N' \Leftrightarrow N \underline{\leftrightarrow}_{pombprf} N'$ [Tar97];
4. $N \sim_i N' \Leftrightarrow N \sim_{pom} N'$ [Tar97].



Merging of the equivalences on sequential nets

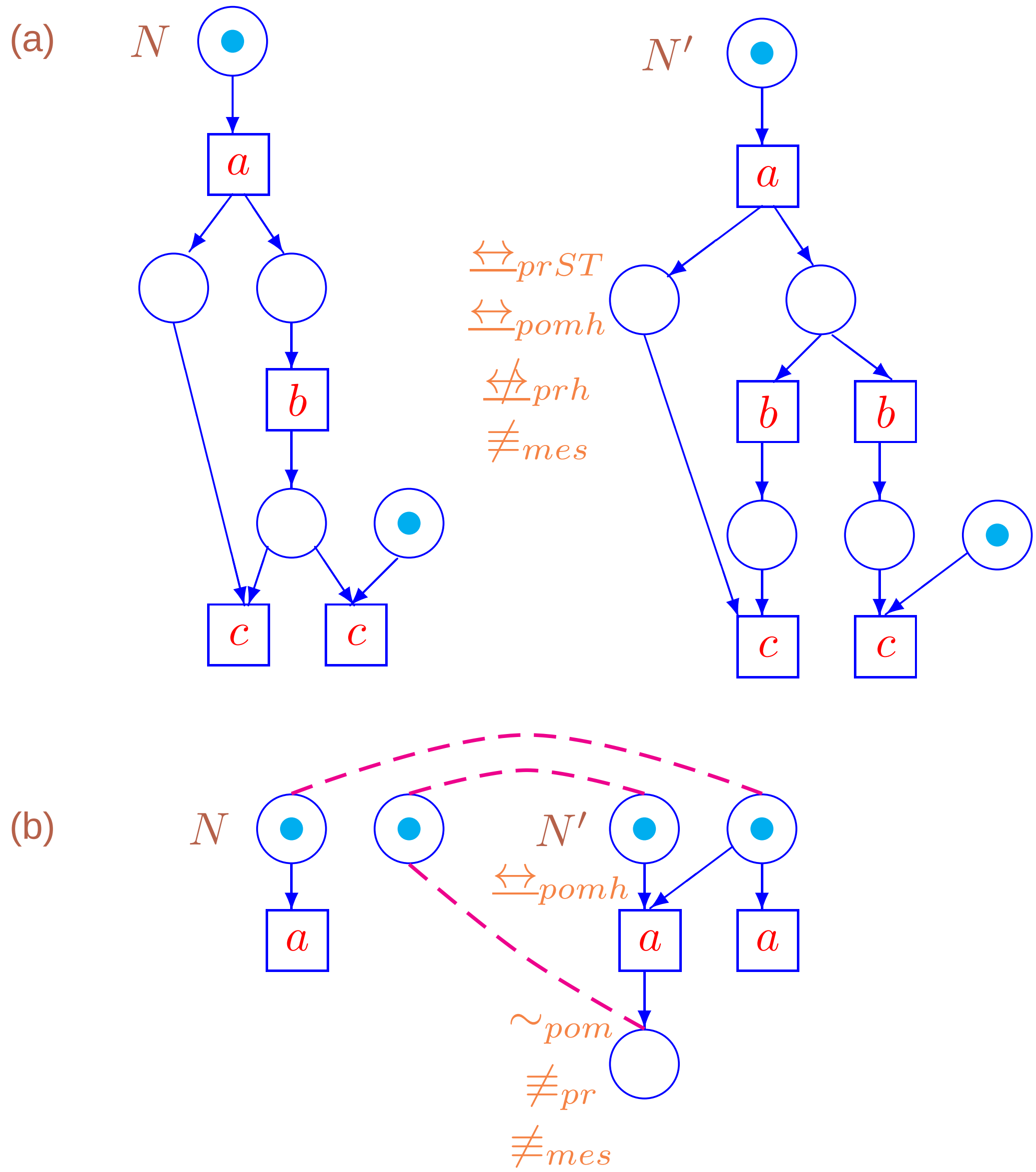


Interrelations of the equivalences on sequential nets

Theorem 10 *Let* $\leftrightarrow, \twoheadleftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \sim, \simeq\}$, $\star, \star\star \in \{\_, i, pr, prST, prh, mes, occ\}$. *For sequential nets* N *and* N'

$$N \leftrightarrow_\star N' \;\Rightarrow\; N \twoheadleftrightarrow_{\star\star} N'$$

iff in the graph above there exists a directed path from $\leftrightarrow_\star$ *to* $\twoheadleftrightarrow_{\star\star}$.



SN: Examples of the equivalences on sequential nets

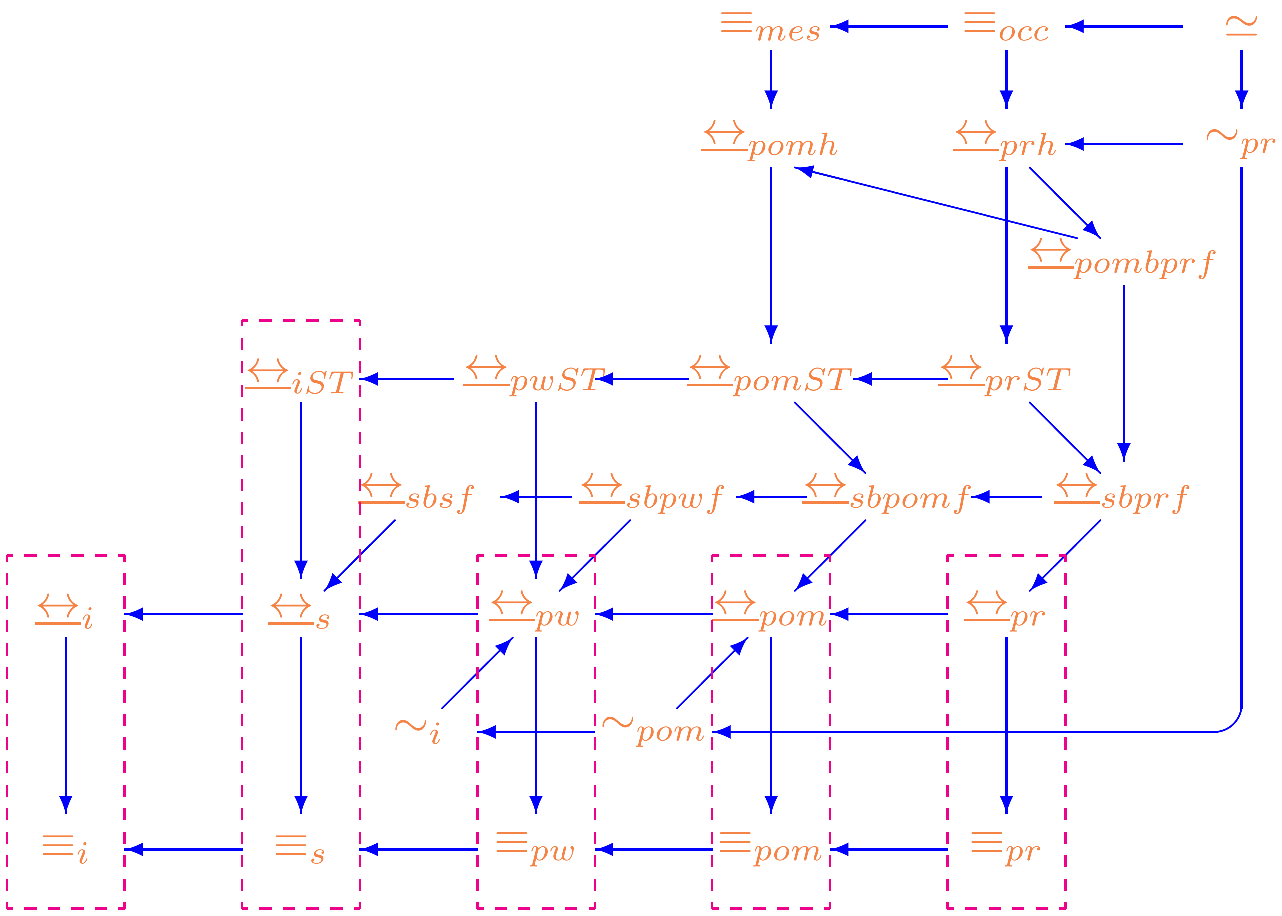
- In Figure B(d), $N \equiv_{mes} N'$, but $N \not\equiv_{pr} N'$.
- In Figure RB(e), $N \equiv_{pr} N'$, but $N \underline{\not\leftrightarrow}_i N'$.
- In Figure BF(c), $N \underline{\leftrightarrow}_{pr} N'$, but $N \underline{\not\leftrightarrow}_{prST} N'$.
- In Figure SN(a), $N \underline{\leftrightarrow}_{prST} N'$, but $N \underline{\not\leftrightarrow}_{prh} N'$, since only in the net N' there is process with actions a and b s.t. it can be extended by process with action c in the only way (so that connection of causal net with action c and a-containing subnet of causal net with actions a and b be unique).
- In Figure B1(c), $N \underline{\leftrightarrow}_{prh} N'$, but $N \not\equiv_{mes} N'$.
- In Figure B1(d), $N \equiv_{occ} N'$, but $N \not\simeq N'$.
- In Figure SN(b), $N \sim_i N'$, but $N \not\equiv_{pr} N'$, since only in the net N' the transition with label a has two input places.
- In Figure P(c), $N \equiv_{occ} N'$, but $N \not\sim_i N'$.
- In Figure B1(c), $N \sim_{pr} N'$, but $N \not\equiv_{mes} N'$.

The equivalences on strictly labeled nets

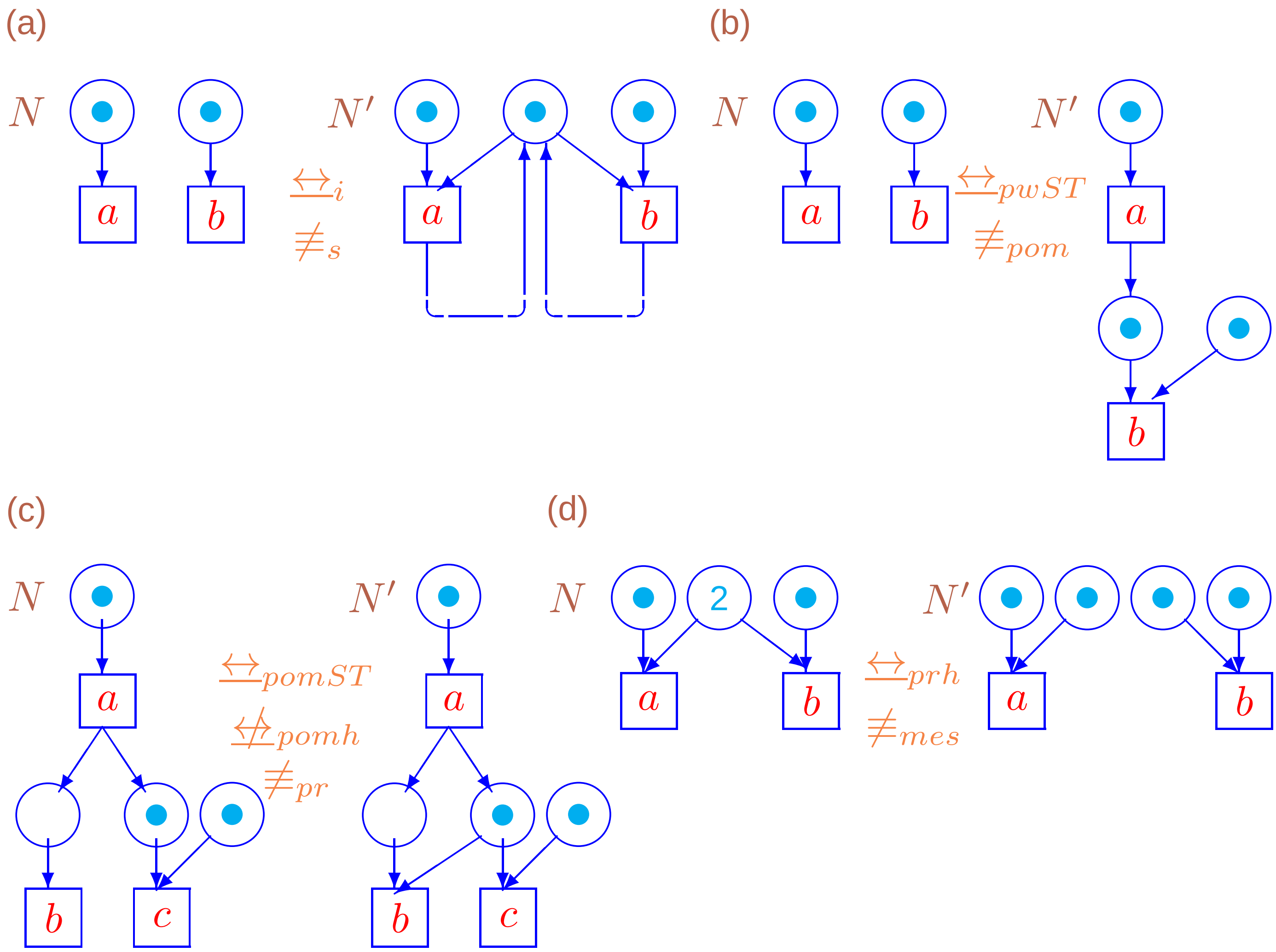
Definition 50 *A net* $N = (P_N, T_N, W_N, L_N)$ *is* strictly labeled (unlabeled) *if* $\forall t, u \in T_N \; L_N(t) \neq L_N(u)$.

Proposition 9 *Let* $\star \in \{i, pw, pom, pr\}$. *For strictly labeled nets* N *and* N'

1. $N \equiv_\star N' \;\Leftrightarrow\; N \underline{\leftrightarrow}_\star N'$ *[PRS92,Tar97];*
2. $N \equiv_s N' \;\Leftrightarrow\; N \underline{\leftrightarrow}_{iST} N'$ *[Tar97].*



Merging of the equivalences on strictly labeled nets



UL: Examples of the equivalences on strictly labeled nets

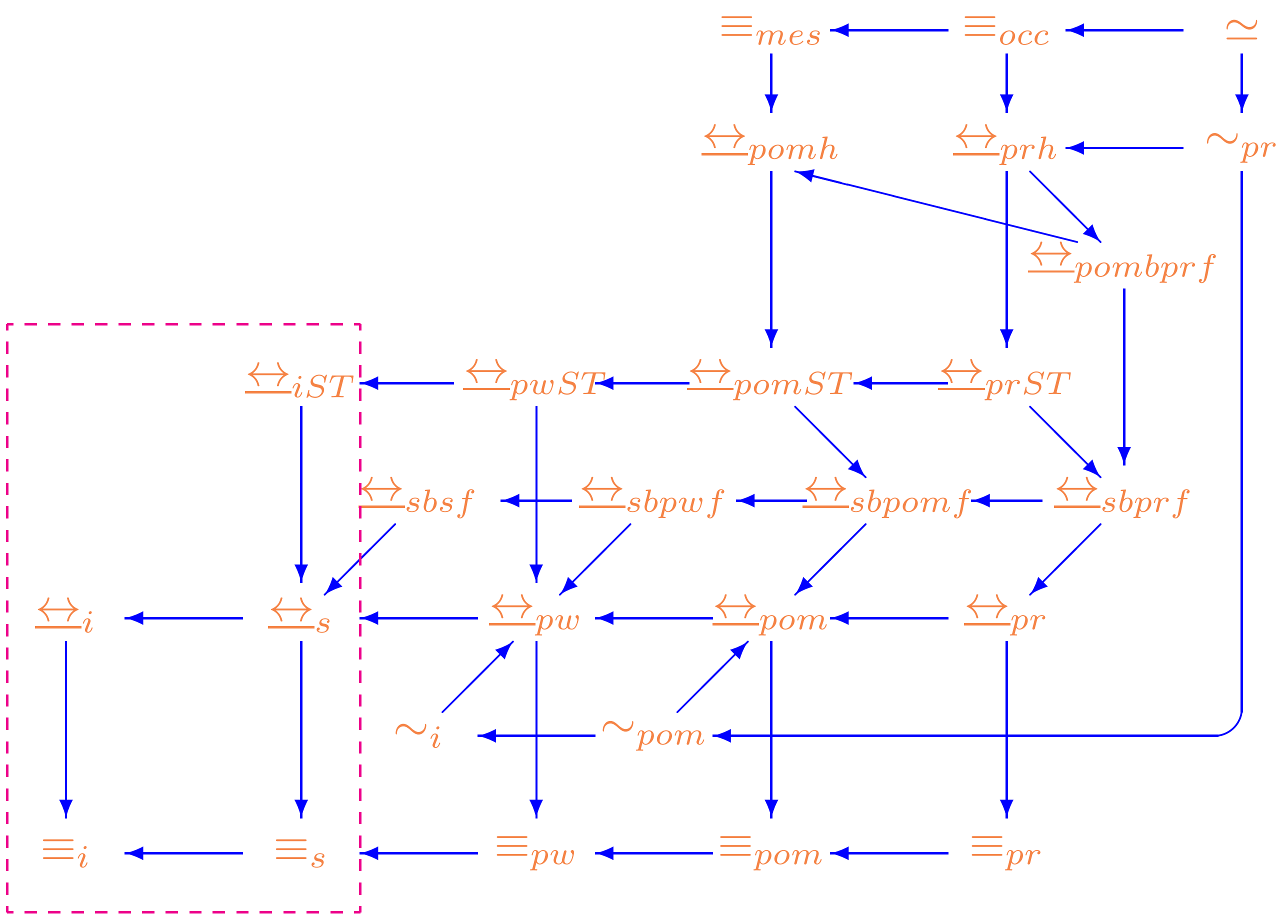
- In Figure UN(a), $N \underline{\leftrightarrow}_i N'$, but $N \not\equiv_s N'$, since only in the net N actions a and b can occur concurrently.
- In Figure UN(b), $N \underline{\leftrightarrow}_{pwh} N'$, but $N \not\equiv_{pom} N'$, since only in the net N' action b can depend on a.
- In Figure B(d), $N \equiv_{mes} N'$, but $N \not\equiv_{pr} N'$.
- In Figure UN(c), $N \underline{\leftrightarrow}_{pomST} N'$, but $N \not\underline{\leftrightarrow}_{pomh} N'$, since only in the net N' a sequence of actions ab can occur so that c must depend on a.
- In Figure UN(d), $N \underline{\leftrightarrow}_{prh} N'$, but $N \not\equiv_{mes} N'$, since only in the unfolding of the net N' transitions with labels a and b have common input place. A MES with conflict actions a and b corresponds to this unfolding.
- In Figure B1(d), $N \equiv_{occ} N'$, but $N \not\simeq N'$.
- In Figure P(c), $N \equiv_{occ} N'$, but $N \not\simeq_i N'$.

The equivalences on T-nets

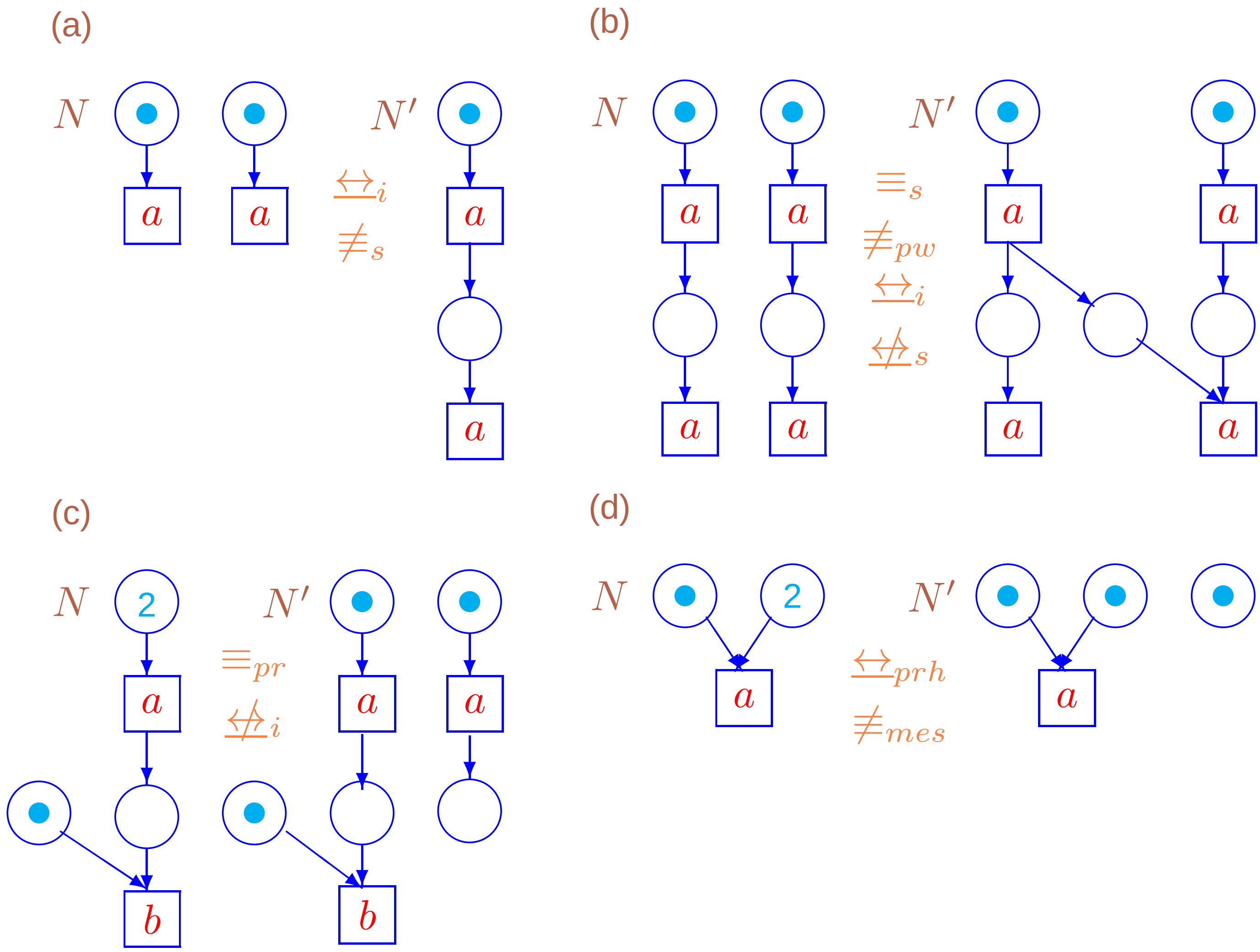
Definition 51 *A net* $N = (P_N, T_N, W_N, L_N)$ *is a* T-net*, if* $\forall p \in P_N \ |{}^\bullet p| \leq 1$ *and* $|p^\bullet| \leq 1$.

Proposition 10 *[Tar97] For auto-concurrency free T-nets* N *and* N'

$$N \equiv_i N' \Leftrightarrow N \underline{\leftrightarrow}_{iST} N'.$$

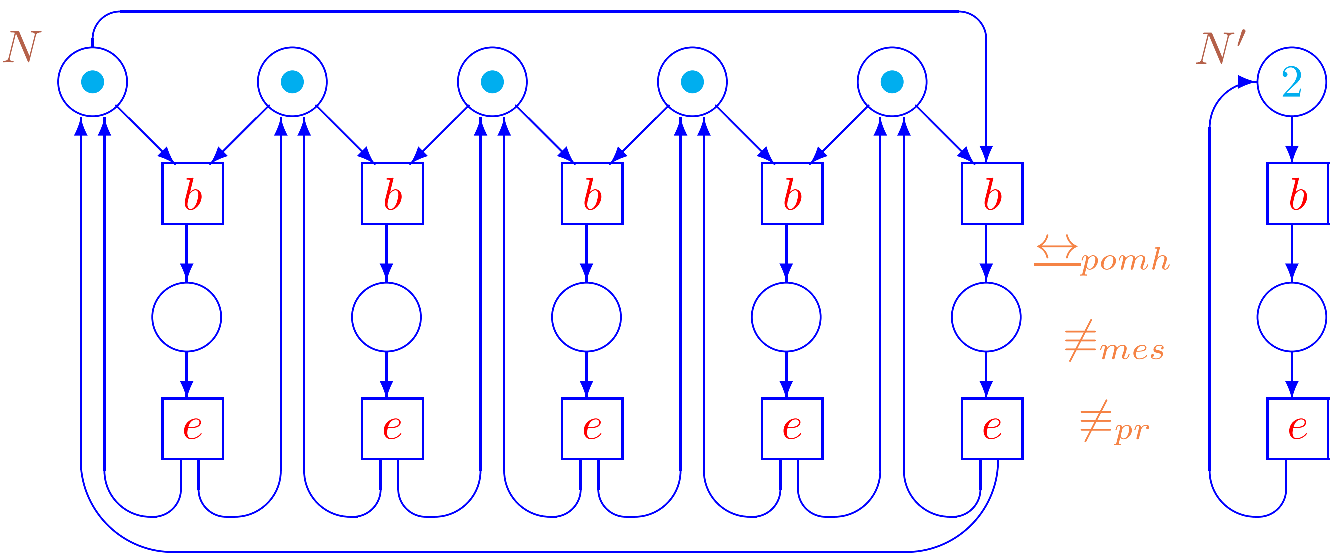


Merging of the equivalences on auto-concurrency free T-nets

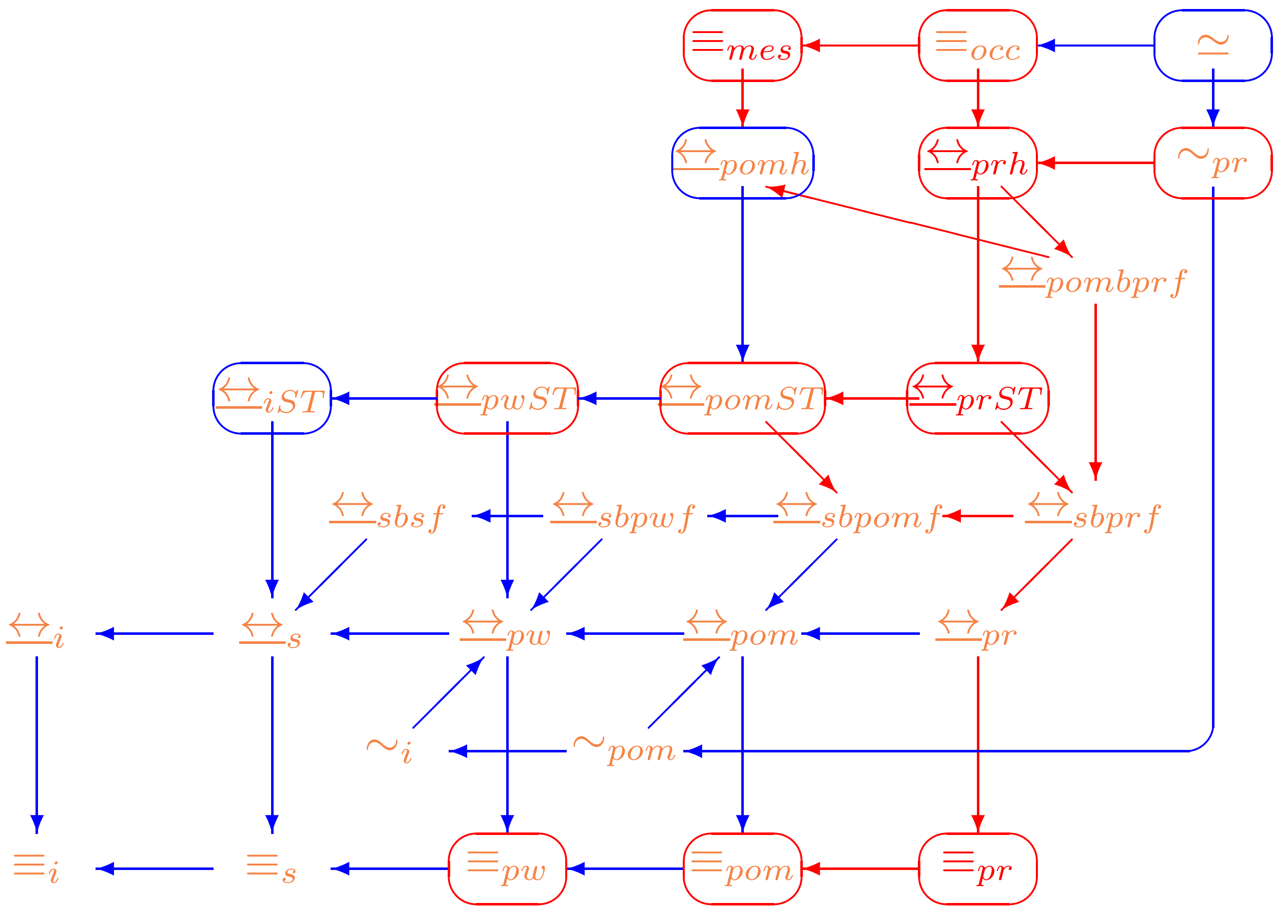


TN: Examples of the equivalences on T-nets

- In Figure TN(a), $N \underline{\leftrightarrow}_i N'$, but $N \not\equiv_s N'$, since only in the net N' an action a cannot occur concurrently with itself (it is not auto-concurrent).
- In Figure TN(b), $N \equiv_s N'$, but $N \not\equiv_{pw} N'$, since the net N structurally represents a pomset s.t. even less sequential one cannot occur in N'.
- In Figure UN(b), $N \underline{\leftrightarrow}_{pwST} N'$, but $N \not\equiv_{pom} N'$.
- In Figure B(d), $N \equiv_{mes} N'$, but $N \not\equiv_{pr} N'$.
- In Figure TN(c), $N \equiv_{pr} N'$, but $N \underline{\not\leftrightarrow}_i N'$, since only in the net N' an action a can occur so that no b is possible afterwards.
- In Figure TN(d), $N \underline{\leftrightarrow}_{prh} N'$, but $N \not\equiv_{mes} N'$, since only in the behaviour of N' there is a MES with two conflict actions a.
- In Figure B1(d), $N \equiv_{occ} N'$, but $N \not\simeq N'$.



The complete and reduced PNs of the abstract dining philosophers system



New results for the equivalences

Decidability

Decidability results for the equivalences

- $\equiv_i$
  - is decidable for:

    unlabeled (strictly labeled) nets [Jan94];

    finite safe nets (EXPSPACE) [JM96].
  - is undecidable for:

    communication free (BPP) nets [CHM93];

    nets with ≥ 2 unbounded places [Jan94].
- $\equiv_s$
  - is decidable for:

    finite safe nets (EXPSPACE) [JM96].
- $\equiv_{pom}$
  - is decidable for:

    unlabeled (strictly labeled) nets [Jan94];

    finite safe nets (EXPSPACE) [JM96];

    communication free (BPP) nets [CHM93].
- $\underline{\leftrightarrow}_i$
  - is decidable for:

    unlabeled (strictly labeled) nets [Jan94];

    finite safe nets (DEXPTIME) [JM96];

    communication free (BPP) nets [CHM93];

    nets s.t. one of them is deterministic up to bisimilarity [Jan94].
  - is undecidable for:

    nets with ≥ 2 unbounded places [Jan94].

- $\underline{\leftrightarrow}_s$
  - is decidable for:

    finite safe nets (DEXPTIME) [JM96].

- $\underline{\leftrightarrow}_{pom}$
  - is decidable for:

    finite safe nets (DEXPTIME / EXPSPACE) [JM96].

- $\underline{\leftrightarrow}_{iST}$
  - is decidable for:

    bounded nets [Dev92];

    finite safe nets (DEXPTIME) [JM96].

- $\underline{\leftrightarrow}_{pomST}$
  - is decidable for:

    finite safe nets (DEXPTIME / EXPSPACE) [JM96].

- $\underline{\leftrightarrow}_{pomh}$
  - is decidable for:

    safe nets (DEXPTIME) [Vog91b].

- $\sim_i$
  - is decidable for:

    arbitrary nets (polynomial, $O(|P_N|^2 \cdot |T_N|^2)$, if $\forall t \in T_N \ |^\bullet t| + |t^\bullet| \leq const$) [AS92].

- $\sim_{pr}$
  - is decidable for:

    arbitrary nets (polynomial, $O(|P_N|^2 \cdot |T_N|^2)$, if $\forall t \in T_N \ |^\bullet t| + |t^\bullet| \leq const$) [AS92].

Equivalences for Petri Nets with Silent Transitions

Abstract: Behavioural equivalences of concurrent systems modeled by Petri nets with silent transitions are considered.

Known basic τ-equivalences and back-forth τ-bisimulation equivalences are supplemented by new ones.

Their interrelations are examined for the general Petri nets as well as for their subclasses of no silent transitions and sequential nets (no concurrent transitions).

A logical characterization of back-forth τ-equivalences in terms of logics with past modalities is proposed.

A preservation of all the equivalences by refinements is investigated to find out their appropriateness for top-down design.

Keywords: Petri nets with silent transitions, sequential nets, basic τ-equivalences, back-forth τ-bisimulation equivalences, logical characterization, refinement.

Contents

- **Back-forth τ-simulation and logics**
  - **Back-forth τ-bisimulation equivalences**
  - **Comparing back-forth τ-bisimulation equivalences**
  - **Comparing back-forth τ-bisimulation equivalences with basic ones**
  - **Logic BFL**
  - **Logic $SPBFL$**
- **Simulation with and without silent actions**
  - **Interrelations of equivalences with τ-equivalences**
- **Refinements**
  - **SM-refinements**
- **Net subclasses**
  - **The τ-equivalences on nets without silent transitions**
  - **The τ-equivalences on sequential nets**
- **Decidability**
  - **Decidability results for the τ-equivalences**
- **Open questions**
  - **Further research**

Introduction

Previous work

Equivalences which abstract of silent actions are τ*-equivalences* (they are labeled by τ). The following basic τ-equivalences are known:

- τ*-trace equivalences* (respect protocols of behavior):

  interleaving ($\equiv_i^\tau$) [Pom86], step ($\equiv_s^\tau$) [Pom86], partial word ($\equiv_{pw}^\tau$) [Vog91a] and pomset ($\equiv_{pom}^\tau$) [PRS92].

- *Usual* τ*-bisimulation equivalences* (respect branching structure of behavior):

  interleaving ($\underline{\leftrightarrow}_i^\tau$) [Mil80], step ($\underline{\leftrightarrow}_s^\tau$) [Pom86], partial word ($\underline{\leftrightarrow}_{pw}^\tau$) [Vog91a] and pomset ($\underline{\leftrightarrow}_{pom}^\tau$) [PRS92].

- *ST-*τ*-bisimulation equivalences* (respect the duration or maximality of events in behavior):

  interleaving ($\underline{\leftrightarrow}_{iST}^\tau$) [Vog91a], partial word ($\underline{\leftrightarrow}_{pwST}^\tau$) [Vog91a] and pomset ($\underline{\leftrightarrow}_{pomST}^\tau$) [Vog91a].

- *History preserving* τ*-bisimulation equivalences* (respect the "history" of behavior):

  pomset ($\underline{\leftrightarrow}_{pomh}^\tau$) [Dev92].

- *History preserving ST-*τ*-bisimulation equivalences* (respect the "history" and the duration or maximality of events in behavior):

  pomset ($\underline{\leftrightarrow}_{pomhST}^\tau$) [Dev92].

- *Usual branching* τ*-bisimulation equivalences* (respect branching structure of behavior with a special care for silent actions):

  interleaving ($\underline{\leftrightarrow}_{ibr}^\tau$) [Gla93].

- *History preserving branching τ-bisimulation equivalences* (respect "history" and branching structure of behavior with a special care for silent actions): pomset ($\underline{\leftrightarrow}^{\tau}_{pomhbr}$) [Dev92].
- *Isomorphism* (coincidence up to renaming of components): ($\simeq$).

Back-forth bisimulation equivalences: bisimulation relation do not only require simulation in the forward direction but also also when going back in history, backward. They connected with equivalences of logics with past modalities.

Interleaving *back* interleaving *forth* τ*-bisimulation equivalence* ($\underline{\leftrightarrow}^{\tau}_{ibif} = \underline{\leftrightarrow}^{\tau}_{ibr}$) [NMV90].

Pomset *back* pomset *forth* τ*-bisimulation equivalence* ($\underline{\leftrightarrow}^{\tau}_{pombpomf} = \underline{\leftrightarrow}^{\tau}_{pomhbr}$) [Pin93].

New τ-equivalences

- Basic τ-equivalences:

  interleaving *ST-branching τ-bisimulation* ($\underline{\leftrightarrow}^{\tau}_{iSTbr}$),

  pomset *history preserving ST-branching τ-bisimulation* ($\underline{\leftrightarrow}^{\tau}_{pomhSTbr}$) and

  multi event structure ($\equiv^{\tau}_{mes}$).

- Back-forth τ-bisimulation equivalences:

  interleaving *back* step *forth* ($\underline{\leftrightarrow}^{\tau}_{ibsf}$),

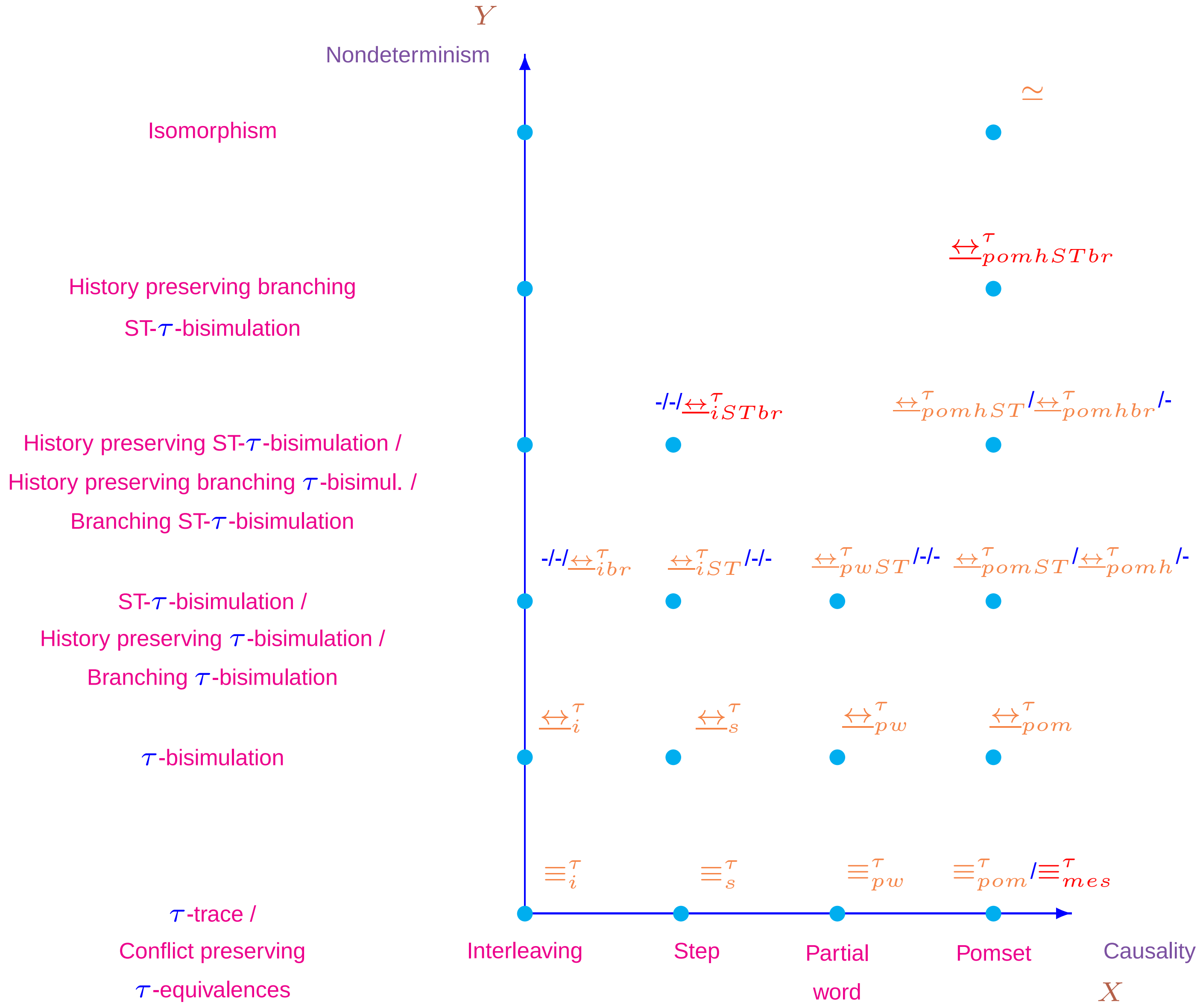
  interleaving *back* partial word *forth* ($\underline{\leftrightarrow}^{\tau}_{ibpwf}$),

  interleaving *back* pomset *forth* ($\underline{\leftrightarrow}^{\tau}_{ibpomf}$),

  step *back* step *forth* ($\underline{\leftrightarrow}^{\tau}_{sbsf}$),

  step *back* partial word *forth* ($\underline{\leftrightarrow}^{\tau}_{sbpwf}$) and

  step *back* pomset *forth* ($\underline{\leftrightarrow}^{\tau}_{sbpomf}$).



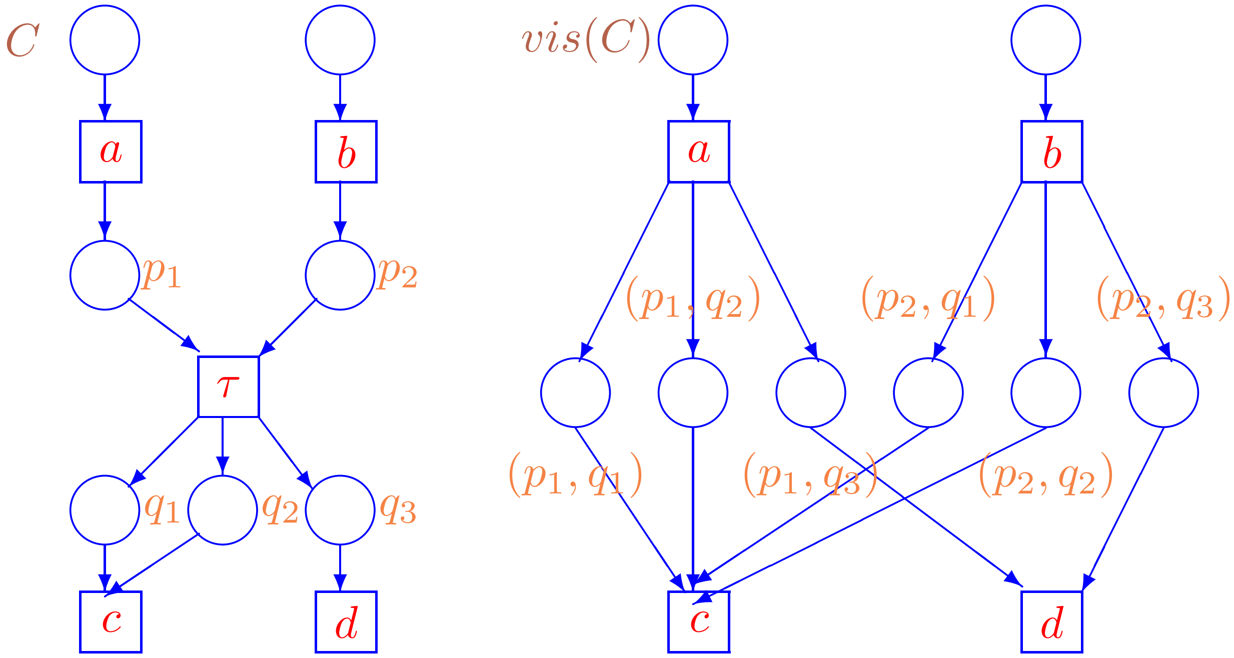
Classification of basic τ-equivalences

Basic τ-equivalences are positioned on coordinate plane.

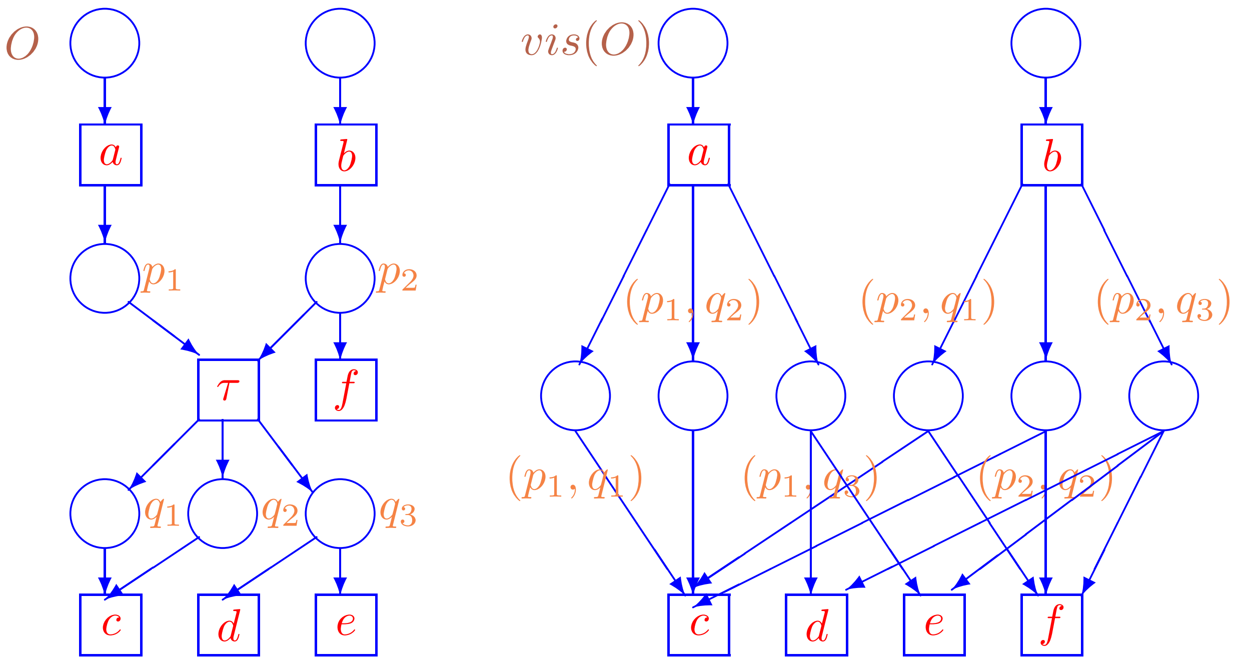
New relations are depicted in red colour.

Moving along X axis: a degree of causality grows.

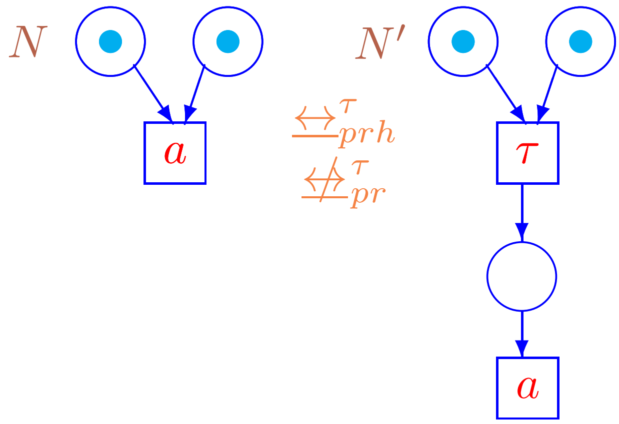
Moving along Y axis: a degree of non-determinism grows.



An application of the mapping vis to a causal net



An application of the mapping vis to an occurrence net



A crash of interrelations of the process τ-bisimulation equivalences comparing with that of the process bisimulation equivalences

Basic τ-simulation

τ-trace equivalences

The empty string is ε.

Let $\sigma = a_1 \cdots a_n \in Act_\tau^*$ and $a \in Act_\tau$. We define $vis(\sigma)$:

1. $vis(\varepsilon) = \varepsilon$;
2. $vis(\sigma a) = \begin{cases} vis(\sigma)a, & a \neq \tau; \\ vis(\sigma), & a = \tau. \end{cases}$

Definition 52 *A* visible interleaving trace *of a net N is a sequence*
$vis(a_1 \cdots a_n) \in Act^*$ *s.t.*
$\pi_N \stackrel{a_1}{\to} \pi_1 \stackrel{a_2}{\to} \ldots \stackrel{a_n}{\to} \pi_n,\ \pi_i \in \Pi(N)\ (1 \leq i \leq n)$.

The set of all visible interleaving traces *of N is $VisIntTraces(N)$.*

N and N' are interleaving τ-trace equivalent*, $N \equiv_i^\tau N'$, if*

$$VisIntTraces(N) = VisIntTraces(N').$$

Let $A \in I\!N_{fin}^{Act_\tau}$. We denote $vis(A) = \sum_{\{a \in A \mid a \in Act\}} a$.

Let $\Sigma = A_1 \cdots A_n \in (I\!N_{fin}^{Act_\tau})^*$ and $A \in I\!N_{fin}^{Act_\tau}$. We define $vis(\Sigma)$:

1. $vis(\varepsilon) = \varepsilon$;
2. $vis(\Sigma A) = \begin{cases} vis(\Sigma)vis(A), & A \cap Act \neq \emptyset; \\ vis(\Sigma), & \text{otherwise}. \end{cases}$

Definition 53 *A visible step trace of a net N is a sequence $vis(A_1 \cdots A_n) \in (\mathbb{N}_{fin}^{Act})^*$ s.t. $\pi_N \stackrel{A_1}{\to} \pi_1 \stackrel{A_2}{\to} \ldots \stackrel{A_n}{\to} \pi_n,\ \pi_i \in \Pi(N)\ (1 \le i \le n)$.*

The set of all visible step traces of N is $VisStepTraces(N)$.

N and N' are step τ-trace equivalent, $N \equiv_s^\tau N'$, if

$$VisStepTraces(N) = VisStepTraces(N').$$

Let $\rho = (X, \prec, l)$ is lposet s.t. $l : X \to Act_\tau$. We denote:

- $vis(X) = \{x \in X \mid l(x) \in Act\}$;
- $vis(\rho) = \rho|_{vis(X)}$.

Definition 54 *A visible pomset trace of a net N is a pomset $vis(\rho)$, an isomorphism class of lposet $vis(\rho_C)$ for $\pi = (C, \varphi) \in \Pi(N)$.*

The set of all visible pomset traces of N is $VisPomsets(N)$.

N and N' are partial word τ-trace equivalent, $N \equiv_{pw}^\tau N'$, if

$$VisPomsets(N) \sqsubseteq VisPomsets(N') \text{ and}$$

$$VisPomsets(N') \sqsubseteq VisPomsets(N).$$

Definition 55 *N and N' are pomset τ-trace equivalent, $N \equiv_{pom}^\tau N'$, if*

$$VisPomsets(N) = VisPomsets(N').$$

Usual τ-bisimulation equivalences

Let $C = (P_C, T_C, W_C, L_C)$ be causal net. We denote:

- $vis(T_C) = \{v \in T_C \mid L_C(v) \in Act\}$;
- $vis(\prec_C) = \prec_C \cap (vis(T_C) \times vis(T_C))$.

Definition 56 $\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ *is a* $\star$-τ-bisimulation *between nets* N *and* N', $\star \in$ {interleaving, step, partial word, pomset}, $\mathcal{R} : N \underline{\leftrightarrow}_\star^\tau N'$, $\star \in \{i, s, pw, pom\}$, *if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.
2. $(\pi, \pi') \in \mathcal{R}$, $\pi \stackrel{\hat{\pi}}{\rightarrow} \tilde{\pi}$,
   (a) $|vis(T_{\widehat{C}})| = 1$, *if* $\star = i$;
   (b) $vis(\prec_{\widehat{C}}) = \emptyset$, *if* $\star = s$;
   $\Rightarrow \exists \tilde{\pi}' : \pi' \stackrel{\hat{\pi}'}{\rightarrow} \tilde{\pi}'$, $(\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$ *and*
   (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, *if* $\star = pw$;
   (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, *if* $\star \in \{i, s, pom\}$.
3. *As item 2, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* $\star$-τ-bisimulation equivalent, $\star \in$ {interleaving, step, partial word, pomset}, $N \underline{\leftrightarrow}_\star^\tau N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_\star^\tau N'$, $\star \in \{i, s, pw, pom\}$.

ST-τ-bisimulation equivalences

Definition 57 *An* ST-τ-process *of a net* N *is a pair* (π_E, π_P)*:*

1. $\pi_E, \pi_P \in \Pi(N),\ \pi_P \xrightarrow{\pi_W} \pi_E$;
2. $\forall v, w \in T_{C_E}\ (v \prec_{C_E} w) \vee (L_{C_E}(v) = \tau)\ \Rightarrow\ v \in T_{C_P}$.

- π_E is a current process;
- π_P is the completed part;
- π_W is the still working part.

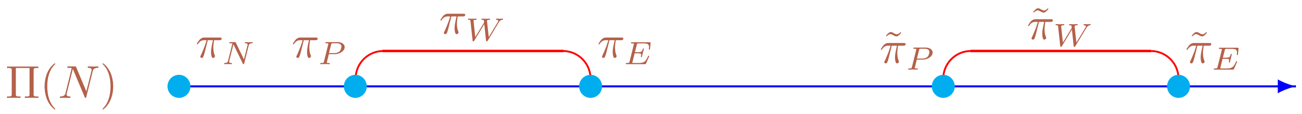
Obviously, $\prec_{C_W} = \emptyset$.

$ST^\tau - \Pi(N)$ is the set of *all ST-τ-processes* of a net N.

(π_N, π_N) is the *initial ST-τ-process* of a net N.

Let $(\pi_E, \pi_P),\ (\tilde{\pi}_E, \tilde{\pi}_P) \in ST^\tau - \Pi(N)$.

We write $(\pi_E, \pi_P) \rightarrow (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \rightarrow \tilde{\pi}_E$ and $\pi_P \rightarrow \tilde{\pi}_P$.



ST-τ-processes

Definition 58 $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, *where*
$\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),$
$\pi' = (C', \varphi') \in \Pi(N')\}$ *is a* $\star$-ST-τ-bisimulation *between nets* N *and* N',
$\star \in$ {interleaving, partial word, pomset}, $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star ST} N',\ \star \in \{i, pw, pom\}$,
if:

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \asymp vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)\ \Rightarrow$
   $\exists \tilde{\beta},\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta,$
   $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$, *and if*
   $\pi_P \stackrel{\pi}{\to} \tilde{\pi}_E,\ \pi'_P \stackrel{\pi'}{\to} \tilde{\pi}'_E,\ \gamma = \tilde{\beta}|_{vis(T_C)}$, *then:*
   (a) $\gamma^{-1} : vis(\rho_{C'}) \sqsubseteq vis(\rho_C)$, *if* $\star = pw$;
   (b) $\gamma : vis(\rho_C) \simeq vis(\rho_{C'})$, *if* $\star = pom$.
4. *As item 3, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* $\star$-ST-τ-bisimulation equivalent, $\star \in$ {interleaving, partial word, pomset}, $N \underline{\leftrightarrow}^\tau_{\star ST} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{\star ST} N',\ \star \in \{i, pw, pom\}$.

History preserving τ-bisimulation equivalences

Definition 59 $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, *where* $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),$ $\pi' = (C', \varphi') \in \Pi(N')\}$, *is a* pomset history preserving τ-bisimulation *between nets* N *and* N', $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomh} N'$, *if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.
2. $(\pi, \pi', \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_C) \simeq vis(\rho_{C'})$.
3. $(\pi, \pi', \beta) \in \mathcal{R},\ \pi \to \tilde{\pi} \ \Rightarrow\ \exists \tilde{\beta},\ \tilde{\pi}' : \pi' \to \tilde{\pi}',\ \tilde{\beta}|_{vis(T_C)} = \beta,$ $(\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of* N *and* N' *are reversed.*

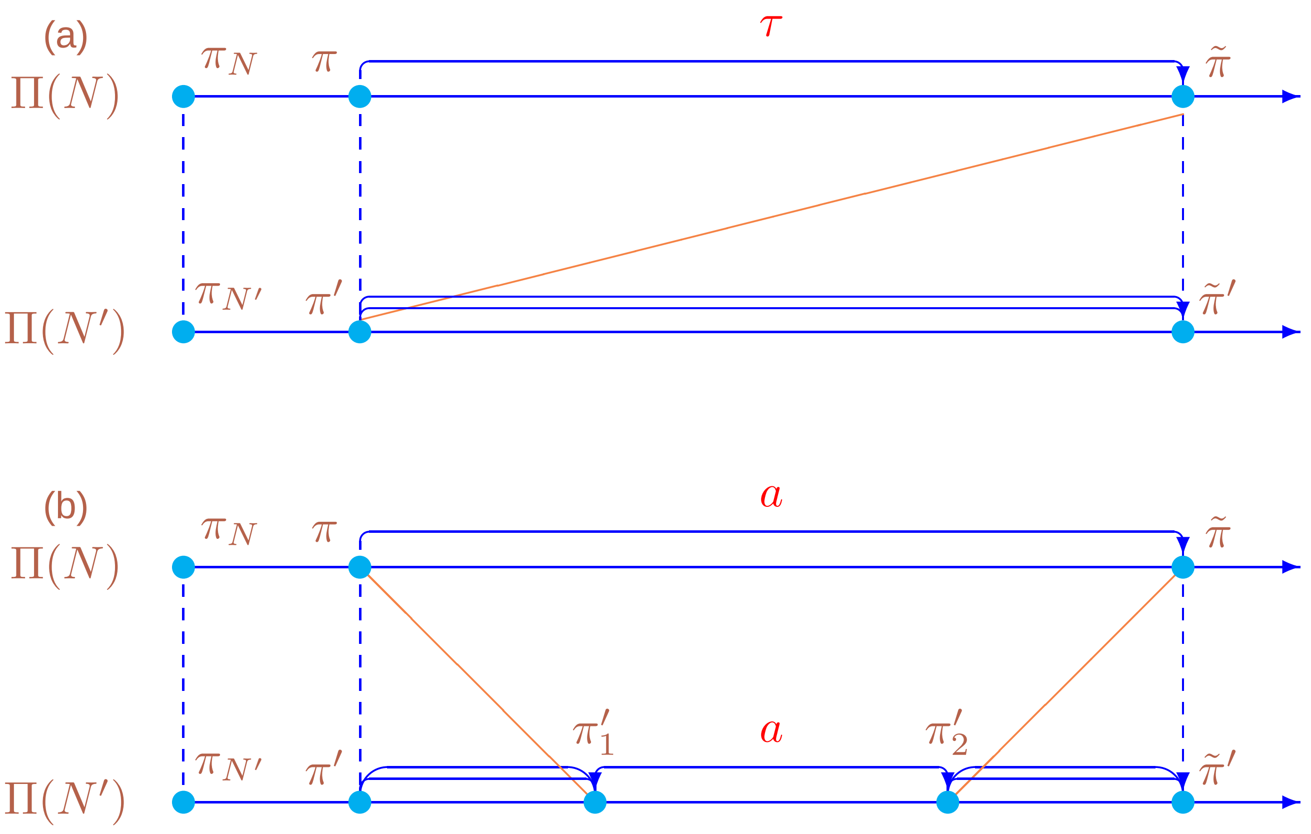
N *and* N' *are* pomset history preserving τ-bisimulation equivalent, $N \underline{\leftrightarrow}^{\tau}_{pomh} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomh} N'$.

History preserving ST-τ-bisimulation equivalences

Definition 60 $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, *where* $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),$ $\pi' = (C', \varphi') \in \Pi(N')\}$, *is a* pomset history preserving ST-τ-bisimulation *between nets* N *and* N', $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$, *if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \simeq vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P)\ \Rightarrow$ $\exists \tilde{\beta},\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),\ \tilde{\beta}|_{vis(T_{C_E})} = \beta,$ $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* pomset history preserving ST-τ-bisimulation equivalent, $N \underline{\leftrightarrow}^\tau_{pomhST} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhST} N'$.



A distinguish ability of the usual and the branching τ-bisimulation equivalences

Usual branching τ-bisimulation equivalences

For a net N and $\pi, \tilde{\pi} \in \Pi(N)$ we write $\pi \Rightarrow \tilde{\pi}$ when $\exists \hat{\pi} = (\widehat{C}, \hat{\varphi})$ s.t. $\pi \xrightarrow{\hat{\pi}} \tilde{\pi}$ and $vis(T_{\widehat{C}}) = \emptyset$.

Definition 61 *$\mathcal{R} \subseteq \Pi(N) \times \Pi(N')$ is an interleaving branching τ-bisimulation between nets N and N', $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{ibr} N'$, if:*

1. $(\pi_N, \pi_{N'}) \in \mathcal{R}$.
2. $(\pi, \pi') \in \mathcal{R}$, $\pi \xrightarrow{a} \tilde{\pi} \ \Rightarrow$
   (a) $a = \tau$ *and* $(\tilde{\pi}, \pi') \in \mathcal{R}$ *or*
   (b) $a \neq \tau$ *and* $\exists \bar{\pi}', \ \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \xrightarrow{a} \tilde{\pi}', \ (\pi, \bar{\pi}') \in \mathcal{R}, \ (\tilde{\pi}, \tilde{\pi}') \in \mathcal{R}$.
3. *As item 2, but the roles of N and N' are reversed.*

N and N' are interleaving branching τ-bisimulation equivalent, $N \underline{\leftrightarrow}^{\tau}_{ibr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{ibr} N'$.

History preserving branching τ-bisimulation equivalences

Definition 62 $\mathcal{R} \subseteq \Pi(N) \times \Pi(N') \times \mathcal{B}$, *where*
$\mathcal{B} = \{\beta \mid \beta : T_C \rightarrow T_{C'},\ \pi = (C, \varphi) \in \Pi(N),\ \pi' = (C', \varphi') \in \Pi(N')\}$,
is a pomset history preserving branching τ-bisimulation *between nets* N *and* N',
$\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$, *if:*

1. $(\pi_N, \pi_{N'}, \emptyset) \in \mathcal{R}$.
2. $(\pi, \pi', \beta) \in \mathcal{R} \ \Rightarrow \ \beta : vis(\rho_C) \simeq vis(\rho_{C'})$.
3. $(\pi, \pi', \beta) \in \mathcal{R},\ \pi \rightarrow \tilde{\pi} \ \Rightarrow$
   (a) $(\tilde{\pi}, \pi', \beta) \in \mathcal{R}$ *or*
   (b) $\exists \tilde{\beta},\ \bar{\pi}',\ \tilde{\pi}' : \pi' \Rightarrow \bar{\pi}' \rightarrow \tilde{\pi}',\ \tilde{\beta}|_{vis(T_C)} = \beta,\ (\pi, \bar{\pi}', \beta) \in \mathcal{R}$,
   $(\tilde{\pi}, \tilde{\pi}', \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* pomset history preserving branching τ-bisimulation equivalent,
$N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$.

ST-branching τ-bisimulation equivalences

Let $(\pi_E, \pi_P), (\tilde{\pi}_E, \tilde{\pi}_P) \in ST^\tau - \Pi(N)$. We write $(\pi_E, \pi_P) \Rightarrow (\tilde{\pi}_E, \tilde{\pi}_P)$, if $\pi_E \Rightarrow \tilde{\pi}_E$ and $\pi_P \Rightarrow \tilde{\pi}_P$.

Definition 63 $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, *where* $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \to vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),$ $\pi' = (C', \varphi') \in \Pi(N')\}$ *is an interleaving ST-branching τ-bisimulation between nets N and N', $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{iSTbr} N'$, if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \asymp vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \to (\tilde{\pi}_E, \tilde{\pi}_P) \ \Rightarrow$
   (a) $((\tilde{\pi}_E, \tilde{\pi}_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ *or*
   (b) $\exists \tilde{\beta},\ (\bar{\pi}'_E, \bar{\pi}'_P),\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \Rightarrow (\bar{\pi}'_E, \bar{\pi}'_P) \to (\tilde{\pi}'_E, \tilde{\pi}'_P),$ $\tilde{\beta}|_{vis(T_{C_E})} = \beta,\ ((\pi_E, \pi_P), (\bar{\pi}'_E, \bar{\pi}'_P), \beta) \in \mathcal{R},$ $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of N and N' are reversed.*

N and N' are interleaving ST-branching τ-bisimulation equivalent, $N \underline{\leftrightarrow}^\tau_{iSTbr} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{iSTbr} N'$.

History preserving ST-branching τ-bisimulation equivalences

Definition 64 $\mathcal{R} \subseteq ST^\tau - \Pi(N) \times ST^\tau - \Pi(N') \times \mathcal{B}$, *where* $\mathcal{B} = \{\beta \mid \beta : vis(T_C) \rightarrow vis(T_{C'}),\ \pi = (C, \varphi) \in \Pi(N),$ $\pi' = (C', \varphi') \in \Pi(N')\}$ *is a* pomset history preserving ST-branching τ-bisimulation *between nets* N *and* N', $\mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$, *if:*

1. $((\pi_N, \pi_N), (\pi_{N'}, \pi_{N'}), \emptyset) \in \mathcal{R}$.
2. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R} \ \Rightarrow\ \beta : vis(\rho_{C_E}) \simeq vis(\rho_{C'_E})$ *and* $\beta(vis(T_{C_P})) = vis(T_{C'_P})$.
3. $((\pi_E, \pi_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R},\ (\pi_E, \pi_P) \rightarrow (\tilde{\pi}_E, \tilde{\pi}_P)\ \Rightarrow$
   (a) $((\tilde{\pi}_E, \tilde{\pi}_P), (\pi'_E, \pi'_P), \beta) \in \mathcal{R}$ *or*
   (b) $\exists \tilde{\beta},\ (\bar{\pi}'_E, \bar{\pi}'_P),\ (\tilde{\pi}'_E, \tilde{\pi}'_P) :\ (\pi'_E, \pi'_P) \Rightarrow (\bar{\pi}'_E, \bar{\pi}'_P) \rightarrow (\tilde{\pi}'_E, \tilde{\pi}'_P),$ $\tilde{\beta}|_{vis(T_{C_E})} = \beta,\ ((\pi_E, \pi_P), (\bar{\pi}'_E, \bar{\pi}'_P), \beta) \in \mathcal{R},$ $((\tilde{\pi}_E, \tilde{\pi}_P), (\tilde{\pi}'_E, \tilde{\pi}'_P), \tilde{\beta}) \in \mathcal{R}$.
4. *As item 3, but the roles of* N *and* N' *are reversed.*

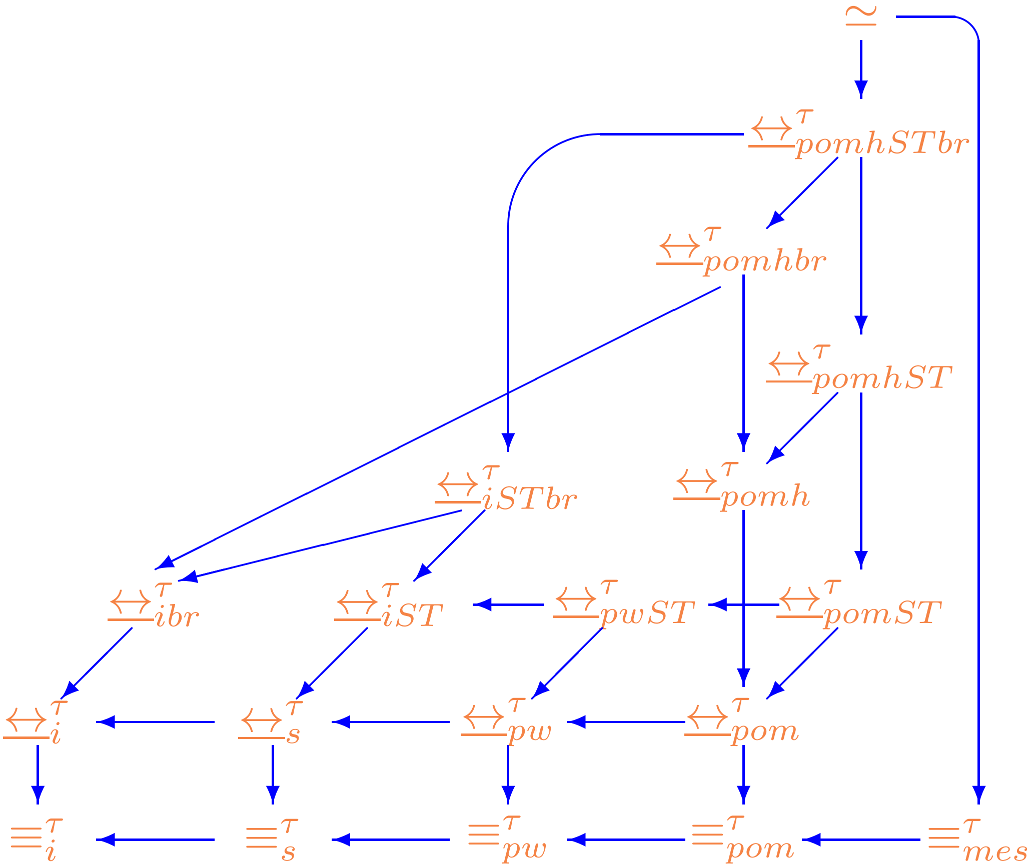
N *and* N' *are* pomset history preserving ST-branching τ-bisimulation equivalent, $N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}^\tau_{pomhSTbr} N'$.

Conflict preserving τ-equivalences

Let $\xi = (X, \prec, \#, l)$ be a LES s.t. $l : X \to Act_\tau$. We denote $vis(X) = \{x \in X \mid l(x) \in Act\}$ and $vis(\xi) = \xi|_{vis(X)}$.

Definition 65 *N and N' are MES-τ-conflict preserving equivalent, $N \equiv^\tau_{mes} N'$, if $vis(\mathcal{E}(N)) = vis(\mathcal{E}(N'))$.*

Comparing basic τ-equivalences

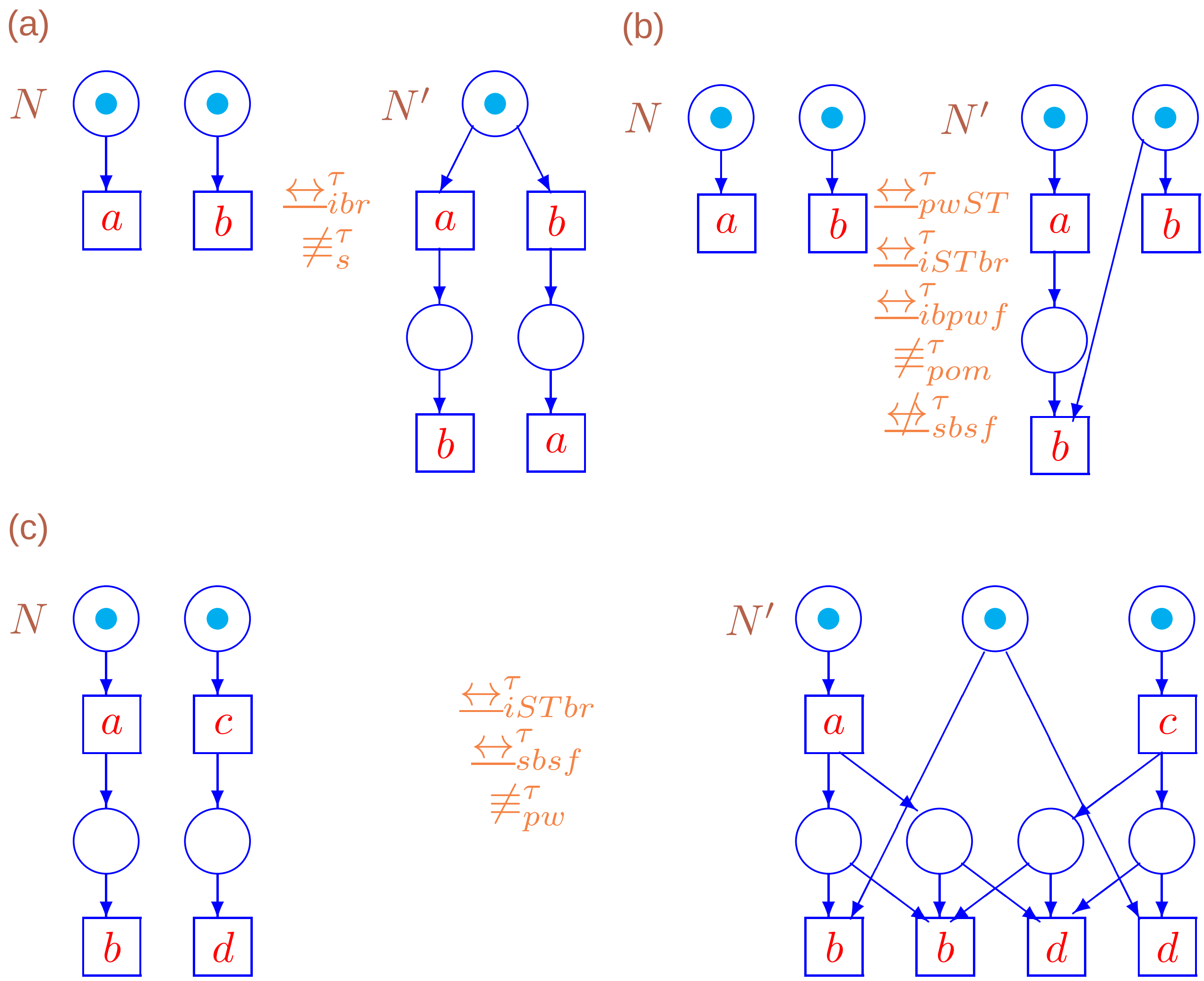


Interrelations of basic τ-equivalences

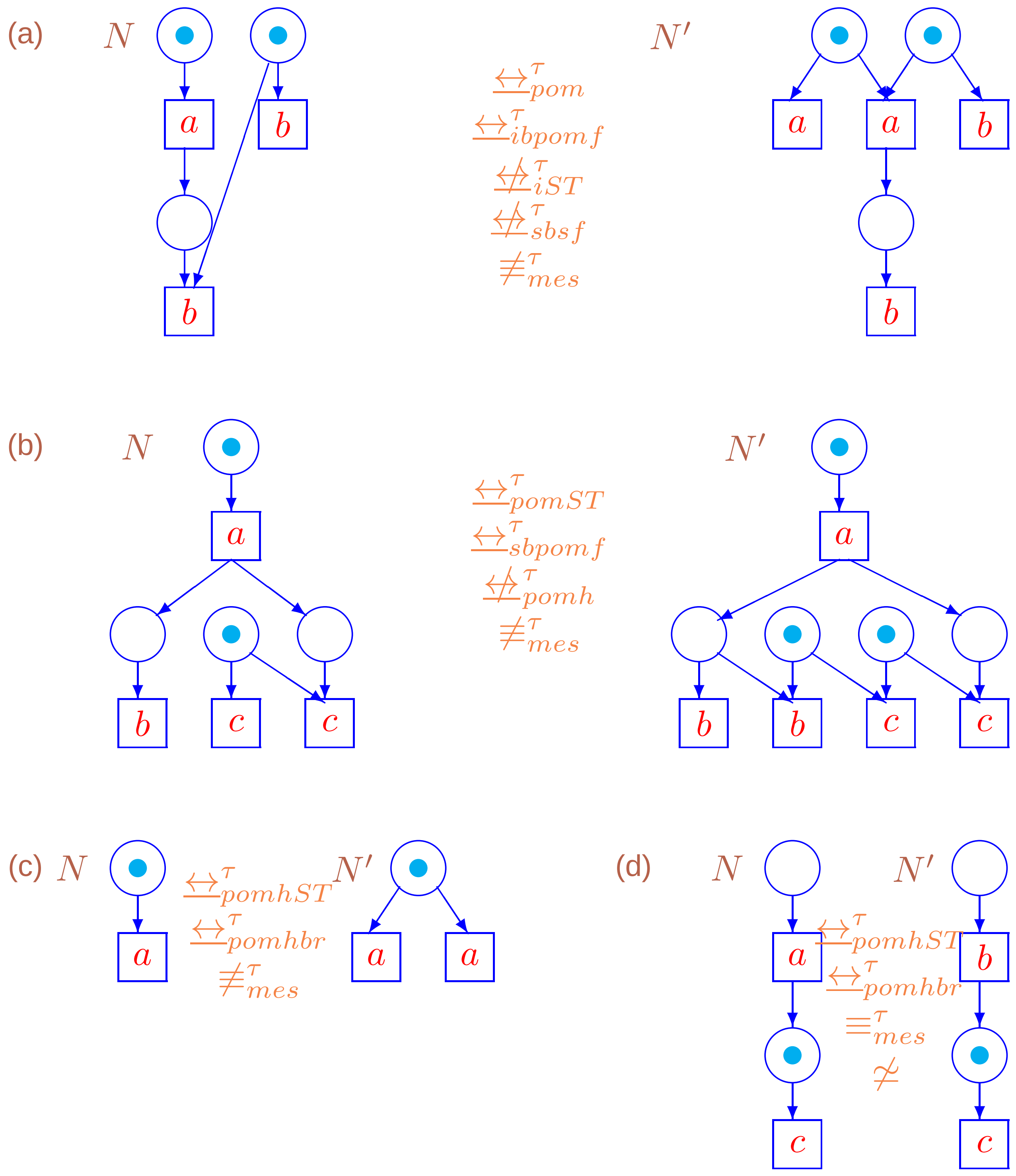
Theorem 11 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\leftrightarrow \in \{\equiv^\tau, \underline{\leftrightarrow}^\tau, \simeq\}$, $\star, \star\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr, pomhbr, iSTbr, pomhSTbr, mes\}$. *For nets* N *and* N'

$$N \leftrightarrow_\star N' \Rightarrow N \leftrightarrow\!\!\!\leftrightarrow_{\star\star} N'$$

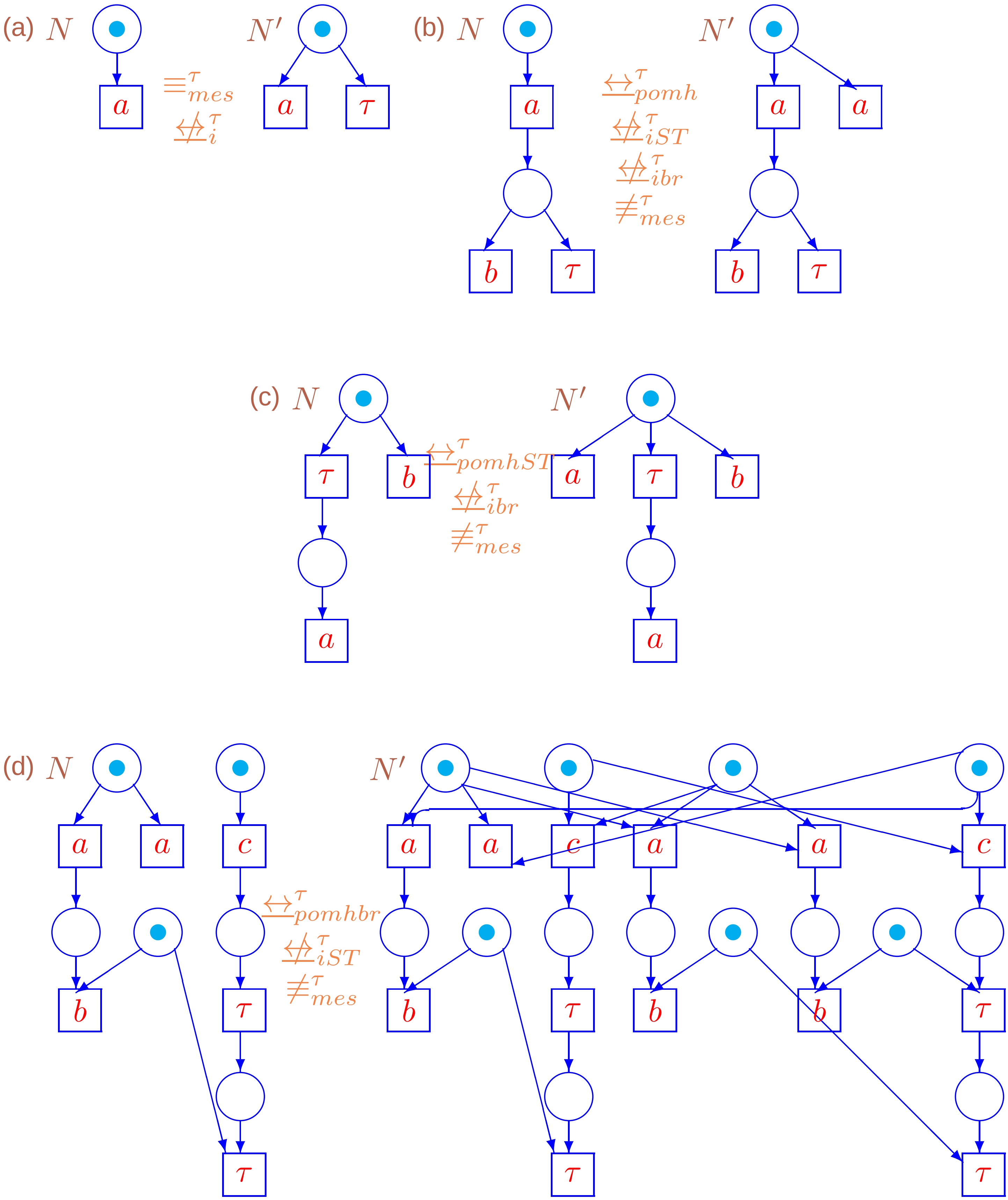
iff in the graph above there exists a directed path from $\leftrightarrow_\star$ *to* $\leftrightarrow\!\!\!\leftrightarrow_{\star\star}$.



BT: Examples of basic τ-equivalences



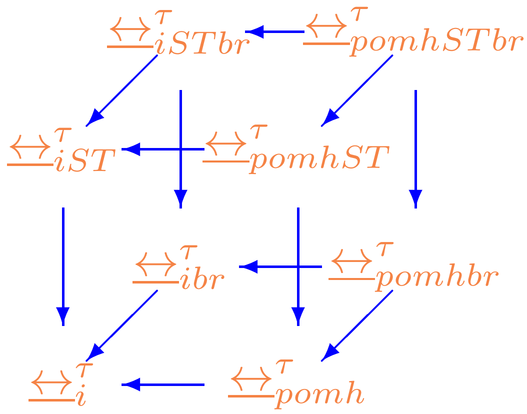
BT1: Examples of basic τ-equivalences (continued)



BT2: Examples of basic τ-equivalences (continued 2)

- In Figure BT(a), $N \underline{\leftrightarrow}^{\tau}_{ibr} N'$, but $N \not\equiv^{\tau}_{s} N'$, since only in the net N' actions a and b cannot occur concurrently.
- In Figure BT(c), $N \underline{\leftrightarrow}^{\tau}_{iSTbr} N'$, but $N \not\equiv^{\tau}_{pw} N'$, since for the pomset corresponding to the net N there is no even less sequential pomset in N'.
- In Figure BT(b), $N \underline{\leftrightarrow}^{\tau}_{pwST} N'$, but $N \not\equiv^{\tau}_{pom} N'$, since only in the net N' action b can depend on a.
- In Figure BT2(a), $N \equiv^{\tau}_{mes} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{i} N'$, since only in the net N' action τ can occur so that in the corresponding initial state of the net N action a cannot occur.
- In Figure BT1(a), $N \underline{\leftrightarrow}^{\tau}_{pom} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{iST} N'$, since only in the net N' action a can start so that no action b can begin to work until finishing a.
- In Figure BT1(b), $N \underline{\leftrightarrow}^{\tau}_{pomST} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{pomh} N'$, since only in the net N' after action a action b can occur so that action c must depend on a.
- In Figure BT2(b), $N \underline{\leftrightarrow}^{\tau}_{pomh} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{iST} N'$, since only in the net N' action a can start so that the action b can never occur.
- In Figure BT2(c), $N \underline{\leftrightarrow}^{\tau}_{pomhST} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ibr} N'$, since in the net N' an action a can occur so that it will be simulated by sequence of actions τa in N. Then the state of the net N reached after τ must be related with the initial state of a net N, but in such a case the occurrence of action b from the initial state of N' cannot be imitated from the corresponding state of N.

- In Figure BT2(d), $N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{iST} N'$, since in the net N' an action c may start so that during work of the corresponding action c in the net N an action a may occur in such a way that the action b never occur.
- In Figure BT1(c), $N \underline{\leftrightarrow}^{\tau}_{pomhSTbr} N'$, but $N \not\equiv^{\tau}_{mes} N'$, since only the MES corresponding to the net N' has two conflict actions a.
- In Figure BT1(d), $N \equiv^{\tau}_{mes} N'$, but $N \not\simeq N'$, since unfireable transitions of the nets N and N' are labeled by different actions (a and b).



Cube of interrelations for basic τ-bisimulation equivalences

Orthogonality of the following parameters:

ST- / history preservation / branching.

Back-forth τ-simulation and logics

Back-forth τ-bisimulation equivalences

Definition 66 $\mathcal{R} \subseteq Runs(N) \times Runs(N')$ *is a* $\star$-back $\star\star$-forth τ-bisimulation *between nets* N *and* N', $\star, \star\star \in$ {interleaving, step, partial word, pomset}, $\mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom\}$, *if:*

1. $((\pi_N, \varepsilon), (\pi_{N'}, \varepsilon)) \in \mathcal{R}$.
2. $((\pi, \sigma), (\pi', \sigma')) \in \mathcal{R}$
   - *(back)* $(\tilde{\pi}, \tilde{\sigma}) \stackrel{\hat{\pi}}{\rightarrow} (\pi, \sigma)$,

     (a) $|vis(T_{\widehat{C}})| = 1$, *if* $\star = i$;

     (b) $vis(\prec_{\widehat{C}}) = \emptyset$, *if* $\star = s$;

     $\Rightarrow\ \exists (\tilde{\pi}', \tilde{\sigma}') :\ (\tilde{\pi}', \tilde{\sigma}') \stackrel{\hat{\pi}'}{\rightarrow} (\pi', \sigma'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*

     (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, *if* $\star = pw$;

     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, *if* $\star \in \{i, s, pom\}$;
   - *(forth)* $(\pi, \sigma) \stackrel{\hat{\pi}}{\rightarrow} (\tilde{\pi}, \tilde{\sigma})$,

     (a) $|vis(T_{\widehat{C}})| = 1$, *if* $\star\star = i$;

     (b) $vis(\prec_{\widehat{C}}) = \emptyset$, *if* $\star\star = s$;

     $\Rightarrow\ \exists (\tilde{\pi}', \tilde{\sigma}') :\ (\pi', \sigma') \stackrel{\hat{\pi}'}{\rightarrow} (\tilde{\pi}', \tilde{\sigma}'),\ ((\tilde{\pi}, \tilde{\sigma}), (\tilde{\pi}', \tilde{\sigma}')) \in \mathcal{R}$ *and*

     (a) $vis(\rho_{\widehat{C}'}) \sqsubseteq vis(\rho_{\widehat{C}})$, *if* $\star\star = pw$;

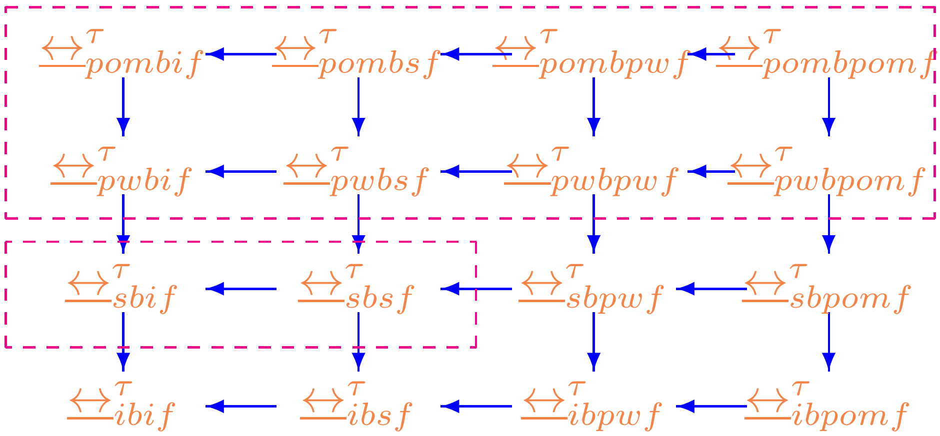
     (b) $vis(\rho_{\widehat{C}}) \simeq vis(\rho_{\widehat{C}'})$, *if* $\star\star \in \{i, s, pom\}$.
3. *As item 2, but the roles of* N *and* N' *are reversed.*

N *and* N' *are* $\star$-back $\star\star$-forth τ-bisimulation equivalent, $\star, \star\star \in$ {interleaving, step, partial word, pomset}, $N \underline{\leftrightarrow}^{\tau}_{\star b \star\star f} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}^{\tau}_{\star b \star\star f} N'$, $\star, \star\star \in \{i, s, pw, pom\}$.

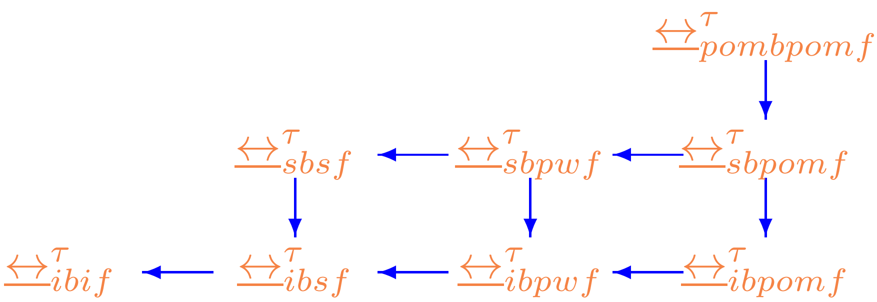
Comparing back-forth τ-bisimulation equivalences

Proposition 11 [Pin93,Tar97] *Let* $\star \in \{i, s, pw, pom\}$*. For nets* N *and* N'

1. $N \underline{\leftrightarrow}^{\tau}_{pwb\star f} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{pomb\star f} N'$;
2. $N \underline{\leftrightarrow}^{\tau}_{\star bif} N' \;\Leftrightarrow\; N \underline{\leftrightarrow}^{\tau}_{\star b\star f} N'$.



Merging of back-forth τ-bisimulation equivalences

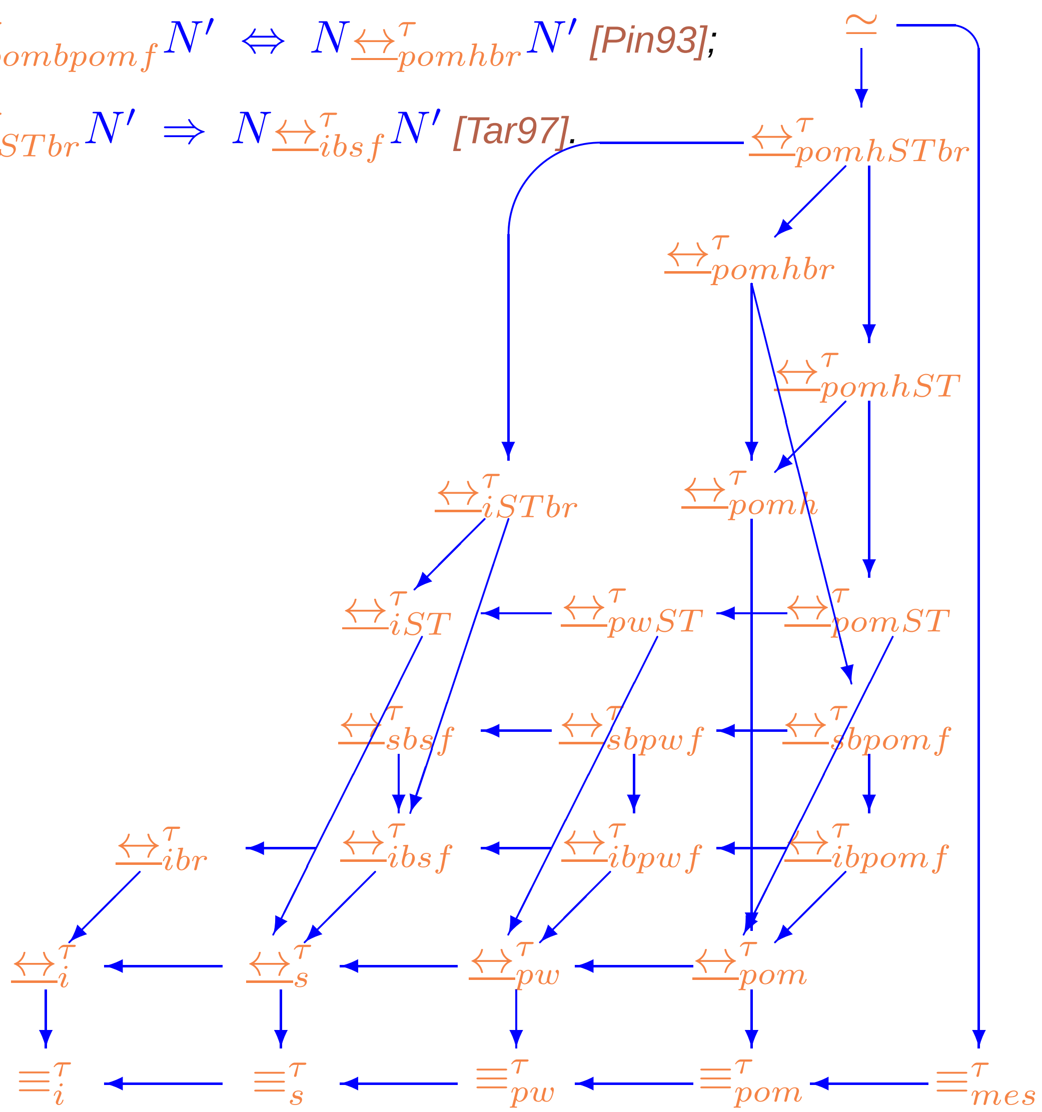


Interrelations of back-forth τ-bisimulation equivalences

Comparing back-forth τ-bisimulation equivalences with basic ones

Proposition 12 *For nets N and N'*

1. $N \underline{\leftrightarrow}^{\tau}_{ibif} N' \Leftrightarrow N \underline{\leftrightarrow}^{\tau}_{ibr} N'$ [Gla93];
2. $N \underline{\leftrightarrow}^{\tau}_{pombpomf} N' \Leftrightarrow N \underline{\leftrightarrow}^{\tau}_{pomhbr} N'$ [Pin93];
3. $N \underline{\leftrightarrow}^{\tau}_{iSTbr} N' \Rightarrow N \underline{\leftrightarrow}^{\tau}_{ibsf} N'$ [Tar97].

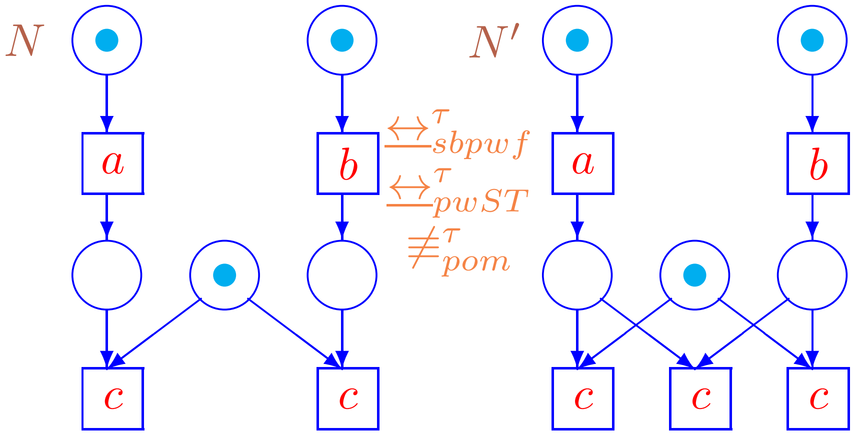


Interrelations of back-forth τ-bisimulation equivalences with basic ones

Theorem 12 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\leftrightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$ *and* $\star, \star\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr, iSTbr, pomhSTbr, pomhbr, mes, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. *For nets N and N'*

$$N \leftrightarrow_{\star} N' \Rightarrow N \leftrightarrow\!\!\!\leftrightarrow_{\star\star} N'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ to $\leftrightarrow\!\!\!\leftrightarrow_{\star\star}$.



BFT: Example of back-forth τ-bisimulation equivalences

- In Figure BT(c), $N \underline{\leftrightarrow}^{\tau}_{sbsf} N'$, but $N \not\equiv^{\tau}_{pw} N'$.
- In Figure BFT, $N \underline{\leftrightarrow}^{\tau}_{sbpwf} N'$, but $N \not\equiv^{\tau}_{pom} N'$.
- In Figure BT1(a), $N \underline{\leftrightarrow}^{\tau}_{ibpomf} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{sbsf} N'$.
- In Figure BT(b), $N \underline{\leftrightarrow}^{\tau}_{iSTbr} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{sbsf} N'$.

Logic BFL [NMV90]

Definition 67 *$\top$ denotes the truth, $a \in Act$.*

A formula of BFL:

$$\Phi ::= \ \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \langle\leftarrow a\rangle\Phi \mid \langle a\rangle\Phi$$

BFL is the *set of all formulas* of BFL.

Definition 68 *Let N be a net and $(\pi, \sigma) \in Runs(N)$. The satisfaction relation $\models_N \in Runs(N) \times \mathbf{BFL}$:*

1. *$(\pi, \sigma) \models_N \top$ — always;*
2. *$(\pi, \sigma) \models_N \neg\Phi$, if $(\pi, \sigma) \not\models_N \Phi$;*
3. *$(\pi, \sigma) \models_N \Phi\wedge\Psi$, if $(\pi, \sigma) \models_N \Phi$ and $(\pi, \sigma) \models_N \Psi$;*
4. *$(\pi, \sigma) \models_N \langle\leftarrow a\rangle\Phi$, if $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$, where $\hat{\pi} = (\widehat{C}, \hat{\varphi}),\ vis(L_{\widehat{C}}(T_{\widehat{C}})) = a$ and $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$;*
5. *$(\pi, \sigma) \models_N \langle a\rangle\Phi$, if $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$, where $\hat{\pi} = (\widehat{C}, \hat{\varphi}),\ vis(L_{\widehat{C}}(T_{\widehat{C}})) = a$ and $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$.*

$[a]\Phi = \neg\langle a\rangle\neg\Phi$, $[\leftarrow a]\Phi = \neg\langle\leftarrow a\rangle\neg\Phi$.

$N \models_N \Phi$, if $(\pi_N, \varepsilon) \models_N \Phi$.

Definition 69 *N and N' are logical equivalent in BFL, $N =_{BFL} N'$, if $\forall\Phi \in \mathbf{BFL}\ N \models_N \Phi \ \Leftrightarrow\ N' \models_{N'} \Phi$.*

Let N be a net and $\pi \in \Pi(N),\ a \in Act$.

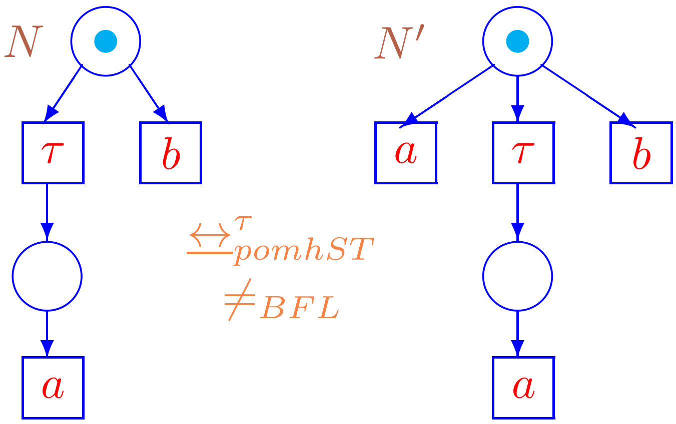
The set of *visible extensions* of a process π by action a (*image set*) is $VisImage(\pi, a) = \{\tilde{\pi} \mid \pi \xrightarrow{\hat{\pi}} \tilde{\pi},\ \hat{\pi} = (\widehat{C}, \hat{\varphi}),\ vis(L_{\widehat{C}}(T_{\widehat{C}})) = a\}$.

A net N is a *image-finite* one, if
$\forall \pi \in \Pi(N)\ \forall a \in Act\ |VisImage(\pi, a)| < \infty$.

Theorem 13 *For image-finite nets N and N'*

$$N \underline{\leftrightarrow}_{ibr}^{\tau} N' \Leftrightarrow N \underline{\leftrightarrow}_{ibif}^{\tau} N' \Leftrightarrow N =_{BFL} N'.$$

Example on logical equivalence of BFL



Differentiating power of $=_{BFL}$

$N \underline{\leftrightarrow}^{\tau}_{pomhST} N'$, but $N \neq_{BFL} N'$, because for $\Phi = \langle a \rangle [\leftarrow a] \langle b \rangle \top$, $N \not\models_N \Phi$, but $N' \models_{N'} \Phi$, since in N' an action a can occur so that it will be simulated by sequence τa in N.

Then the state of the net N reached after τ must be related with the initial state of a net N, but in such a case the occurrence of action b from the initial state of N' cannot be imitated from the corresponding state of N.

Logic $SPBFL$ [Pin93]

Definition 70 $\top$ *denotes the truth,* ρ *is a pomset with labeling into* Act*.*

A formula *of* $SPBFL$*:*

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \langle\leftarrow \rho\rangle\Phi \mid \langle a\rangle\Phi$$

$\mathbf{SPBFL}$ is the *set of all formulas* of $SPBFL$.

Definition 71 *Let* N *be a net and* $(\pi, \sigma) \in Runs(N)$*. The* satisfaction relation $\models_N \in Runs(N) \times \mathbf{SPBFL}$*:*

1. $(\pi, \sigma) \models_N \top$ *— always;*
2. $(\pi, \sigma) \models_N \neg\Phi$*, if* $(\pi, \sigma) \not\models_N \Phi$*;*
3. $(\pi, \sigma) \models_N \Phi\wedge\Psi$*, if* $(\pi, \sigma) \models_N \Phi$ *and* $(\pi, \sigma) \models_N \Psi$*;*
4. $(\pi, \sigma) \models_N \langle\leftarrow \rho\rangle\Phi$*, if* $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\tilde{\pi}, \tilde{\sigma}) \xrightarrow{\hat{\pi}} (\pi, \sigma)$*, where* $\hat{\pi} = (\widehat{C}, \hat{\varphi}),\ vis(\rho_{\widehat{C}}) \in \rho$ *and* $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$*;*
5. $(\pi, \sigma) \models_N \langle a\rangle\Phi$*, if* $\exists(\tilde{\pi}, \tilde{\sigma}) \in Runs(N)\ (\pi, \sigma) \xrightarrow{\hat{\pi}} (\tilde{\pi}, \tilde{\sigma})$*, where* $\hat{\pi} = (\widehat{C}, \hat{\varphi}),\ vis(L_{\widehat{C}}(T_{\widehat{C}})) = a$ *and* $(\tilde{\pi}, \tilde{\sigma}) \models_N \Phi$*.*

$[a]\Phi = \neg\langle a\rangle\neg\Phi$, $[\leftarrow \rho]\Phi = \neg\langle\leftarrow \rho\rangle\neg\Phi$.

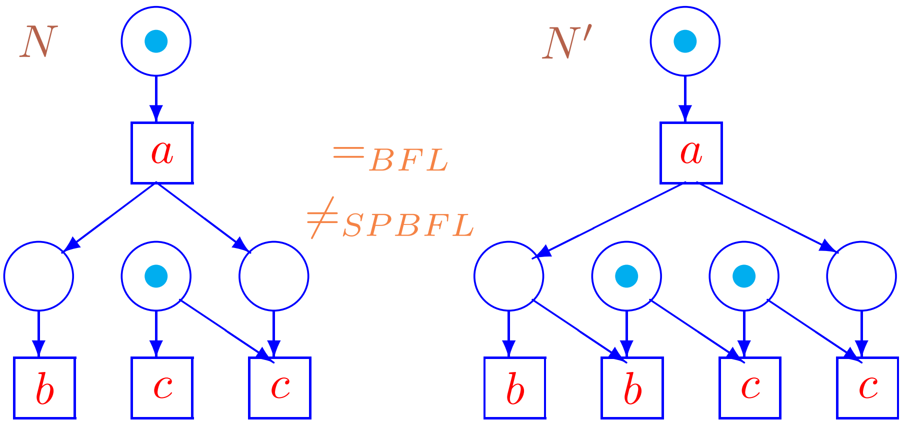
$N \models_N \Phi$, if $(\pi_N, \varepsilon) \models_N \Phi$.

Definition 72 N *and* N' *are* logical equivalent *in* $SPBFL$*,* $N =_{SPBFL} N'$*, if* $\forall\Phi \in \mathbf{SPBFL}\ N \models_N \Phi \Leftrightarrow N' \models_{N'} \Phi$*.*

Theorem 14 *For image-finite nets* N *and* N'

$$N \underline{\leftrightarrow}^{\tau}_{pomhbr} N' \Leftrightarrow N \underline{\leftrightarrow}^{\tau}_{pombpomf} N' \Leftrightarrow N =_{SPBFL} N'.$$

Example on logical equivalence of $SPBFL$



Differentiating power of $=_{SPBFL}$

$N =_{BFL} N'$, but $N \neq_{SPBFL} N'$, because for $\Phi = [a][b]\langle c\rangle\langle\leftarrow (a;b)\|c\rangle\top$, $N \models_N \Phi$, but $N' \not\models_{N'} \Phi$ since only in N' after a action b can occur so that c must depend on a.

Here $(a;b)\|c$ denotes the pomset where b depends on a, and $a,\ b$ are independent with c.

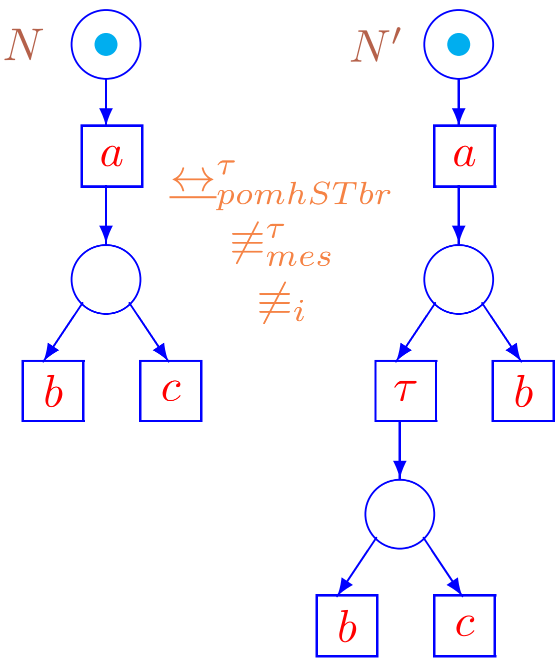
Simulation with and without silent actions

Interrelations of equivalences with τ-equivalences

Theorem 15 *Let* $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$, $\star \in \{i, s, pw, pom, iST, pwST, pomST, mes, sbsf, sbpwf, sbpomf\}$, $\star\star \in \{s, pw, pom\}$. *For nets* N *and* N'

1. $N \leftrightarrow_\star N' \;\Rightarrow\; N \leftrightarrow_\star^\tau N'$;
2. $N \underline{\leftrightarrow}_i N' \;\Rightarrow\; N \underline{\leftrightarrow}_{ibr}^\tau N'$;
3. $N \underline{\leftrightarrow}_{iST} N' \;\Rightarrow\; N \underline{\leftrightarrow}_{iSTbr}^\tau N'$;
4. $N \underline{\leftrightarrow}_{pomh} N' \;\Rightarrow\; N \underline{\leftrightarrow}_{pomhSTbr}^\tau N'$;
5. $N \underline{\leftrightarrow}_{\star\star} N' \;\Rightarrow\; N \underline{\leftrightarrow}_{ib\star\star f}^\tau N'$.

and all the implications are strict.

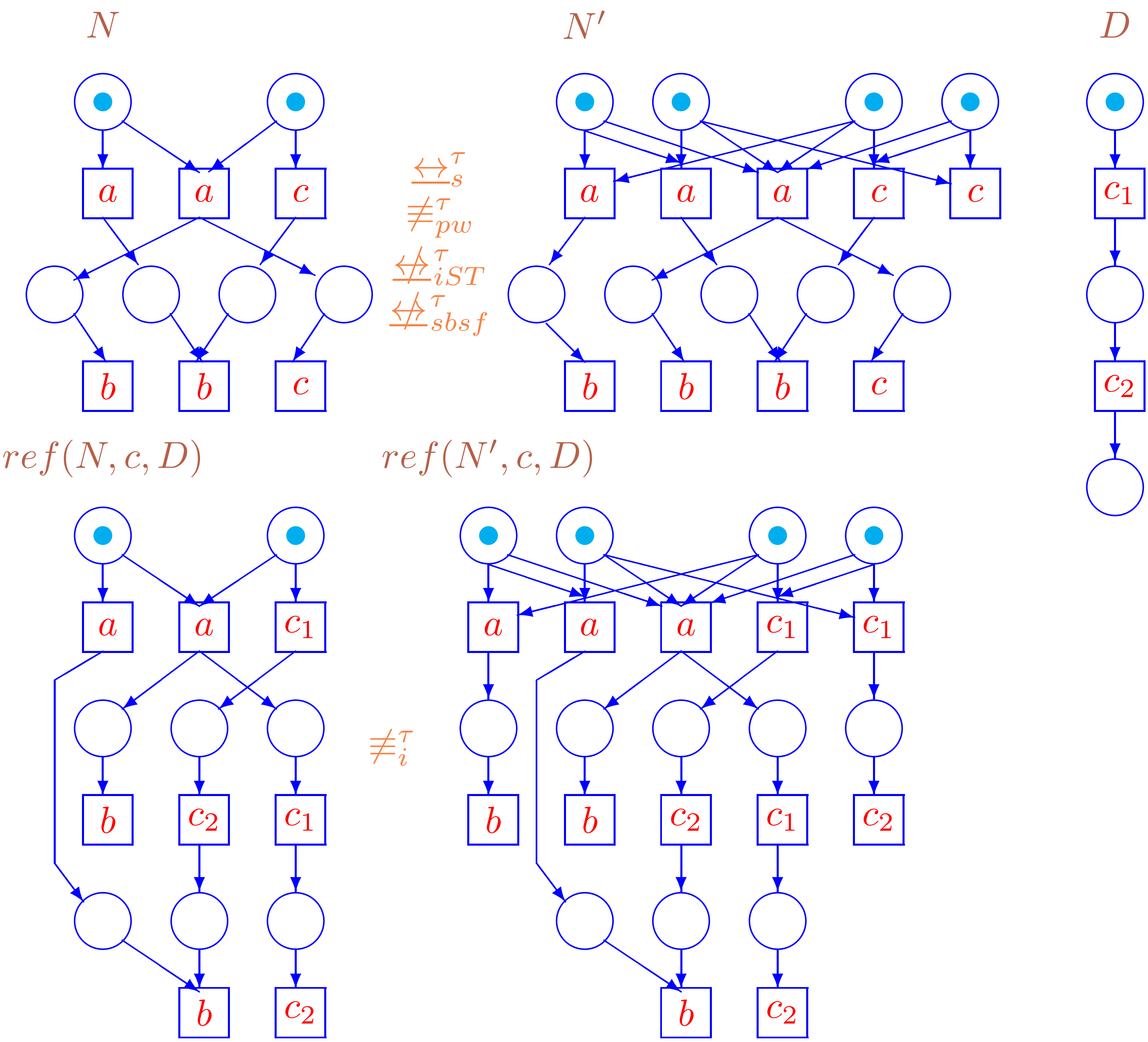


ETE: Example of interrelations of equivalences and τ-equivalences

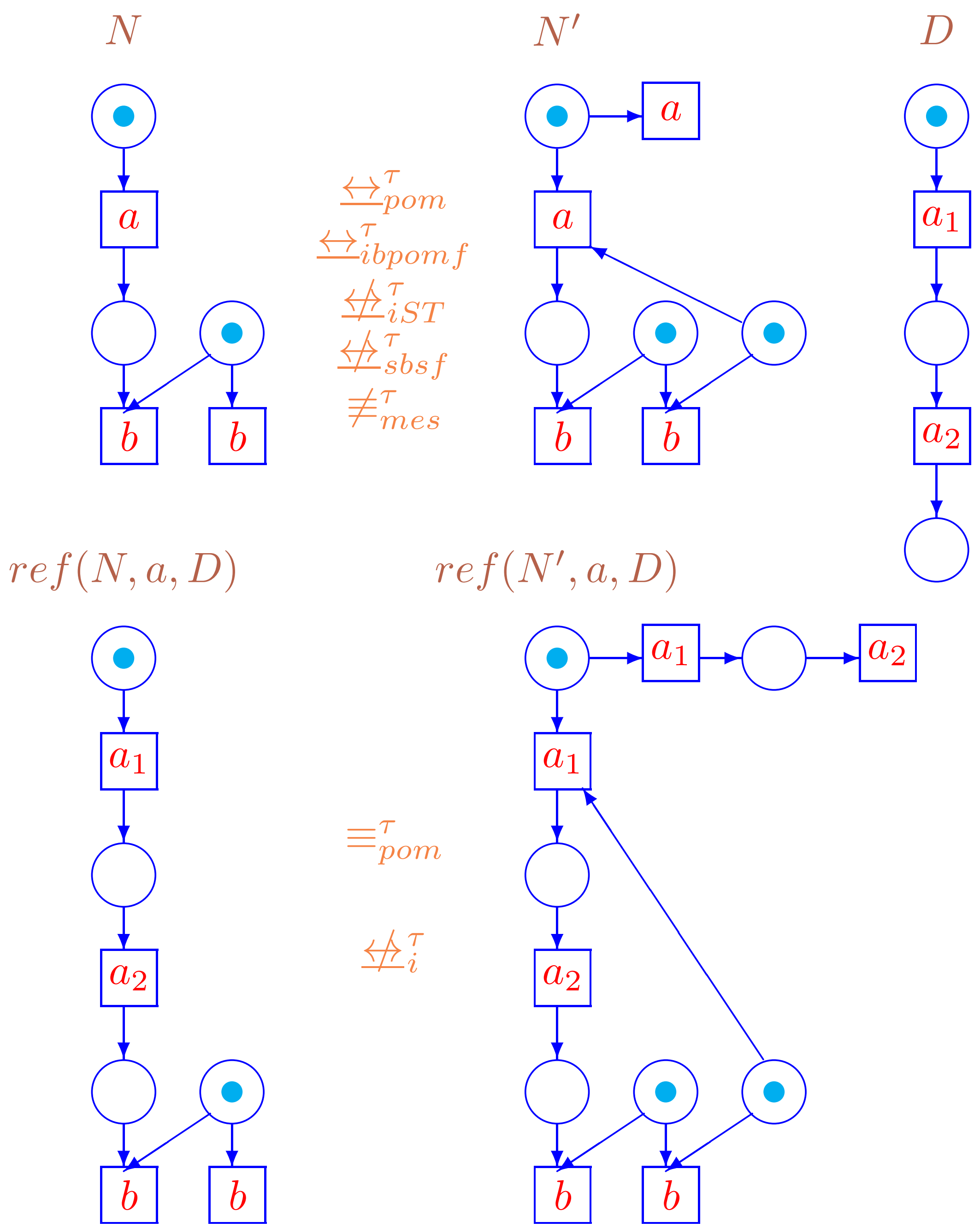
- In Figure ETE, $N \underline{\leftrightarrow}^{\tau}_{pomhSTbr} N'$, but $N \not\equiv_i N'$, since only in the net N' an action a can occur in the initial state.
- In Figure BT2(a), $N \equiv^{\tau}_{mes} N'$, but $N \not\equiv_i N'$, since only in the net N' an action τ can occur in the initial state.

Refinements

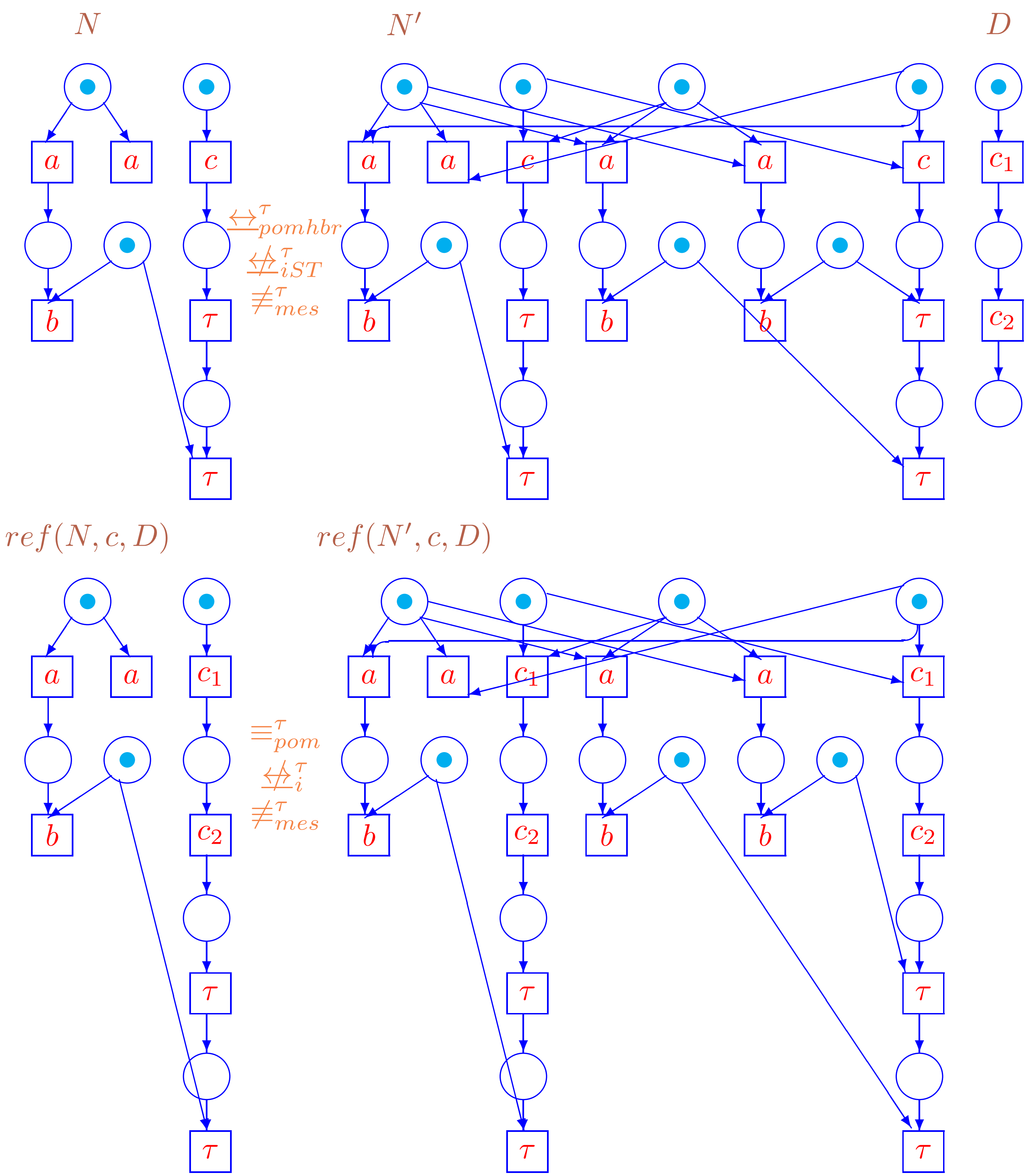
SM-refinements



RBT: The τ-equivalences between $\equiv_i^\tau$ and $\underline{\leftrightarrow}_s^\tau$ are not preserved by SM-refinements



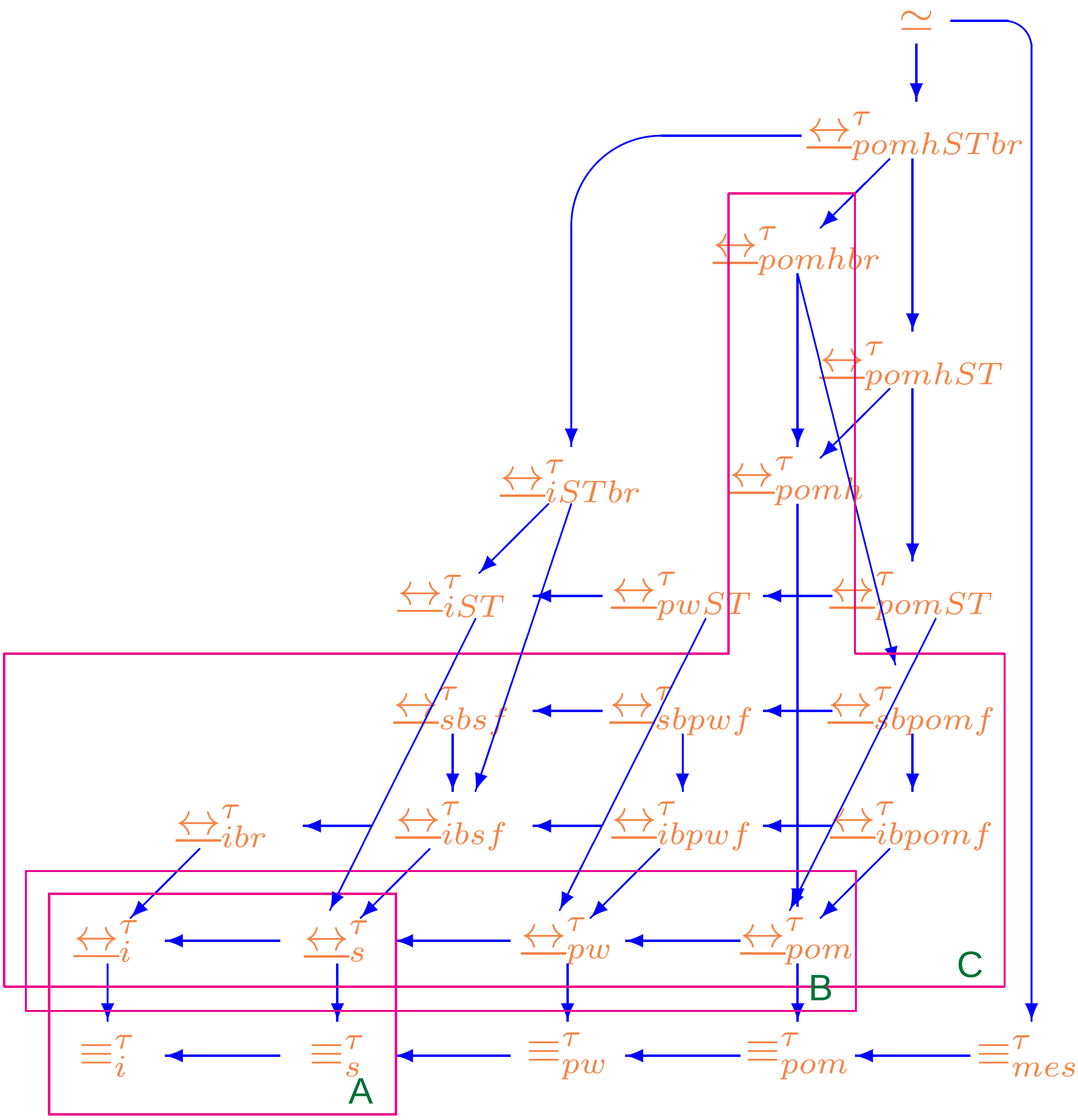
RBT1: The τ-equivalences between $\underline{\leftrightarrow}^{\tau}_{i}$ and $\underline{\leftrightarrow}^{\tau}_{pom}$ are not preserved by SM-refinements



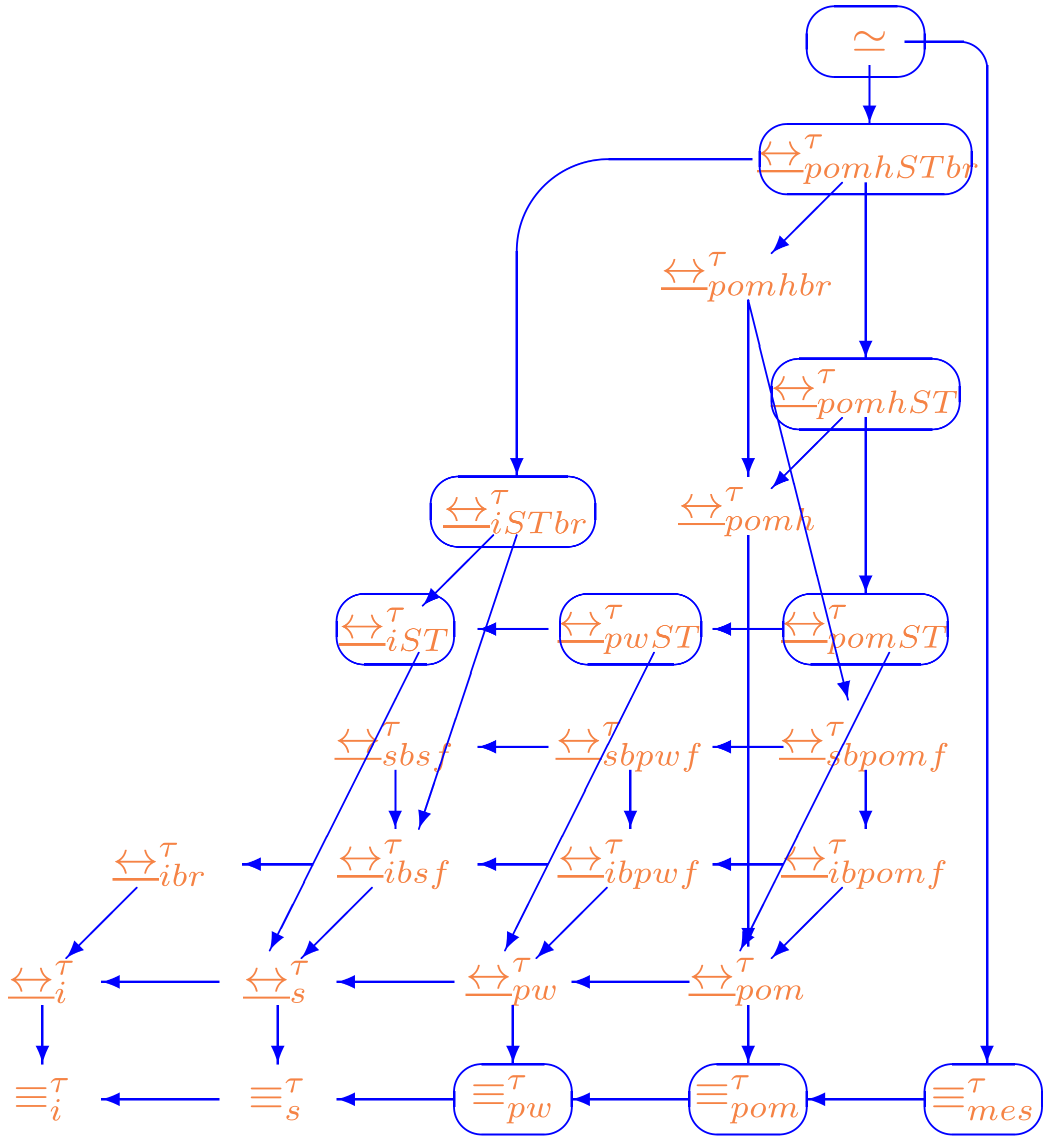
RBT2: The τ-equivalences between $\underline{\leftrightarrow}^{\tau}_{i}$ and $\underline{\leftrightarrow}^{\tau}_{pomhbr}$ are not preserved by SM-refinements

- In Figure RBT, $N \underline{\leftrightarrow}_s^\tau N'$, but $ref(N, c, D) \not\equiv_i^\tau ref(N', c, D)$, since only in $ref(N', c, D)$ the sequence of actions $c_1 abc_2$ can occur.
- In Figure RBT1, $N \underline{\leftrightarrow}_{pom}^\tau N'$, but $ref(N, a, D) \not\underline{\leftrightarrow}_i^\tau ref(N', a, D)$, since only in $ref(N', a, D)$ after occurrence of action a_1 action b can not occur.
- In Figure RBT2, $N \underline{\leftrightarrow}_{pomhbr}^\tau N'$, but $ref(N, a, D) \not\underline{\leftrightarrow}_i^\tau ref(N', a, D)$, since only in $ref(N', a, D)$ an action c_1 may occur so that after the corresponding action c_1 in the net N an action a may occur in such a way that the action b never occur.

Proposition 13 *[BDKP91,Dev92,Tar97] Let* $\star \in \{i, s\}$, $\star\star \in \{i, s, pw, pom, pomh, ibr, pomhbr, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$*. Then the* τ*-equivalences* $\equiv^\tau_\star$, $\underline{\leftrightarrow}^\tau_{\star\star}$ *are not preserved by SM-refinements.*



The τ-equivalences which are not preserved by SM-refinements



Preservation of the τ-equivalences by SM-refinements

Theorem 16 *Let* $\leftrightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$ *and* $\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, pomhST, ibr, pomhbr, iSTbr, pomhSTbr, mes, ibsf, ibpwf, ibpomf, sbsf, sbpwf, sbpomf\}$. *For nets* $N,\ N'$ *s.t.* $a \in L_N(T_N) \cap L_{N'}(T_{N'}) \cap Act$ *and SM-net* D

$$N \leftrightarrow_{\star} N' \ \Rightarrow\ ref(N, a, D) \leftrightarrow_{\star} ref(N', a, D)$$

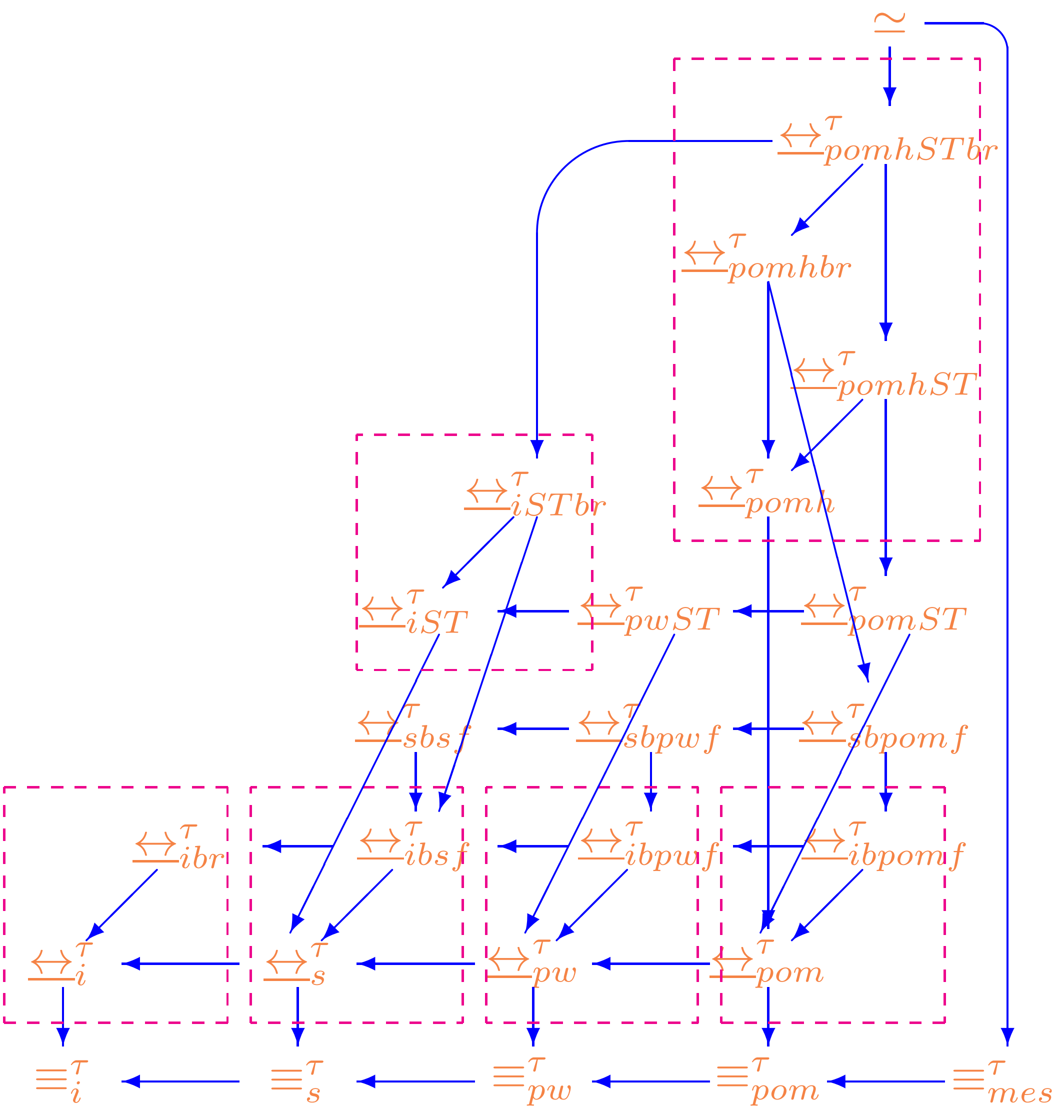
iff the equivalence $\leftrightarrow_{\star}$ *is in oval in the figure above.*

Net subclasses

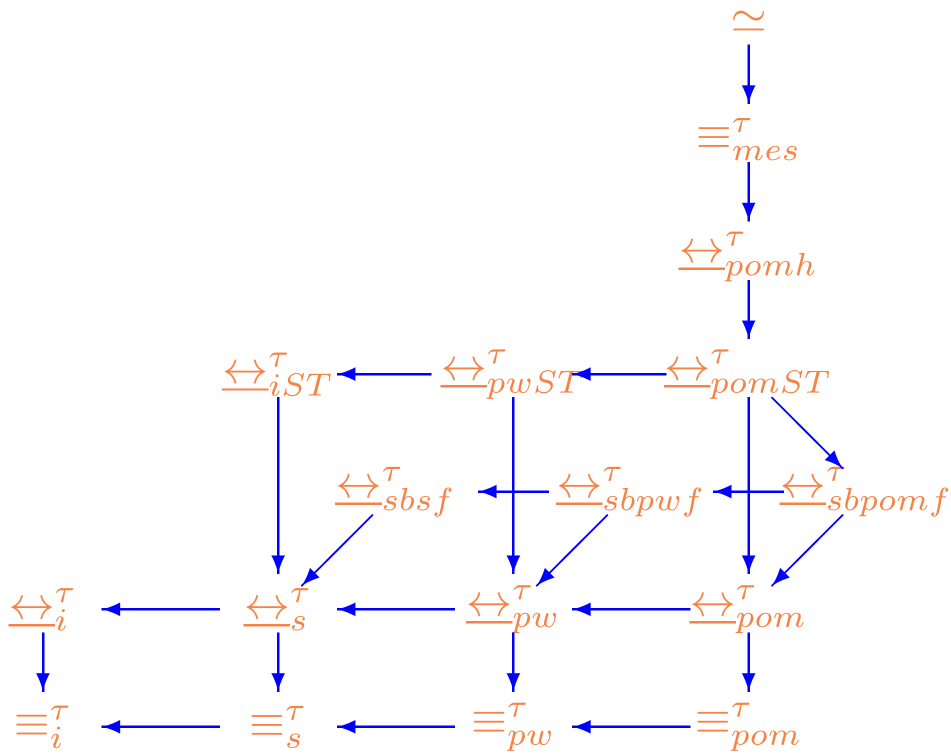
The τ-equivalences on nets without silent transitions

Proposition 14 *Let* $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$, $\star \in \{i, s, pw, pom, iST, pwST, pomST, mes, sbsf, sbpwf, sbpomf\}$, $\star\star \in \{s, pw, pom\}$. *For nets without silent transitions* N *and* N'

1. $N \leftrightarrow_\star N' \Leftrightarrow N \leftrightarrow_\star^\tau N'$ [Gla93,Tar97];
2. $N \underline{\leftrightarrow}_i N' \Leftrightarrow N \underline{\leftrightarrow}_{ibr}^\tau N'$ [Gla93];
3. $N \underline{\leftrightarrow}_{iST} N' \Leftrightarrow N \underline{\leftrightarrow}_{iSTbr}^\tau N'$ [Tar97];
4. $N \underline{\leftrightarrow}_{pomh} N' \Leftrightarrow N \underline{\leftrightarrow}_{pomhSTbr}^\tau N'$ [Tar97];
5. $N \underline{\leftrightarrow}_{\star\star} N' \Leftrightarrow N \underline{\leftrightarrow}_{ib\star\star f}^\tau N'$ [Tar97].



Merging of the τ-equivalences on nets without silent transitions



Interrelations of the τ-equivalences on nets without silent transitions

Theorem 17 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, \simeq\}$, $\star, \star\star \in \{\_, i, s, pw, pom, iST, pwST, pomST, pomh, ibr, mes, sbsf, sbpwf, sbpomf\}$. *For nets without silent transitions* N *and* N'

$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \twoheadleftarrow\!\!\twoheadrightarrow_{\star\star} N'$$

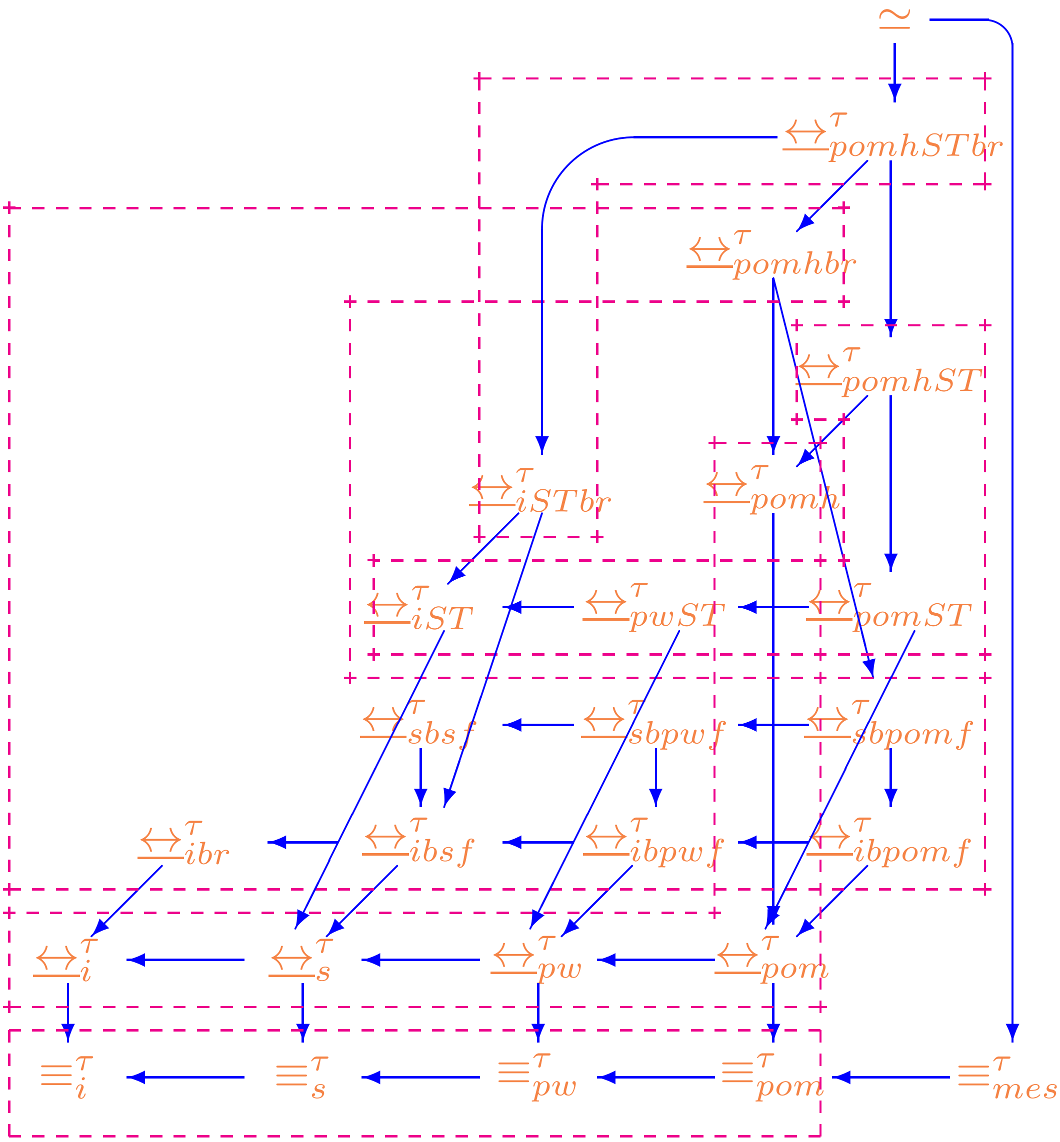
iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\twoheadrightarrow_{\star\star}$.

The τ-equivalences on sequential nets

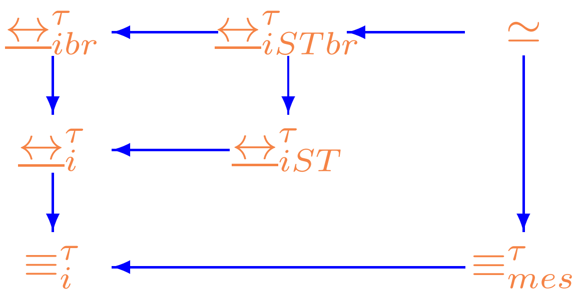
Definition 73 *A net* $N = (P_N, T_N, W_N, L_N, M_N)$ *is* sequential*, if* $\forall M \in RS(N)\ \neg\exists t, u \in T_N :\ {}^\bullet t + {}^\bullet u \subseteq M$.

Proposition 15 *For sequential nets* N *and* N'

1. $N \equiv_i^\tau N' \ \Leftrightarrow\ N \equiv_{pom}^\tau N'$ [Eng85];
2. $N \underline{\leftrightarrow}_i^\tau N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{pomh}^\tau N'$ [BDKP91];
3. $N \underline{\leftrightarrow}_{iST}^\tau N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{pomhST}^\tau N'$ [Tar98a];
4. $N \underline{\leftrightarrow}_{ibr}^\tau N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{pomhbr}^\tau N'$ [Tar98a];
5. $N \underline{\leftrightarrow}_{iSTbr}^\tau N' \ \Leftrightarrow\ N \underline{\leftrightarrow}_{pomhSTbr}^\tau N'$ [Tar98a].



Merging of the τ-equivalences on sequential nets



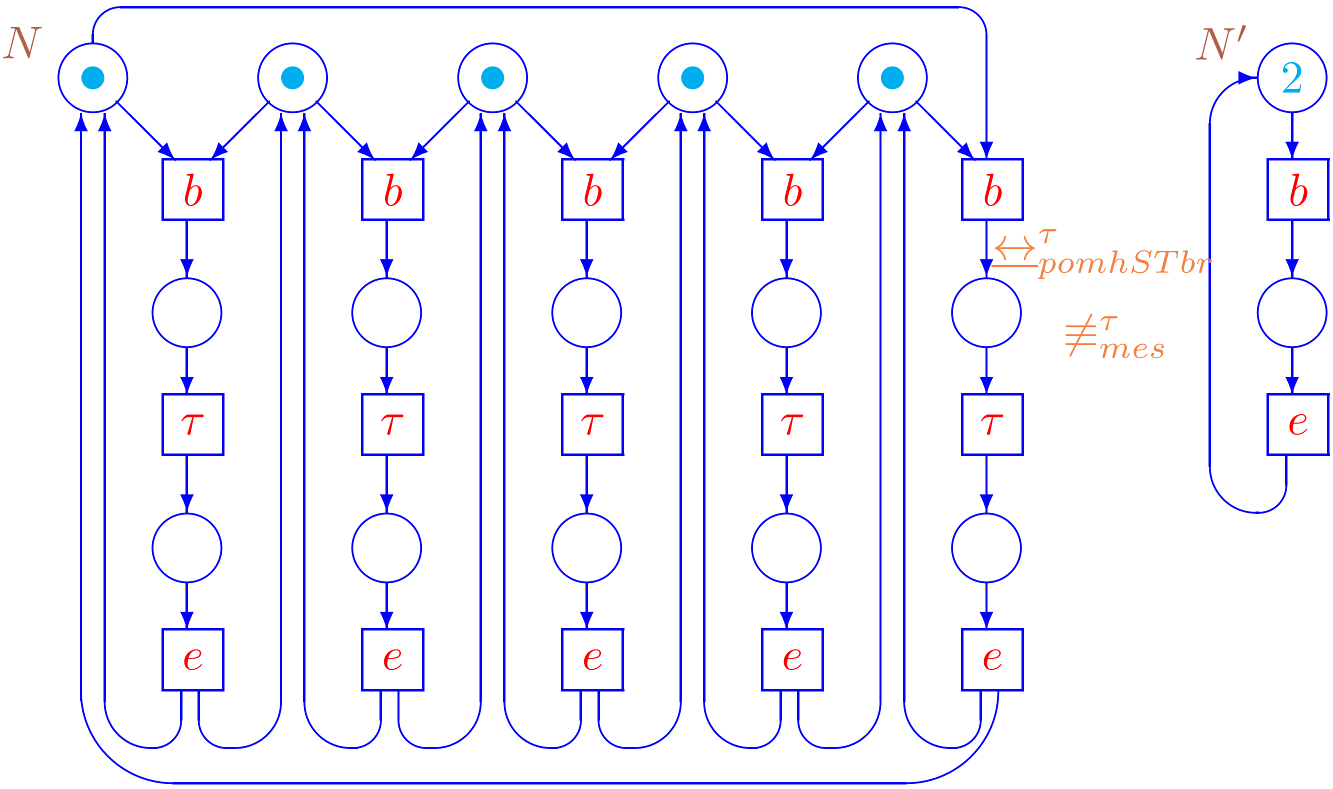
Interrelations of the τ-equivalences on sequential nets

Theorem 18 *Let* $\leftrightarrow, \twoheadleftrightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$, $\star, \star\star \in \{\_, i, iST, ibr, iSTbr, mes\}$. *For sequential nets* N *and* N'

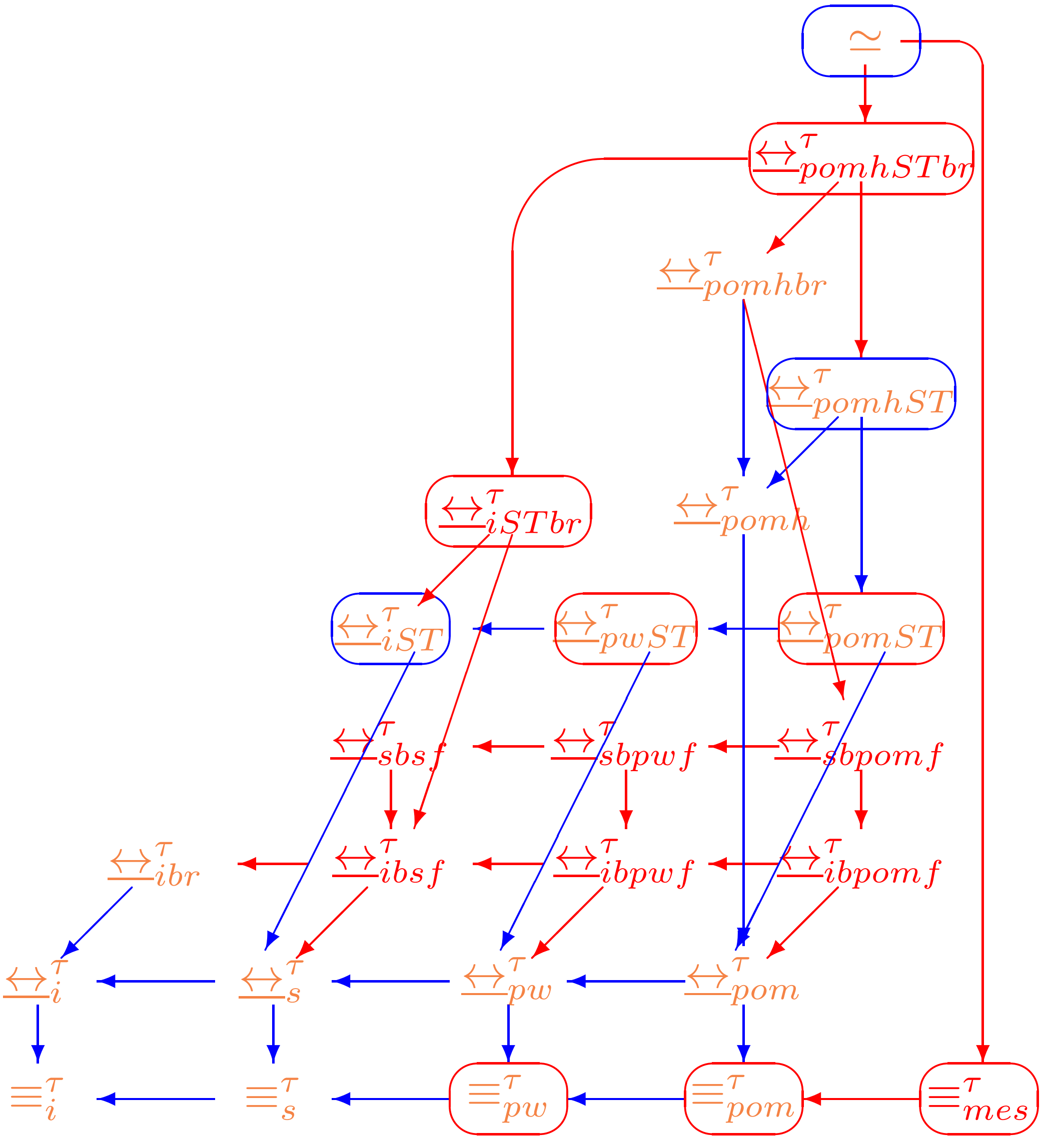
$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \twoheadleftrightarrow_{\star\star} N'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftrightarrow_{\star\star}$.

- In Figure BT2(a), $N \equiv^{\tau}_{mes} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{i} N'$.
- In Figure BT2(c), $N \underline{\leftrightarrow}^{\tau}_{i} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ibr} N'$.
- In Figure BT2(b), $N \underline{\leftrightarrow}^{\tau}_{i} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{iST} N'$.
- In Figure BT1(c), $N \underline{\leftrightarrow}^{\tau}_{iSTbr} N'$, but $N \not\equiv^{\tau}_{mes} N'$.



The complete and reduced PNs with invisible transitions of the abstract dining philosophers system



New results for the τ-equivalences

Decidability

Decidability results for the τ-equivalences

- $\equiv_i^\tau$
  - is decidable for:

    finite safe nets (EXPSPACE) [JM96].
  - is undecidable for:

    labeled nets [Jan95].
- $\equiv_s^\tau$
  - is decidable for:

    finite safe nets (EXPSPACE) [JM96].
- $\equiv_{pom}^\tau$
  - is decidable for:

    finite safe nets (EXPSPACE) [JM96].
- $\underline{\leftrightarrow}_i^\tau$
  - is decidable for:

    finite safe nets (DEXPTIME) [JM96].
  - is undecidable for:

    labeled nets [Jan95].
- $\underline{\leftrightarrow}_s^\tau$
  - is decidable for:

    finite safe nets (DEXPTIME) [JM96].
- $\underline{\leftrightarrow}_{pom}^\tau$
  - is decidable for:

    finite safe nets (DEXPTIME / EXPSPACE)[JM96].

- $\underline{\leftrightarrow}^{\tau}_{iST}$
  - is decidable for:

    bounded nets [Dev92];

    finite safe nets (DEXPTIME) [JM96].

- $\underline{\leftrightarrow}^{\tau}_{pomST}$
  - is decidable for:

    finite safe nets (DEXPTIME / EXPSPACE) [JM96].

- $\underline{\leftrightarrow}^{\tau}_{pomh}$
  - is decidable for:

    finite safe nets (DEXPTIME) [Vog91b,JM96].

- $\underline{\leftrightarrow}^{\tau}_{pomhST}$
  - is decidable for:

    finite safe nets (DEXPTIME) [Vog91b,JM96].

- $\underline{\leftrightarrow}^{\tau}_{ibr}$
  - is decidable for:

    finite safe nets (DEXPTIME) [JM96].

Open questions

Further research

τ-variants of place bisimulation equivalences.

- New equivalences.

  Interleaving *place τ-bisimulation equivalence* ($\sim_i^\tau$).

  Behavior preserving reduction of Petri nets with silent transitions [Aut93,APS94].

- Interleaving *branching place τ-bisimulation equivalence* ($\sim_{ibr}^\tau$).

- Non-interleaving variants of *place τ-bisimulations* ($\sim_s^\tau, \sim_{pw}^\tau$ and $\sim_{pom}^\tau$).

- Interrelations of the place τ-bisimulations.

  Whether any two of $\sim_i^\tau, \sim_s^\tau$ and $\sim_{pw}^\tau$ coincide?

  We have only counterexamples showing that

  $\sim_{ibr}^\tau$ and $\sim_{pom}^\tau$ do not imply each other and

  do not merge with any of three mentioned τ-equivalences.

- Interrelations of the place τ-bisimulations with the other τ-equivalences we proposed.

  We compared place equivalences with other ones on Petri nets without silent transitions [Tar98b].

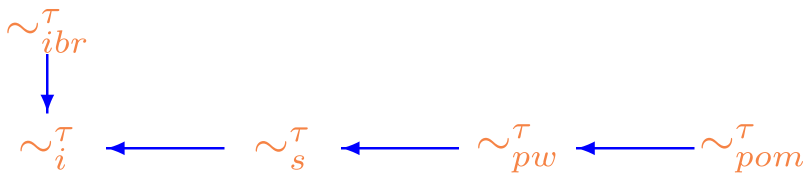
- Preservation of place τ-bisimulations by SM-refinements.

  We can show that no place τ-bisimulation relation is preserved by SM-refinements [Tar98b].

- Interrelations of place τ-bisimulations on net subclasses.

  On nets without silent transitions place τ-equivalences coincide with the corresponding relations that do not abstract of silent actions. In particular, $\sim^{\tau}_{ibr}$ merges with $\sim_i$.

  On sequential nets, all non-interleaving place relations coincide with interleaving ones: only $\sim^{\tau}_{i}$ and $\sim^{\tau}_{ibr}$ are remained.



Interrelations of place τ-bisimulation equivalences

Review of Stochastic Petri Nets

Abstract: Stochastic Petri nets (SPNs) are an extension of Petri nets (PNs) with an ability of performance (quantitative) analysis.

Behavior analysis is accomplished via stochastic process built on the basis of an SPN.

Kinds of SPNs: discrete and continuous timing, various time transition delays, inhibitor arcs and transition priorities.

Four well-known SPN classes are described: Discrete Time SPNs (DTSPNs), Continuous Time SPNs (CTSPNs), Generalized SPNs (GSPNs) and Deterministic SPNs (DSPNs).

Application examples and areas are presented.

Defining of labeling and equivalences is discussed.

Keywords: Inhibitor and priority Petri nets, stochastic Petri nets, probability distributions, Markov processes and chains, transient and stationary behaviour, labeling, equivalences.

Contents

- **Generalized stochastic Petri nets**
  - **Formal model of GSPNs**
  - **Analysis methods for GSPNs**
  - **Example of GSPNs**
  - **Summary for GSPNs**
- **Deterministic stochastic Petri nets**
  - **Formal model of DSPNs**
  - **Analysis methods for DSPNs**
  - **Example of DSPNs**
  - **Summary for DSPNs**
- **Overview and discussion**
  - **The results obtained**
  - **Advantages and disadvantages of stochastic Petri nets**

Introduction

Previous work

- Continuous time (subsets of $I\!R_+$): interleaving semantics
  - *Continuous time stochastic Petri nets (CTSPNs)* [Mol82,FN85]:
    exponential transition firing delays,
    Continuous time Markov chain (CTMC).
  - *Generalized stochastic Petri nets (GSPNs)* [MCB84,CMBC93]:
    exponential and zero transition firing delays,
    Semi-Markov chain (SMC).
  - *Extended generalized stochastic Petri nets (EGSPNs)* [HS89,MBBCCC89]:
    hyper-exponential or Erlang or phase and zero transition firing delays.
  - *Deterministic stochastic Petri nets (DSPNs)* [MC87,MCF90]:
    exponential and deterministic transition firing delays,
    Semi-Markov process (SMP), if no two deterministic transitions are enabled in any marking.
  - *Extended deterministic stochastic Petri nets (EDSPNs)* [GL94]:
    non-exponential and deterministic transition firing delays.
  - *Extended stochastic Petri nets (ESPNs)* [DTGN85]:
    arbitrary transition firing delays.

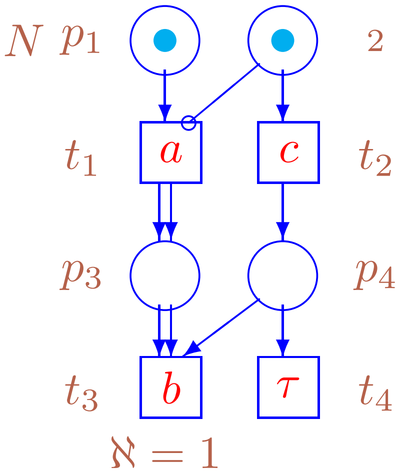
- Discrete time (subsets of $I\!N$): step semantics
  - *Discrete time stochastic Petri nets (DTSPNs)* [Mol85,ZG94]:
    geometric transition firing delays,
    Discrete time Markov chain (DTMC).
  - *Discrete time deterministic and stochastic Petri nets (DTDSPNs)* [ZFH01]:
    geometric and fixed transition firing delays,
    Semi-Markov chain (SMC).
  - *Discrete deterministic and stochastic Petri nets (DDSPNs)* [ZCH97]:
    phase and fixed transition firing delays,
    Semi-Markov chain (SMC).

Basic definitions

Petri nets with inhibitor arcs and priorities

Definition 74 *A* Petri net with inhibitor arcs and priorities (IPPN) *is a tuple* $N = (P_N, T_N, W_N, L_N, H_N, \aleph_N, M_N)$*:*

- $(P_N, T_N, W_N, L_N, M_N)$ *is a marked net;*
- $H_N : (P_N \times T_N) \to I\!N$ *is the* inhibitor arc weight *function;*
- $\aleph_N : T_N \to I\!N$ *is the* transition priority *function.*



Petri net with inhibitor arcs and priorities (IPPN)

Let N be an IPPN and $t \in T_N$. The *negative precondition* ${}^\circ t$ of t is the multiset $({}^\circ t)(p) = H_N(p, t)$.

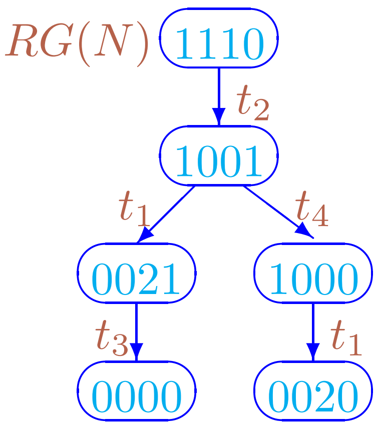
Let M be a marking of IPPN N. A transition t *has concession* in M, if ${}^\bullet t \subseteq M$ and $\forall p \in P_N \ ({}^\circ t)(p) > M(p)$.

$Concess(M)$ is the set of *all transitions having concession* in M.

A transition t is *enabled* in M, if $\forall u \in Concess(M) \ \aleph_N(t) \geq \aleph_N(u)$.

$Ena(M)$ is the set of *all transitions enabled* in M.

Marking change and other notions related to reachability, boundness, liveness and reversibility are defined as for marked nets.



Reachability graph of the IPPN

Foundations of probability theory

Probability theory: [Gne69,Bor86]. Formal methods: [Mar90,Her00,Hav00].

V is a set of *elementary events*. 2^V is the *set of all subsets (powerset)* of V.

Field of random events over V (σ-algebra of subsets of V) is a set $G \subseteq 2^V$:

1. $V \in G$;
2. $A \in G \Rightarrow \overline{A} \in G$ ($\overline{A}$ is a completion of A);
3. $A_1, A_2, \ldots \in G \Rightarrow \cap_{i=1}^{\infty} A_i, \cup_{i=1}^{\infty} A_i \in G$.

Probabilistic space is a triple $\Sigma = (V, G, \mathsf{P})$:

- V is a set of elementary events;
- $G \subseteq 2^V$ is a field of random events over V;
- $\mathsf{P} : G \to [0; 1]$ is a *probabilistic measure* on G.

Definition 75 *Let $\Sigma = (V, G, \mathsf{P})$ be a probabilistic space.* Random value (RV) *is a function $\xi : V \to \mathbb{R}$, s.t. $\forall x \in \mathbb{R}\ \{v \in V \mid \xi(v) < x\} \in G$ and $\forall x \in \mathbb{R}\ \mathsf{P}(\xi < x)$ is defined.*

Random values: discrete or continuous.

It depends on domain area (usually, $\mathbb{N}$ or $\mathbb{R}_+$).

Definition 76 Probability distribution function (PDSF) *of a RV ξ is:*

$$F_\xi(x) = \mathsf{P}(\xi < x).$$

PDSF of a continuous RV is a nonnegative nondecreasing function s.t. $\lim_{x\to-\infty} F_\xi(x) = 0$ and $\lim_{x\to\infty} F_\xi(x) = 1$.

Definition 77 Probability mass function (PMF) *of a discrete RV:*

$$p_\xi(x_i) = \mathsf{P}(\xi = x_i) \ (i \in I\!N).$$

Probability density function (PDF) *of a continuous RV ξ:*

$$f_\xi(x) = \frac{d}{dx}F_\xi(x),$$

if F_ξ is absolute continuous or could be differentiated on the whole its domain.

PMF of a discrete RV in vector form: $p_\xi = (p_\xi(x_1), p_\xi(x_2), \ldots)$.

PDF of a continuous RV is nonnegative and $\int_{-\infty}^{\infty} f_\xi(x)dx = 1$.

For discrete RV ξ PMF is

$$F_\xi(x_n) = \sum_{i=0}^{n-1} p_\xi(x_i).$$

For continuous RV ξ PDF is

$$F_\xi(x) = \int_{-\infty}^{x} f_\xi(y)dy.$$

Definition 78 Mean value (MV) *of a discrete RV* ξ *is*

$$\mathsf{M}(\xi) = \sum_{i=0}^{\infty} x_i p_\xi(x_i),$$

if the series is absolute summarizable.

Mean value (MV) *of a continuous RV* ξ *is*

$$\mathsf{M}(\xi) = \int_{-\infty}^{\infty} x f_\xi(x) dx,$$

if there exists the integral $\int_{-\infty}^{\infty} |x| f_\xi(x) dx$.

Definition 79 Variance *of RV* ξ *is*

$$\mathsf{D}(\xi) = \mathsf{M}((\xi - \mathsf{M}(\xi))^2).$$

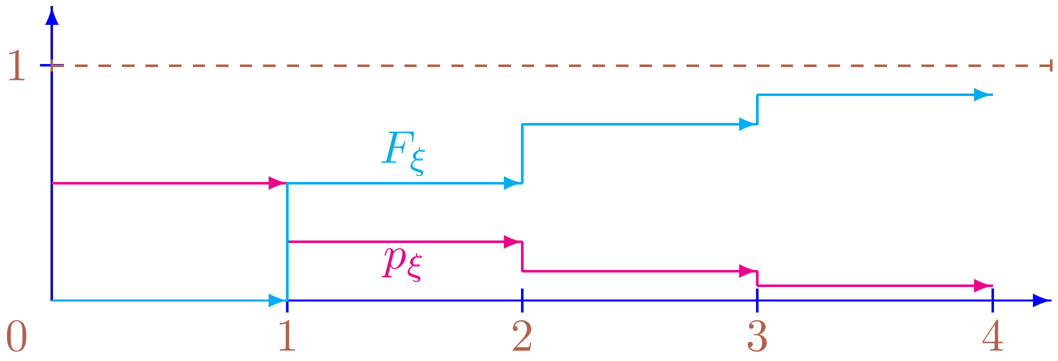
For discrete RV ξ its variance is

$$\mathsf{D}(\xi) = \sum_{i=0}^{\infty} (x_i - \mathsf{M}(\xi))^2 p_\xi(x_i).$$

For continuous RV ξ its variance is

$$\mathsf{D}(\xi) = \int_{-\infty}^{\infty} (x - \mathsf{M}(\xi))^2 f_\xi(x) dx.$$

The following holds: $\mathsf{D}(\xi) = \mathsf{M}(\xi^2) - (\mathsf{M}(\xi))^2$.



PDSF and PMF graphics of geometric distribution with $\rho = \frac{1}{2}$

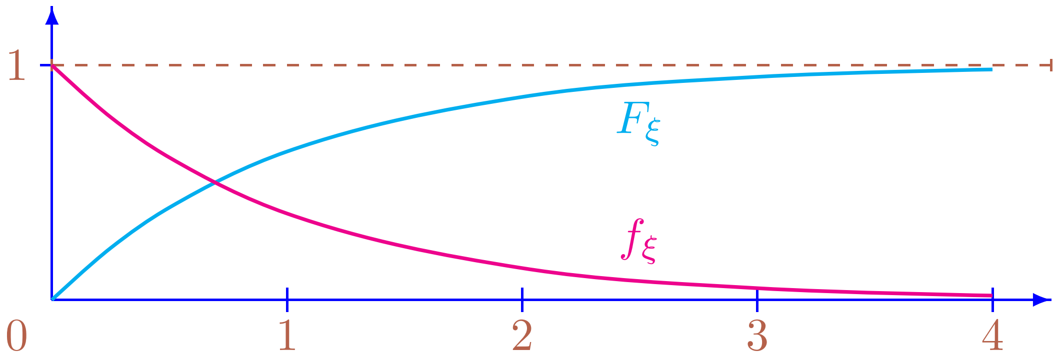
Discrete *geometric distribution*:

$$F_\xi(n) = \mathsf{P}(\xi < n) = 1 - \rho^n \ (\rho \in (0;1),\ n \in \mathbb{N})$$

$$p_\xi(i) = \mathsf{P}(\xi = i) = \rho^i(1-\rho) \ (i \in \mathbb{N})$$

$$\mathsf{M}(\xi) = \sum_{i=0}^{\infty} i p_\xi(i) = \frac{\rho}{1-\rho}$$

$$\mathsf{D}(\xi) = \sum_{i=0}^{\infty} (i - \mathsf{M}(\xi))^2 p_\xi(i) = \frac{\rho}{(1-\rho)^2}$$



PDSF and PDF graphics of exponential distribution with $\lambda = 1$

Continuous *exponential distribution*:

$$F_\xi(x) = \mathsf{P}(\xi < x) = 1 - e^{-\lambda x} \ (\lambda \in \mathbb{R},\ x \geq 0)$$

$$f_\xi(x) = \frac{d}{dx} F_\xi(x) = \lambda e^{-\lambda x} \ (x \geq 0)$$

$$\mathsf{M}(\xi) = \int_0^\infty x f_\xi(x) dx = \frac{1}{\lambda}$$

$$\mathsf{D}(\xi) = \int_0^\infty (x - \mathsf{M}(\xi))^2 f_\xi(x) dx = \frac{1}{\lambda^2}$$

Stochastic processes

Definitions of Stochastic processes and Markov chains: [Gne69,Bor86,Mar90,Her00].

Let $(\xi_1, \ldots, \xi_n)$ be a vector of n RVs.

Joint PDSF is

$$F_\xi(x) = \mathsf{P}(\xi_1 < x_1, \ldots, \xi_n < x_n).$$

Joint PDF is

$$f_\xi(x) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} F_\xi(x).$$

Definition 80 *Let Δ be a set of parameters (indices) and S be a set of states. Stochastic process is a set of RVs $\{\xi(\delta) \mid \delta \in \Delta\} \subseteq S$.*

Usual interpretation: δ is time, Δ is a time scale (discrete $\mathbb{N}$ or continuous $\mathbb{R}_+$), S is a set of all states of RV $\xi(\delta)$.

Stochastic processes: discrete or continuous by type of set of states.

Stochastic chain is a stochastic process with discrete set of states.

Stochastic chains: discrete or continuous, depends on time scale.

Stochastic process is *stationary*, if its properties do not change with simultaneous shift of all states along time scale.

Probabilistic characterization of stochastic processes: hard task.

Special classes of stochastic processes:

- Gauss: multi-factor processes of nature;
- Markov: dynamic of resource sharing systems.

Definition 81 *Let for sets of indices Δ, states S and numbers $i \in I\!N$ holds $\delta_0, \ldots, \delta_{i-1}, \delta_i \in \Delta\ (\delta_0 < \ldots < \delta_{i-1} < \delta_i),\ s_0, \ldots, s_{i-1}, s_i \in S$.*
Markov process (MP) *is a stochastic process with*
Markov property (post-effect absence, memoryless)

$$\mathsf{P}(\xi(\delta_i) = s_i \mid \xi(\delta_0) = s_0, \ldots, \xi(\delta_{i-1}) = s_{i-1}) =$$

$$\mathsf{P}(\xi(\delta_i) = s_i \mid \xi(\delta_{i-1}) = s_{i-1}).$$

Markov chain (MC) is a MP with a discrete set of states.

Discrete time MC (DTMC) is a MC with state changes on finite of countable sets.

Continuous time MC (CTMC) is a MC with state changes on intervals.

MC is *(time-)homogeneous*, if state change probabilities do not depend on moments when they happen ($\delta \in I\!N$ for DTMCs or $\delta \in I\!R_+$ for CTMCs):

$$\mathsf{P}(\xi(\delta_i) = s_i \mid \xi(\delta_j) = s_j) = \mathsf{P}(\xi(\delta_i + \delta) = s_i \mid \xi(\delta_j + \delta) = s_j).$$

Furthermore, all MCs are considered to be homogeneous.

Discrete time Markov chains

Geometric distribution is the only discrete one with memoryless property

$$\mathsf{P}(\xi = i + j \mid \xi > j) = \mathsf{P}(\xi = i)\ (i, j \in \mathbb{N},\ i \geq 1).$$

Complete probabilistic description of a DTMC: PMF over set of states $S = \{s_1, \ldots, s_n\}$ at the initial time moment and one-step (along discrete time scale) transition probabilities $\rho_{ij}\ (1 \leq i, j \leq n)$ from s_i to s_j.

(One-step) transition probability diagram (TPD) of a DTMC is a labeled oriented graph with vertices corresponding to states from S, and arcs labeled by one-step transition probabilities $\rho_{ij}\ (1 \leq i, j \leq n)$. TPD is a graphical representation of a DTMC.

(One-step) transition probability matrix (TPM) of a DTMC is a matrix $\mathbf{P}$ of $n \times n$ over $[0; 1]$ with one-step transition probabilities $\rho_{ij} = \mathsf{P}(\xi(1) = s_j \mid \xi(0) = s_i)\ (1 \leq i, j \leq n)$ as elements.

Matrix $\mathbf{P}^k$ has k-step transition probabilities as elements $\rho_{ij}(k) = \mathsf{P}(\xi(k) = s_j \mid \xi(0) = s_i)\ (1 \leq i, j \leq n)$. $\mathbf{P}^0 = \mathbf{I}$.

Chapman-Kolmogorov equation establishes a relation between $k + l$-step probabilities $(k, l \in \mathbb{N})$ and k-step and l-step ones:

$$\mathbf{P}^{k+l} = \mathbf{P}^k \mathbf{P}^l.$$

Probability to stay in s_i during k steps and state change at step $k + 1$ is $\rho_{ii}^k (1 - \rho_{ii})$.

Change a state: success. Stay in a state: failure.

Sojourn time in states of a DTMC is geometrically distributed.

A *DTMC solution*: PMF calculation at arbitrary time moment or

at equilibrium conditions.

Transient behaviour: transient states.

Let $\psi_i(k) = \mathsf{P}(\xi(k) = s_i)\ (1 \le i \le n)$ be probability to enter into s_i during k steps, $\psi(k) = (\psi_1(k), \ldots, \psi_n(k))$ be its PMF at the moment k, its (*transient PMF*), and $\mathbf{P}$ be TPM.

Transient PMF is a solution of equation system

$$\psi(k) = \psi(0)\mathbf{P}^k.$$

Long time system behaviour: state probabilities could stabilize (equilibrate).

Stationary behaviour: steady states.

DTMC is *ergodic*, if steady state PMF exists.

Let $\psi_i = \lim_{k\to\infty} \psi_i(k)\ (1 \le i \le n)$ be a probability for an ergodic DTMC to be in steady state s_i, $\psi = (\psi_1, \ldots, \psi_n)$ be its *steady-state PMF*, and $\mathbf{P}$ be TPM.

Steady state PMF is a solution of equation system

$$\begin{cases} \psi(\mathbf{P} - \mathbf{I}) = \mathbf{0} \\ \psi\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of order n, $\mathbf{0}$ is a vector of n values 0, $\mathbf{1}$ is that of n values 1.

Steady state existence (ergodicity) conditions for a DTMC.

A state $s_i\ (1 \leq i \leq n)$ of a DTMC *non-essential*, if $\exists j\ (1 \leq j \leq n)\ \exists k \in I\!N\ \rho_{ij}(k) > 0$ and $\forall l \in I\!N\ \rho_{ji}(l) = 0$. Otherwise s_i is *essential*.

Essential states s_i and $s_j\ (1 \leq i, j \leq n)$ of a DTMC are *communicating*, if $\exists k, l \in I\!N\ \rho_{ij}(k) > 0$ and $\rho_{ji}(l) > 0$. The set of essential states is partitioned by non-intersecting classes of communicating states $S_1, \ldots, S_m$.

If a class of communicating states $S_c\ (1 \leq c \leq m)$ contains the only state s_i, it is *absorbing*. In this case $\lim_{k\to\infty} \rho_{ii}(k) = 1$ and $\forall j \neq i\ (1 \leq j \leq n)\ \lim_{k\to\infty} \rho_{ij}(k) = 0$.

A DTMC is *irreducible*, if its state set is the only class of communicating essential states, and *reducible* otherwise.

A probability for system starting from state $s_i\ (1 \leq i \leq n)$ to *return to it first after k steps* is

$$Return_i(k) = \mathsf{P}(\xi(k) = s_i, \xi(k-1) \neq s_i, \ldots, \xi(1) \neq s_i \mid \xi(0) = s_i).$$

A probability for system starting from state $s_i\ (1 \leq i \leq n)$ to *return to it eventually* is

$$Return_i = \sum_{k=1}^{\infty} Return_i(k).$$

A state $s_i\ (1 \leq i \leq n)$ of an irreducible DTMC is *recurrent*, if $Return_i = 1$, and *non-recurrent (transient)*, if $Return_i < 1$.

A state $s_i\ (1 \leq i \leq n)$ of an irreducible DTMC is *null*, if $\lim_{k\to\infty} \rho_{ii}(k) = 0$, and *non-null (positive)* otherwise.

A state $s_i \ (1 \leq i \leq n)$ of an irreducible DTMC is *periodic* with period $d_i \in I\!N \ (d_i \geq 2)$, if d_i is a maximal common divisor (MCD) of numbers $\{k \in I\!N \mid Return_i(k) > 0\}$. A state s_i is *aperiodic* otherwise.

For an irreducible DTMC there exists $2^3 = 8$ types of states.

The following theorem: only 6 types exists.

Theorem 19 *[Bor86] In an irreducible DTMC non-recurrent state is null.*

State classification by two parameters.

1. *Asymptotic properties*: non-recurrent, recurrent null, non-null.

2. *Arithmetic properties*: periodic, aperiodic.

Theorem 20 (Solidarity) *[Bor86] In an irreducible DTMC all states are of the same type: recurrent or null or periodic with period $d \in I\!N \ (d \geq 2)$.*

An irreducible DTMS is *periodic*, if all its states are periodic with period $d \in I\!N \ (d \geq 2)$, and *aperiodic* otherwise.

Theorem 21 (Ergodicity) *[Bor86] There is a state s_i of an irreducible and aperiodic DTMC s.t. $\sum_{k=1}^{\infty} kReturn_i(k) < \infty$ iff*

$\forall i, j \ (1 \leq i, j \leq n)$ there is an independent from $\psi(0)$ and a unique steady-state PMF $\psi = (\psi_1, \ldots, \psi_n)$:

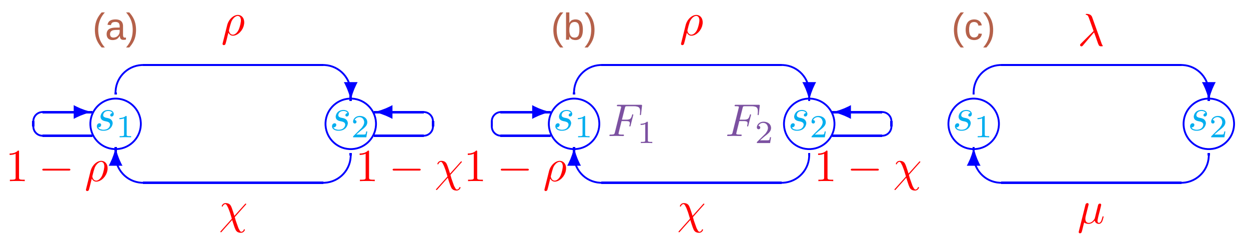
$$\lim_{k\to\infty} \rho_{ij}(k) = \lim_{k\to\infty} \psi_j(k) = \psi_j > 0.$$

A DTMC is *ergodic*, if it has steady-state PMF.

A finite DTMC is ergodic iff it is irreducible and aperiodic.

TPM of DTMC in Figure MC(a):

$$\mathbf{P} = \begin{pmatrix} 1-\rho & \rho \\ \chi & 1-\chi \end{pmatrix}.$$



MC: DTMC, SMC and CTMC

Semi-Markov chains

Semi-Markov chains (SMCs) are an extension of DTMCs: positive sojourn time with PDSF $F_i(\delta)$ and PDF $f_i(\delta)$ are associated with each state s_i.

Complete probabilistic description of a SMC: PMF over set of states $S = \{s_1, \ldots, s_n\}$ at the initial time moment, one-step (along discrete time scale) transition probabilities ρ_{ij} $(1 \leq i, j \leq n)$ from s_i to s_j and vector of PDSFs for sojourn time in states $F(\delta) = (F_1(\delta), \ldots, F_n(\delta))$.

(One-step) transition probability diagram (TPD) of an SMC is a labeled oriented graph with vertices corresponding to states from S, with the information on PDSFs for sojourn time in the states, and arcs labeled by one-step transition probabilities p_{ij} $(1 \leq i, j \leq n)$. TPD is a graphical representation of an SMC.

Interpretation of SMCs.

State change moments: as DTMC with TPM $\mathbf{P}$.

Coming in a state s_i: the next state change is only possible after time distributed with PDSF $F_i(\delta)$.

SMC solution: PMF calculation at arbitrary time or at equilibrium conditions.

Calculation of steady-state PMF for SMC.

1. Find steady-state PMF $\psi = (\psi_1, \ldots, \psi_n)$ for *embedded DTMC (EDTMC)* with TPM $\mathbf{P}$.

2. Find average sojourn time in states s_i $(1 \leq i \leq n)$ as

$$SJ(s_i) = \int_0^\infty \delta f_i(\delta) d\delta.$$

3. Find steady-state PMF $\varphi = (\varphi_1, \ldots, \varphi_n)$ for SMC as

$$\varphi_i = \frac{\psi_i SJ(s_i)}{\sum_{j=1}^n \psi_j SJ(s_j)}.$$

TPM of SMC in Figure MC(b):

$$\mathbf{P} = \begin{pmatrix} 1-\rho & \rho \\ \chi & 1-\chi \end{pmatrix}.$$

Continuous time Markov chains

Exponential distribution is the only continuous one with memoryless property

$$\mathsf{P}(\xi \geq x + d \mid \xi \geq d) = \mathsf{P}(\xi \geq x) \ (x, d \in I\!\!R_+).$$

A parameter λ is a rate of a CTMC transition.

Complete probabilistic description of a CTMC: PMF over set of states $S = \{s_1, \ldots, s_n\}$ at the initial time moment and transition rates $q_{ij} \ (1 \leq i, j \leq n)$ from s_i to s_j.

Transition rate diagram (TRD) of a CTMC is a labeled oriented graph with vertices corresponding to states from S, and arcs labeled by transition rates $q_{ij} \ (1 \leq i, j \leq n)$. TRD is a graphical representation of a CTMC.

Transition rate matrix (TRM) or *infinitesimal generator* of a CTMC is a matrix $\mathbf{Q}$ of $n \times n$ over $I\!\!R_+$ with transition rates $\rho_{ij} = \mathsf{P}(\xi(1) = s_j \mid \xi(0) = s_i) \ (1 \leq i, j \leq n)$ as non-main-diagonal elements. Each main-diagonal element is a negative sum of all other elements of the corresponding line.

A *CTMC solution*: PMF calculation at arbitrary time moment or at equilibrium conditions.

Let $\varphi_i(\delta) = \mathsf{P}(\xi(\delta) = s_i) \ (1 \leq i \leq n)$ be a probability for a CTMC to be in s_i at the moment δ, $\varphi(\delta) = (\varphi_1(\delta), \ldots, \varphi_n(\delta))$ be its PMF at the moment δ (*transient PMF*), and $\mathbf{Q}$ be TRM.

Transient PMF is calculated as

$$\varphi(\delta) = \varphi(0)e^{\mathbf{Q}\delta},$$

where $e^{\mathbf{Q}\delta}$ is matrix exponential $e^{\mathbf{Q}\delta} = \sum_{k=0}^{\infty} \frac{(\mathbf{Q}\delta)^k}{k!}$.

A CTMC is *ergodic*, if its steady-state PMF exists.

Let $\varphi_i = \lim_{\delta\to\infty} \varphi_i(\delta)\ (1 \le i \le n)$ be probability for an ergodic CTMC to be in steady state s_i, $\varphi = (\varphi_1, \ldots, \varphi_n)$ be its *steady-state (equilibrium) PMF*, and $\mathbf{Q}$ be TRM.

Stationary PMF is a solution of equation system

$$\begin{cases} \varphi\mathbf{Q} = \mathbf{0} \\ \varphi\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{0}$ is a row vector of n values 0, $\mathbf{1}$ is that of n values 1.

Steady state existence (ergodicity) conditions for a CTMS: as for DTMC.

TRM of CTMC in Figure MC(c):

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

General analysis of Markov chains

1. Find all states s_i $(1 \leq i \leq n)$ from S.

2. DTMC: calculate one-step transition probabilities ρ_{ij} from its state s_i to s_j $(1 \leq i, j \leq n)$.

   SMC: calculate one-step transition probabilities ρ_{ij} from state of EDTMC s_i to s_j $(1 \leq i, j \leq n)$.

   CTMC: calculate transition rates q_{ij} from its state s_i to s_j $(1 \leq i, j \leq n)$.

3. DTMC: iteration system of linear equations to analyze its transient behaviour.

   SMC: iteration system of linear equations to analyze transient behaviour of EDTMC.

   CTMC: matrix exponent system of linear equations to analyze its transient behaviour.

4. DTMC: fixpoint system of linear equations to analyze its stationary behaviour

   SMC: fixpoint system of linear equations to analyze stationary behaviour of EDTMC.

   CTMC: equilibrium system of linear equations to analyze its stationary behaviour.

5. DTMC and CTMC: calculate state probabilities analytically or with numerical methods.

   SMC: calculate state probabilities of EDTMC analytically or with numerical methods, weight them with average sojourn time in states and normalize. The result are state probabilities of SMC.

6. Calculate standard performance indices using state probabilities (throughout, waiting, response time, etc.).

Solution methods for Markov chains [Hav01]

Let a MC has n states.

- Transient state probabilities
  - Runge-Kutta methods
  - Uniformization (randomization, Jensen's method): $O(\lambda t n)$ (sparse matrix, λ is the *uniformization rate*, t is a current time) or $O(n^2)$ (general case)
- Stationary state probabilities
  - Direct
    - ∗ Gaussian elimination: $O(n^3)$
    - ∗ LU decomposition: $O(n^3)$
  - Iterative
    - ∗ The power method: $O(n^2)$
    - ∗ The Jakobi method: $O(n^2)$
    - ∗ The Gauss-Seidel method: $O(n^2)$
    - ∗ The successive over-relaxation (SOR): $O(n^2)$

Discrete time stochastic Petri nets

Formal model of DTSPNs

Definition 82 *A discrete time SPN (DTSPN) is a tuple* $N = (P_N, T_N, W_N, \Omega_N, M_N)$*:*

- (P_N, T_N, W_N, M_N) *ia an unlabeled PN;*
- $\Omega_N : T_N \rightarrow (0; 1)$ *is the transition conditional probability function.*

Concurrent transition firings at discrete time moments.

DTSPNs have *step* semantics.

Let M be a marking of a DTSPN $N = (P_N, T_N, W_N, \Omega_N, M_N)$. Then $t \in Ena(M)$ fires in the next time moment with probability $\Omega_N(t)$, if no other transition is enabled in M.

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U is ready for firing in M*:

$$PF(U, M) = \prod_{t \in U} \Omega_N(t) \cdot \prod_{u \in Ena(M) \setminus U} (1 - \Omega_N(u)).$$

In the case $U = \emptyset$ we define

$$PF(\emptyset, M) = \begin{cases} \prod_{u \in Ena(M)} (1 - \Omega_N(u)) & Ena(M) \neq \emptyset; \\ 1 & \text{otherwise}. \end{cases}$$

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U fires in M*:

$$PT(U, M) = \frac{PF(U, M)}{\sum_{\{V \mid {}^\bullet V \subseteq M\}} PF(V, M)}.$$

If $U = \emptyset$ then $M = \widetilde{M}$ and

$$PT(\emptyset, M) = \frac{PF(\emptyset, M)}{\sum_{\{V \mid {}^\bullet V \subseteq M\}} PF(V, M)}.$$

Analysis methods for DTSPNs

For all DTSPN $N = (P_N, T_N, W_N, \Omega_N, M_N)$ we have
$RS(N) = RS(P_N, T_N, W_N, M_N)$: reachability sets of a DTSPN and its underlying PN coincide.

Qualitative properties of a DTSPNs: analysis of reachability graphs for underlying PNs.

Quantitative properties of a DTSPNs: analysis of DTMCs for *bounded* and *live* DTSPNs.

DTMC $DTMC(N)$ *corresponding* to a DTSPN N:

1. Set of states $S = RS(N)$.
2. Probability ρ_{ij} $(1 \leq i, j \leq n = |S|)$ of state change from M_i to M_j is

$$\rho_{ij} = \sum_{\{U \mid M_i \stackrel{U}{\to} M_j\}} PT(U, M_i);$$

3. the initial state $s_1 = M_N$.

(One-step) TPM $\mathbf{P}$ for $DTMC(N)$ with elements ρ_{ij}.

Transient (k-step) PMF for DTMC $DTMC(N)$:

$$\psi(k) = \psi(0)\mathbf{P}^k,$$

where $k \in I\!N$ and $\psi(0) = (\psi_1(0), \ldots, \psi_n(0))$ is a probability of the initial distribution, $\psi_i(0)$ $(1 \le i \le n)$:

$$\psi_i(0) = \begin{cases} 1 & M_i = M_N \\ 0 & \text{otherwise} \end{cases}.$$

Here $\psi(k) = (\psi_1(k), \ldots \psi_n(k))$ is a transient PMF over k-step reachable markings, and $\psi_i(k)$ $(1 \le i \le n)$ are transient probabilities of M_i.

Steady state PMF for DTMC $DTMC(N)$:

$$\begin{cases} \psi(\mathbf{P} - \mathbf{I}) = \mathbf{0} \\ \psi\mathbf{1}^T = 1 \end{cases}.$$

Here $\psi = (\psi_1, \ldots, \psi_n)$ is a steady-state PMF over reachable markings, and ψ_i $(1 \le i \le n)$ are steady-state probabilities of M_i.

Performance indices for DTSPNs.

- *Average sojourn time in a marking* M_i is the mean of the residence time RV ξ with PMF $p_\xi(k) = \rho_{ii}^{k-1}(1-\rho_{ii})\ (k \geq 1)$.

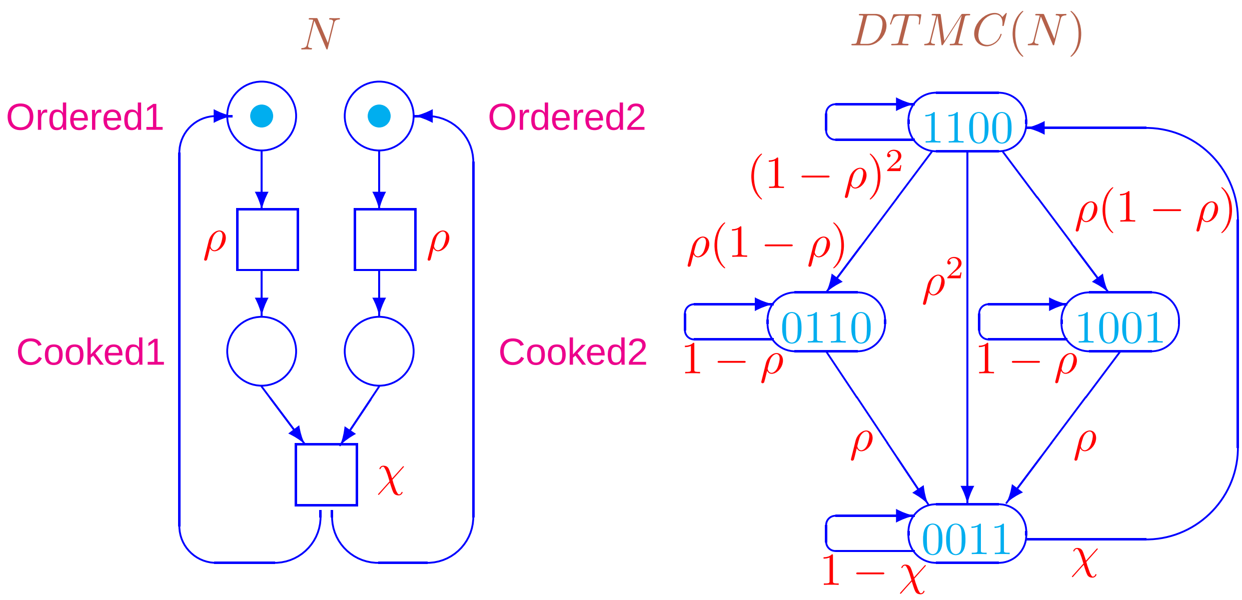
$$SJ(M_i) = \mathsf{M}(\xi) = \frac{1}{1-\rho_{ii}}.$$

- *Fraction of residence time in a marking* M_i is ψ_i.

- *Average recurrence time in a marking* M_i is inverse to the fraction of residence time in it:

$$RC(M_i) = \frac{1}{\psi_i}.$$

Example of DTSPNs



DTSPN of restaurant and its DTMC

Restaurant with two-course dinner: DTSPN N.

First, the dinner is ordered.

When *both* dishes have been cooked, they are served.

Cooking processes of the dishes are *independent*.

Cooking time is about *equal*.

Places: $P_N = \{p_1, p_2, p_3, p_4\}$.

Transitions: $T_N = \{t_1, t_2, t_3\}$.

Conditional probabilities: $\Omega_N(t_1) = \Omega_N(t_2) = \rho$, $\Omega_N(t_3) = \chi$.

Interpretation of places.

p_1: first dish has been ordered (Ordered1).

p_2: second dish has been ordered (Ordered2).

p_3: first dish has been cooked (Cooked1).

p_4: second dish has been cooked (Cooked2).

Interpretation of markings.

$M_1 = (1, 1, 0, 0)$: both dishes have been ordered (Ordered).

$M_2 = (0, 1, 1, 0)$: first dish has been cooked (Cooked1).

$M_3 = (1, 0, 0, 1)$: second dish has been cooked (Cooked2).

$M_4 = (0, 0, 1, 1)$: both dishes have been cooked (Cooked).

Interpretation of transitions and their conditional probabilities.

1. When both dishes have been ordered, first dish is cooked:

   t_1 with probability ρ.

2. When both dishes have been ordered, second dish is cooked:

   t_2 with probability ρ.

3. When both dishes have been cooked, they are served:

   t_3 with probability χ.

One-step TPM for DTMC $DTMC(N)$ is

$$\mathbf{P} = \begin{pmatrix} (1-\rho)^2 & \rho(1-\rho) & \rho(1-\rho) & \rho^2 \\ 0 & 1-\rho & 0 & \rho \\ 0 & 0 & 1-\rho & \rho \\ \chi & 0 & 0 & 1-\chi \end{pmatrix}$$

Steady-state PMF for DTMC $DTMC(N)$ is a solution of equation system

$$\begin{cases} \rho(2-\rho)\psi_1 = \chi\psi_4 \\ \rho(1-\rho)\psi_1 = \rho\psi_2 \\ \rho(1-\rho)\psi_1 = \rho\psi_3 \\ \rho^2\psi_1 + \rho\psi_2 + \rho\psi_3 = \chi\psi_4 \\ \psi_1 + \psi_2 + \psi_3 + \psi_4 = 1 \end{cases}$$

The result is

$$\psi = \frac{1}{\chi(3-2\rho)+\rho(2-\rho)}(\chi, \chi(1-\rho), \chi(1-\rho), \rho(2-\rho)).$$

The case $\rho = \chi = \frac{1}{2}$:

$$\psi = \frac{1}{7}(2, 1, 1, 3).$$

Performance indices.

- *Average dinner delivery time* is $SJ(M_4) = \frac{1}{1-PT(\emptyset, M_4)} = \frac{1}{1-\frac{1}{2}} = 2$.
- *Dinner delivery time fraction* is $\psi_4 = \frac{3}{7}$.
- *Average service time for a visitor* is $RC(M_1) = \frac{1}{\psi_1} = \frac{7}{2} = 3\frac{1}{2}$.

Summary for DTSPNs

DTSPNs are standardly unlabeled:

acceptable to model logically different activities:

transitions t_1 and t_3 of DTSPN from restaurant example;

not acceptable to model logically equal activities:

transitions t_1 and t_2 of DTSPN from restaurant example.

Transition labeling:

$L_N(t_1) = L_N(t_2) = Cook,\ L_N(t_3) = Serve$.

Conditional probabilities are associated with actions:

$Cook$ has probability ρ, and $Serve$ has χ.

Transition concurrency in DTSPNs: step semantics for labeled DTSPNs.

Definition of DTSPN transition labeling: [BT00].

Continuous time stochastic Petri nets

Formal model of CTSPNs

Definition 83 *A continuous time SPN (CTSPN) is a tuple* $N = (P_N, T_N, W_N, \Omega_N, M_N)$*:*

- (P_N, T_N, W_N, M_N) *is an unlabeled PN;*
- $\Omega_N : T_N \to \mathbb{R}_+$ *is the transition rate function.*

Each transition $t \in T_N$ of a CTSPN N has rate $\Omega_N(t)$, a parameter of exponential distribution.

When a transition becomes enabled, its timer is set up to the corresponding arbitrary delay.

Then the timer is decreased with a constant rate.

When timer reaches zero, the transition fires.

Transitions that enabled in the same marking and share tokens. The transition that will fire is chosen with conflict resolving rules.

- Preselection According to a metric (for example, priority).
- Race The one with minimal firing delay.

CTSPNs: race rule.

Keeping track of the past by a transition firing: continue and restart mechanisms.

- Resampling The timers of all transitions are discarded. New values of the timers are set for the transitions that are enabled in the new marking.

  Memory of the past: no.

- Enabling memory The timers of the transitions that are disabled are restarted. The timers of the transitions that are not disabled hold their values.

  Memory of the past: *enabling memory variable*, associated with each transition. The variable measures the enabling time of a transition since the last time it became enabled.

- Age memory The timers of all transitions hold their values.

  Memory of the past: *age memory variable*, associated with each transition. The variable measures the cumulative enabling time of a transition since the last time it fired.

CTSPNs: all the three concepts are equivalent.

Resampling: parallel execution, hypothesis test, theoretical viewpoint.

Enabling and age memory: practical, application viewpoint.

Further: CTSPNs with race and resampling.

Sojourn time in a marking M is exponentially distributed with parameter $\sum_{u\in Ena(M)}\Omega_N(u)$.

PDSF of sojourn time in M is that of minimal firing delay of transitions from $Ena(M)$.

Probability to fire (first) in a marking M of $t\in Ena(M)$ is

$$PE(t,M)=\frac{\Omega_N(t)}{\sum_{u\in Ena(M)}\Omega_N(u)}.$$

Average sojourn time in a marking M is

$$SJ(M)=\frac{1}{\sum_{t\in Ena(M)}\Omega_N(t)}.$$

Continuous time PDSF: zero probability of simultaneous transition firing.

CTSPNs have *interleaving* semantics, unlike DTSPNs.

Analysis methods for CTSPNs

For all CTSPNs $N = (P_N, T_N, W_N, \Omega_N, M_N)$ we have
$RS(N) = RS(P_N, T_N, W_N, M_N)$: reachability sets of a CTSPN and its underlying PN coincide.

Qualitative properties of a CTSPNs: analysis of reachability graphs for underlying PNs.

Quantitative properties of a CTSPNs: analysis of CTMCs for *bounded* CTSPNs.

CTMC $CTMC(N)$ *corresponding* to a CTSPN N:

1. Set of states $S = RS(N)$.
2. Rate r_{ij} $(1 \leq i, j \leq n = |S|)$ of transition from M_i to M_j is

$$r_{ij} = \begin{cases} \sum_{\{t | M_i \xrightarrow{t} M_j\}} \Omega_N(t) & i \neq j \\ 0 & i = j \end{cases}.$$

3. the initial state $s_1 = M_N$.

TRM $\mathbf{Q}$ for CTMC $CTMC(N)$ with elements

$$q_{ij} = \begin{cases} \sum_{\{t | M_i \xrightarrow{t} M_j\}} \Omega_N(t) & i \neq j \\ -\sum_{t \in Ena(M_i)} \Omega_N(t) & i = j \end{cases}.$$

TRM $\mathbf{Q}$ could be defined as

$$q_{ij} = \begin{cases} r_{ij} & i \neq j \\ -\sum_{\{k | 1 \leq k \leq n,\ k \neq i\}} r_{ik} & i = j \end{cases}.$$

Transient PMF for CTMC $CTMC(N)$ is calculated as

$$\varphi(\delta) = \varphi(0)e^{\mathbf{Q}\delta},$$

where $\varphi(0) = (\varphi_1(0), \ldots, \varphi_n(0))$ is the probability of the initial distribution with elements $\varphi_i(0)$ $(1 \le i \le n)$:

$$\varphi_i(0) = \begin{cases} 1 & M_i = M_N \\ 0 & \text{otherwise} \end{cases}.$$

Here $\varphi(\delta) = (\varphi_1(\delta), \ldots \varphi_n(\delta))$ is transient PMF over reachable markings, and $\varphi_i(\delta)$ $(1 \le i \le n)$ are transient probabilities of markings M_i.

Steady state PMF for CTMC $CTMC(N)$ is a solution of equation system

$$\begin{cases} \varphi\mathbf{Q} = \mathbf{0} \\ \varphi\mathbf{1}^T = 1 \end{cases}.$$

Here $\varphi = (\varphi_1, \ldots, \varphi_n)$ is steady-state PMF over reachable markings, and φ_i $(1 \le i \le n)$ are steady-state probabilities of markings M_i.

Performance indices for CTSPNs.

- *Probability of an event* defined through markings. An event $\mathcal{A}$ is defined through a condition that holds for the markings $Mark_{\mathcal{A}} \subseteq RS(N)$.

  The steady-state probability of $\mathcal{A}$ is

$$\mathsf{P}(\mathcal{A}) = \sum_{\{i \mid M_i \in Mark_{\mathcal{A}}\}} \varphi_i.$$

- *Probability to have k tokens in a place $p \in P_N$* is

$$Tokens(p,k) = \sum_{\{i \mid M_i(p)=k\}} \varphi_i.$$

- *Average number of tokens in a place $p \in P_N$* is

$$Tokens(p) = \sum_{\{i \mid p \in M_i\}} M_i(p)\varphi_i = \sum_{k \geq 1} Tokens(p,k)k.$$

- *Firing frequency (average number of firings per unit of time) of a transition $t \in T_N$* is

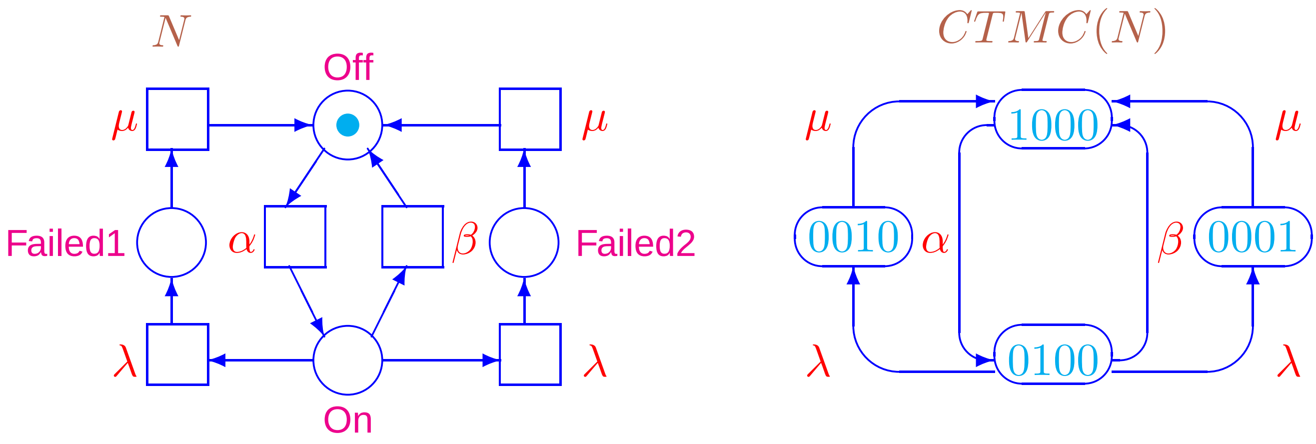
$$Freq(t) = \sum_{\{i \mid t \in Ena(M_i)\}} \Omega_N(t)\varphi_i.$$

- $TravNum$ is the average token number in traversing a subnet of the CTSPN. $Rate$ is the average input (output) token rate into (out) the subnet.

  Average delay of a token in traversing the subnet in steady state is

$$Delay = \frac{TravNum}{Rate}.$$

Example of CTSPNs



CTSPN of garland and its CTMC

Garland with two lamps: CTSPN N.

The lamps are sequentially connected and about *equal*.

One can turn the garland on and off.

When the garland is turned on, one lamp can fail (but not both).

The failed lamp is replaced immediately.

Places: $P_N = \{p_1, p_2, p_3, p_4\}$.

Transitions: $T_N = \{t_1, t_2, t_3, t_4, t_5, t_6\}$.

Rates: $\Omega_N(t_1) = \alpha,\ \Omega_N(t_2) = \beta,\ \Omega_N(t_3) = \Omega_N(t_5) = \lambda,\ \Omega_N(t_4) = \Omega_N(t_6) = \mu$.

Interpretation of places.

p_1: the garland is off (Off).

p_2: the garland is on (On).

p_3: first lamp failed (Failed1).

p_4: second lamp failed (Failed2).

Interpretation of markings.

$M_1 = (1, 0, 0, 0)$: the garland is off (Off).

$M_2 = (0, 1, 0, 0)$: the garland is on (On).

$M_3 = (0, 0, 1, 0)$: first lamp failed (Failed1).

$M_4 = (0, 0, 0, 1)$: second lamp failed (Failed2).

Interpretation of transitions and their rates.

1. When the garland is turned off, after time with exponential distribution parameter α, it could be turned on:

   t_1 with rate α.

2. When the garland is turned on, after time with exponential distribution parameter β, it could be turned off:

   t_2 with rate β.

   or after time with exponential distribution parameter λ first lamp is failed:

   t_3 with rate λ,

   or second lamp is failed:

   t_5 with rate λ.

3. When the garland is failed, after time with exponential distribution parameter μ, first lamp is replaced:

   t_4 with rate μ.

   or second lamp is replaced:

   t_6 with rate μ.

TRM for CTMC $CTMC(N)$ is

$$\mathbf{Q} = \begin{pmatrix} -\alpha & \alpha & 0 & 0 \\ \beta & -(\beta+2\lambda) & \lambda & \lambda \\ \mu & 0 & -\mu & 0 \\ \mu & 0 & 0 & -\mu \end{pmatrix}$$

Steady state PMF for CTMC $CTMC(N)$ is a solution of equation system

$$\begin{cases} \alpha\varphi_1 = \beta\varphi_2 + \mu\varphi_3 + \mu\varphi_4 \\ (\beta+2\lambda)\varphi_2 = \alpha\varphi_1 \\ \mu\varphi_3 = \lambda\varphi_2 \\ \mu\varphi_4 = \lambda\varphi_2 \\ \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 1 \end{cases}$$

The result is:

$$\varphi = \frac{1}{\mu(\beta+2\lambda)+\alpha(\mu+2\lambda)}(\mu(\beta+2\lambda), \alpha\mu, \alpha\lambda, \alpha\lambda).$$

Performance indices.

- *Fraction of time when the garland is on* is φ_2.
- *Fraction of time when the garland is failed* is $\varphi_3 + \varphi_4 = 2\varphi_3 = 2\varphi_4$.
- Average rate if firing one of t_3 or t_5 is
  $Freq(t_3, t_5) = Freq(t_3) + Freq(t_5)$, where
  $Freq(t_3) = \lambda\varphi_2 = Freq(t_5)$.

  Average rate if firing one of t_4 or t_6 is
  $Freq(t_4, t_6) = Freq(t_4) + Freq(t_6)$, where
  $Freq(t_4) = \mu\varphi_3 = \mu\varphi_4 = Freq(t_6)$.

  Average time between two consecutive failures (repairs) is
  $\frac{1}{Freq(t_3,t_5)} = \frac{1}{2\lambda\varphi_2} = \frac{1}{2\mu\varphi_3} = \frac{1}{2\mu\varphi_4} = \frac{1}{Freq(t_4,t_6)}$.
- Average rate if firing of t_1 is $Freq(t_1) = \alpha\varphi_1$.

  Average rate if firing one of t_2 or t_3 or t_5 is
  $Freq(t_2, t_3, t_5) = Freq(t_2) + Freq(t_3) + Freq(t_5)$, where
  $Freq(t_2) = \beta\varphi_2,\ Freq(t_3) = \lambda\varphi_2 = Freq(t_5)$.

  Average time between two consecutive turning on (off) is
  $\frac{1}{Freq(t_1)} = \frac{1}{\alpha\varphi_1} = \frac{1}{(\beta+2\lambda)\varphi_2} = \frac{1}{Freq(t_2,t_3,t_5)}$.

Steady state PMF for garland with n lamps is

$$\varphi = \frac{1}{\mu(\beta + n\lambda) + \alpha(\mu + n\lambda)}(\mu(\beta + n\lambda), \alpha\mu, \underbrace{\alpha\lambda, \ldots, \alpha\lambda}_{n}).$$

Summary for CTSPNs

CTSPNs are standardly unlabeled:

acceptable to model logically different activities:

transitions t_1 and t_2 of CTSPN from garland example;

not acceptable to model logically equal activities:

transitions t_3 and t_5 (t_4 and t_6) of CTSPN from garland example.

Transition labeling:

$$L_N(t_1) = TurnOn,\ L_N(t_2) = TurnOff,$$
$$L_N(t_3) = L_N(t_5) = LampFailure,$$
$$L_N(t_4) = L_N(t_6) = LampChange.$$

Rates are associated with actions:

$TurnOn$ has rate α, $TurnOff$ has β, $LampFailure$ has λ, and $LampChange$ has μ.

Transition interleaving in CTSPNs: interleaving semantics for labeled CTSPNs.

Definition of CTSPN transition labeling: [Buc95].

Generalized stochastic Petri nets

Formal model of GSPNs

Definition 84 *A* generalized SPN (GSPN) *is a tuple* $N = (P_N, T_N, W_N, H_N, \Omega_N, \aleph_N, M_N)$*:*

- $(P_N, T_N, W_N, H_N, \aleph_N, M_N)$ *is an unlabeled IPPN with* T_N *consisting of* *exponential* *and* *immediate* *transitions and* $\aleph_N$ *having value* 0 *for* *exponential* *transitions and* 1 *for* *immediate* *ones;*
- $\Omega_N : T_N \to \mathbb{R}_+$ *is a function of* *exponential* *transition* *rates* *and* *immediate* *transition* *weights.*

Marking M is *tangible*, if $Ena(M)$ contains exponential transitions only.

Marking M is *vanishing*, if $Ena(M)$ contains at least one immediate transition.

$RS_T(N)$ is the set of *all tangible markings* of a GSPN N.

$RS_V(N)$ is the set of *all vanishing markings* of a GSPN N.

$$RS(N) = RS_T(N) \cup RS_V(N),\ RS_T(N) \cap RS_V(N) = \emptyset.$$

Probability to fire (first) in a marking M of $t \in Ena(M)$ is

$$PE(t, M) = \frac{\Omega_N(t)}{\sum_{u \in Ena(M)} \Omega_N(u)}.$$

In a tangible marking (t is exponential), $\Omega_N(t)$ is the rate of t.

In a vanishing marking (t is immediate), $\Omega_N(t)$ is the weight of t.

Average sojourn time in a marking M is

$$SJ(M) = \begin{cases} \frac{1}{\sum_{t \in Ena(M)} \Omega_N(t)} & M \in RS_T(N) \\ 0 & M \in RS_V(N) \end{cases}.$$

Transitions fire one by one, even simultaneously enabled immediate ones.

Concurrent firing of simultaneously enabled immediate transitions does not change the behaviour.

GSPNs have *interleaving* semantics, like CTSPNs.

Analysis methods for GSPNs

For all IPPNs $N = (P_N, T_N, W_N, H_N, \aleph_N, M_N)$ we have $RS(N) \subseteq RS(P_N, T_N, W_N, M_N)$: reachability set of an IPPN contains in that of PN.

Adding inhibitor arcs and transition priorities reduces reachability set of a PN.

For all GSPNs $N = (P_N, T_N, W_N, H_N, \Omega_N, \aleph_N, M_N)$ we have $RS(N) = RS(P_N, T_N, W_N, H_N, \aleph_N, M_N)$: reachability sets of a GSPN and its underlying IPPN coincide.

Qualitative properties of a GSPNs: analysis of reachability graphs for underlying IPPNs.

Quantitative properties of a GSPNs: analysis of SMCs for *bounded reversible* GSPNs.

Embedded DTMC (EDTMC) $EDTMC(N)$ *corresponding* to GSPN N:

1. Set of states $S = RS(N)$.
2. Probability ρ_{ij} $(1 \leq i, j \leq n = |S|)$ of a transition from M_i to M_j is

$$\rho_{ij} = \sum_{\{t \mid M_i \xrightarrow{t} M_j\}} PE(t, M_i);$$

3. the initial state $s_1 = M_N$.

(One-step) TPM $\mathbf{P}$ for EDTMC $EDTMC(N)$ has elements ρ_{ij}.

Transient (k-step) PMF for EDTMC $EDTMC(N)$ is a solution of equation system

$$\psi(k) = \psi(0)\mathbf{P}^k,$$

where $k \in \mathbb{N}$, and $\psi(0) = (\psi_1(0), \ldots, \psi_n(0))$ is probability of the initial distribution with elements $\psi_i(0)$ $(1 \leq i \leq n)$:

$$\psi_i(0) = \begin{cases} 1 & M_i = M_N \\ 0 & \text{otherwise} \end{cases}.$$

Here $\psi(k) = (\psi_1(k), \ldots \psi_n(k))$ is transient PMF over k-step reachable markings, and $\psi_i(k)$ $(1 \leq i \leq n)$ transient probabilities of markings M_i.

Steady state PMF for EDTMC $EDTMC(N)$ is a solution of equation system

$$\begin{cases} \psi(\mathbf{P} - \mathbf{I}) = \mathbf{0} \\ \psi\mathbf{1}^T = 1 \end{cases}.$$

Here $\psi = (\psi_1, \ldots, \psi_n)$ is steady-state PMF over reachable markings, and ψ_i $(1 \leq i \leq n)$ are steady-state probabilities of markings M_i.

Steady state PMF for SMC corresponding to GSPN N is $\varphi = (\varphi_1, \ldots, \varphi_n)$: multiplication of each ψ_i $(1 \leq i \leq n)$ by average sojourn time $SJ(M_i)$ and normalization of the distribution.

Marking M is vanishing: $SJ(M) = 0$.

Marking M is tangible: only exponential transitions are enabled, and sojourn time is calculated as for CTSPNs.

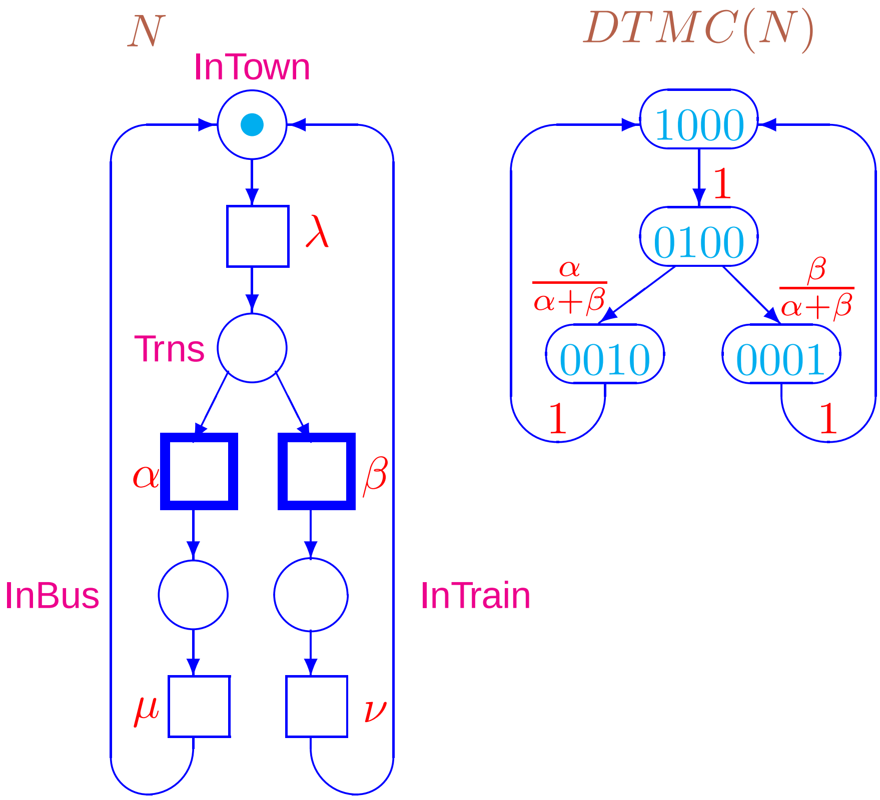
Thus, for $1 \leq i \leq n$:

$$\varphi_i = \begin{cases} \frac{\psi_i SJ(M_i)}{\sum_{j=1}^{n} \psi_j SJ(M_j)} & M_i \in RS_T(N) \\ 0 & M_i \in RS_V(N) \end{cases}.$$

The method above: appropriate by small number of vanishing markings.

Eliminating of vanishing markings: appropriate by big number of vanishing markings [MCB84,Mar90].

Example of GSPNs



GSPN of traveller and its EDTMC

Traveller that visit new towns: GSPN N.

After looking town, the traveller goes to another by the next train of bus.

Buses depart not so frequent as trains, but they go quicker.

Time of stay in town, number of train and bus departures and their velocities *do not depend* on particular town.

Distances between all pairs consisting of current and the next town are about *equal*.

Places: $P_N = \{p_1, p_2, p_3, p_4\}$.

Transitions: $T_N = \{t_1, t_2, t_3, t_4, t_5\}$, where t_1, t_4, t_5 are exponential, and t_2, t_3 are immediate ones.

Rates / weights:

$\Omega_N(t_1) = \lambda,\ \Omega_N(t_2) = \alpha,\ \Omega_N(t_3) = \beta,\ \Omega_N(t_4) = \mu,\ \Omega_N(t_5) = \nu$.

Interpretation of places.

p_1: to be in current town (InTown).

p_2: transport departs to the next town (Trsp).

p_3: to be in bus (InBus).

p_4: to be in train (InTrain).

Interpretation of markings.

$M_1 = (1, 0, 0, 0)$: to be in current town (InTown).

$M_2 = (0, 1, 0, 0)$: transport departs to the next town (Trsp).

$M_3 = (0, 0, 1, 0)$: to be in bus (InBus).

$M_4 = (0, 0, 0, 1)$: to be in train (InTrain).

Marking M_2 is vanishing, time of stay is 0: enter the transport immediately after it comes.

$RS_T(N) = \{M_1, M_3, M_4\}$ and $RS_V(N) = \{M_2\}$.

Interpretation of transitions and their rates / weights.

1. When traveller comes to town, after time that exponentially distributed with parameter λ, (s)he looks the town and waits for transport to the next place:

   t_1 with rate λ.

2. Transport that departs is with probability α bus:

   t_2 with weight α,

   or is with probability β train:

   t_3 with weight β.

   Another interpretation of weights: for α bus departures we have β train departures.

   Buses depart less frequently: $\alpha \leq \beta$.

3. When traveller enters bus, after time that exponentially distributed with parameter μ, (s)he comes by bus to the next town:

   t_4 with rate μ.

4. When traveller enters train, after time that exponentially distributed with parameter ν, (s)he comes by train to the next town:

   t_5 with rate ν .

   Buses go quicker: $\mu \geq \nu$.

TRM for EDTMC $EDTMC(N)$ is

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Steady state PMF for EDTMC $EDTMC(N)$ is a solution of equation system

$$\begin{cases} \psi_1 = \psi_3 + \psi_4 \\ \psi_1 = \psi_2 \\ \frac{\alpha}{\alpha+\beta}\psi_2 = \psi_3 \\ \frac{\beta}{\alpha+\beta}\psi_2 = \psi_4 \\ \psi_1 + \psi_2 + \psi_3 + \psi_4 = 1 \end{cases}$$

The result is

$$\psi = \frac{1}{3}\left(1, 1, \frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right).$$

Vector of average sojourn time in markings is

$$SJ = \left(\frac{1}{\lambda}, 0, \frac{1}{\mu}, \frac{1}{\nu}\right).$$

Steady state PMF ψ weighted by SJ is

$$\frac{1}{3}\left(\frac{1}{\lambda}, 0, \frac{\alpha}{\mu(\alpha+\beta)}, \frac{\beta}{\nu(\alpha+\beta)}\right).$$

Normalized weighted steady-state PMF is

$$\psi SJ^T = \frac{1}{3}\left(\frac{1}{\lambda} + \frac{\alpha\nu+\beta\mu}{\mu\nu(\alpha+\beta)}\right).$$

Steady state PMF for SMC corresponding to GSPN N is

$$\varphi = \frac{1}{\frac{1}{\lambda} + \frac{\alpha\nu+\beta\mu}{\mu\nu(\alpha+\beta)}}\left(\frac{1}{\lambda}, 0, \frac{\alpha}{\mu(\alpha+\beta)}, \frac{\beta}{\nu(\alpha+\beta)}\right).$$

When buses and trains depart with equal frequency ($\alpha = \beta$) and

go with equal velocity ($\mu = \nu$), we have

$$\varphi = \frac{1}{2(\lambda+\mu)}(2\mu, 0, \lambda, \lambda).$$

Then average time of stay in transport w.r.t. that of in town is $\frac{\lambda}{\mu}$.

Summary for GSPNs

GSPNs are standardly unlabeled:

acceptable to model logically different activities:

transitions t_1 and t_4 of GSPN from traveller example;

not acceptable to model logically equal activities:

transitions t_2 and t_3 (t_4 and t_5) of GSPN from traveller example.

Transition labeling:

$L_N(t_1) = SeeTown,\ L_N(t_2) = L_N(t_3) = \tau,$
$L_N(t_4) = L_N(t_5) = BusTravel.$

Weights are associated with transitions:

t_2 has weight α, t_3 has β.

Rates are associated with actions:

$SeeTown$ has rate λ, $BusTravel$ has μ.

Transition interleaving in GSPNs: interleaving semantics for labeled GSPNs.

Definition of GSPN transition labeling: [Buc98].

Eliminating of vanishing markings: do not take into account M_2.

Steady state analysis based on reduced EDTMC:

redirect outgoing arcs from M_2 to M_1,

and delete arc between M_1 and M_2.

Deterministic stochastic Petri nets

Formal model of DSPNs

Definition 85 *A deterministic time SPN (DSPN) is a tuple* $N = (P_N, T_N, W_N, H_N, \Omega_N, \aleph_N, M_N)$*:*

- $(P_N, T_N, W_N, H_N, \aleph_N, M_N)$ *is an unlabeled IPPN with* T_N *consisting of exponential and deterministic transitions and* $\aleph_N$ *having value* 0 *for exponential transitions and value* 1 *for immediate ones (deterministic transitions with zero delay);*
- $\Omega_N : T_N \to \mathbb{R}_+$ *is a function of exponential transition rates and deterministic transition delays.*

Behaviour of DSPNs: race with enabling memory.

DSPNs have *interleaving* semantics, like CTSPNs and GSPNs

Analysis methods for DSPNs

Transitions of a DSPN.

1. *Exclusive*: for all markings enabling it, this is the only enabled one.

2. *Competitive*: it is not exclusive, and for all markings enabling it, all enabled transitions are in conflict with it.

3. *Concurrent*: it is not exclusive, and for some marking enabling it, some enabled transition is not in conflict with it.

Consider only DSPNs s.t. in all markings, *at most one concurrent deterministic transition* is enabled.

Then reachability graph structure is independent of time constraints.

In addition, semi-Markov process can be associated with a DSPN.

Concurrent deterministic transitions:

independent, that cannot be disabled, and

preemptable, that can be disabled.

Possibilities for behaviour of a DSPN N.

1. In $M_i\ (1 \leq i \leq n)$ *no deterministic* transition is enabled or an *exclusive deterministic* is enabled.

   No deterministic transition: average sojourn time in M_i is

   $$SJ(M_i) = \frac{1}{\sum_{t \in Ena(M_i)} \Omega_N(t)}.$$

   If $\exists t \in T_N\ M_i \xrightarrow{t} M_j$, probability of state change from M_i to M_j is

   $$\rho_{ij} = \frac{\sum_{\{t \mid M_i \xrightarrow{t} M_j\}} \Omega_N(t)}{\sum_{t \in Ena(M_i)} \Omega_N(t)}.$$

2. In M_i an *independent deterministic* transition $t_d \in T_N$ is enabled *together with exponential* ones.

   The next state of EDTMC is sampled only at the instant of firing of t_d, with no respect of state changes due to firings of exponential transitions during the enabling interval $\Omega_N(t_d) = \theta_d$.

   The state changes are “delayed” to the instant of firing of t_d.

   State change probability for EDTMC: Chapman-Kolmogorov equation.

3. In M_i a *competitive* or a *preemptable deterministic* transition $t_d \in T_N$ is enabled.

   The next state of EDTMC is sampled at the instant of firing of t_d or the instant of disabling of t_d.

   Probability of firing of t_d is computed based on transient evolution of the stochastic part of process during enabling interval θ_d.

Solution technique: one deterministic transition,

otherwise repeat the analysis step.

For a DSPN N, $RS(N)$ consists of two marking classes:

$MD(N)$: t_d is enabled,

$ME(N)$: t_d is not enabled.

States of EDTMC for DSPN N are reordered: markings from $MD(N)$ come first.

TRM for CTMC is

$$\mathbf{Q} = \begin{pmatrix} \mathbf{D} & \mathbf{K} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix}.$$

Submatrix $\mathbf{D}$: rates of exponential transitions not conflicting with t_d (transition rates between markings of $MD(N)$).

Submatrix $\mathbf{K}$: rates of exponential transitions conflicting with t_d (transition rates from markings of $MD(N)$ to $ME(N)$).

Submatrices $\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$: rates of exponential transitions enabled in markings of $MD(N)$.

TRM reduction: respect only rates of exponential transitions enabled in the same markings as deterministic one.

Reduced TRM for CTMC is

$$\mathbf{Q}' = \begin{pmatrix} \mathbf{D} & \mathbf{K} \\ 0 & 0 \end{pmatrix}.$$

Let t_d be preemptable determinstic transition and $M_i \in MD(N),\ M_j \in ME(N)$.

Probability of EDTMC state change from M_i to M_j, if t_d is preemptable, is

$$\mathbf{u}_i e^{\mathbf{Q}'\theta_d} \mathbf{u}_j^T,$$

where $\mathbf{u}_i$ $(1 \le i \le n)$ is a vector of length n with i-th element be 1, and all other be 0-s.

Probability of EDTMC state change from M_i to M_j, if t_d fires, is

$$\mathbf{u}_i e^{\mathbf{Q}'\theta_d} \Delta_d \mathbf{u}_j^T,$$

where Δ_d is TPM resulting by firing of t_d defined as

$$\Delta_d = \begin{pmatrix} \Delta_{DD} & \Delta_{DE} \\ 0 & \mathbf{I} \end{pmatrix}.$$

Hence, i-th (corresponding to M_i) line of TPM for EDTMC is

$$\mathbf{P}(i) = \mathbf{u}_i e^{\mathbf{Q}'\theta_d} \Delta_d.$$

Average sojourn time in M_i is

$$SJ(M_i) = \int_0^{\theta_d} \mathbf{u}_i e^{\mathbf{Q}'x} \begin{pmatrix} \mathbf{1}^T \\ \mathbf{0}^T \end{pmatrix} dx,$$

where $\mathbf{1}$ is a row vector of $|MD(N)|$ values 1, and $\mathbf{0}$ is that of $|ME(N)|$ values 0.

If t_d is independent then $SJ(M_i) = \theta_d$.

If t_d is preemptable then

$$SJ(M_i) = \sum_{\{j \mid M_j \in MD(N)\}} \mathbf{u}_i \begin{pmatrix} \mathbf{D}^{-1}(e^{\mathbf{D}\theta_d} - \mathbf{I}) & 0 \\ 0 & 0 \end{pmatrix} \mathbf{u}_j^T.$$

Steady-state PMF for DSPN is constructed from that for EDTMC.

First, weighting of steady-state marking probabilities by average sojourn time in that makings.

Second, converting probabilities of markings enabling concurrent deterministic transitions with

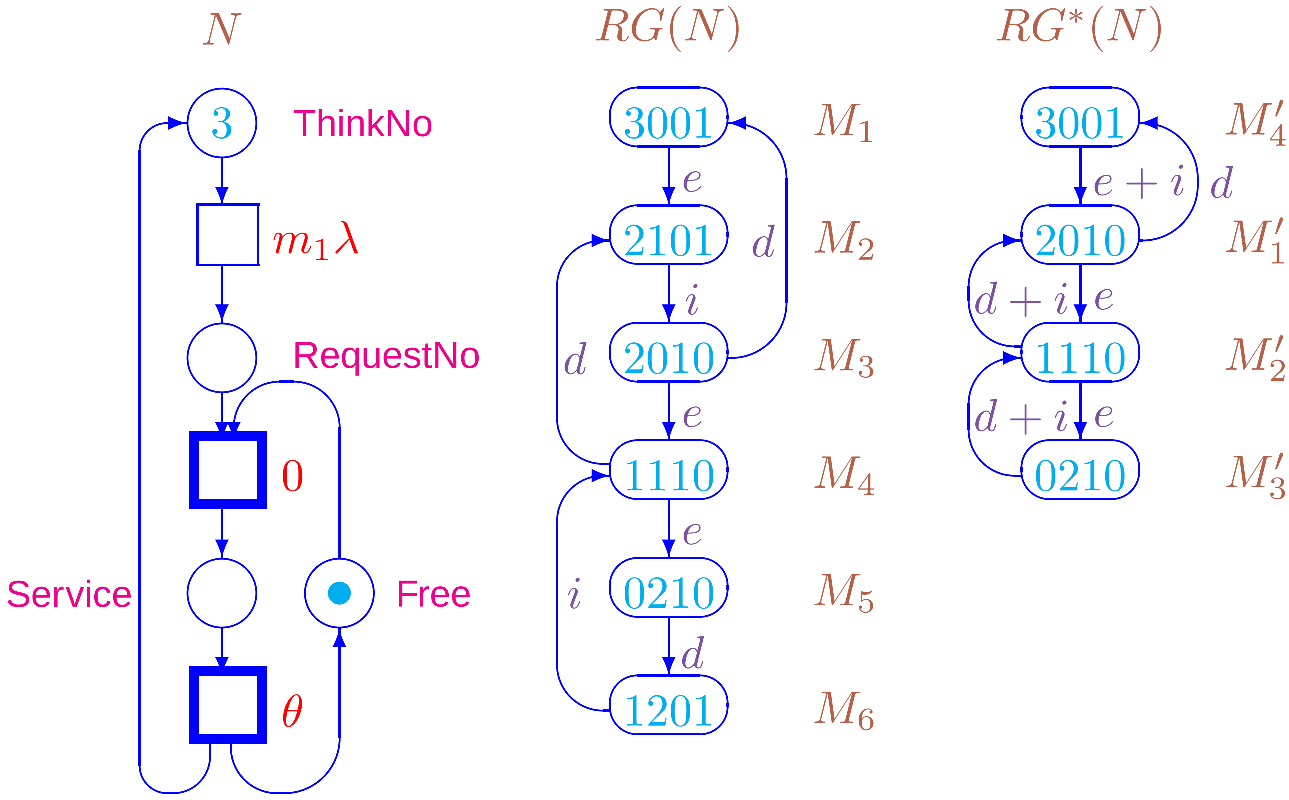
Conversion matrix $\mathbf{C}_d$: difference between average sojourn time in a marking of DSPN and in a state of EDTMC.

Elements (i, j) of conversion matrix $\mathbf{C}_d$ s.t. $M_i, M_j \in MD(N)$ are defined as

$$\mathbf{C}_d(i, j) = \frac{1}{SJ(M_i)} \mathbf{u}_i \int_0^{\theta_d} e^{\mathbf{Q}' x} dx \mathbf{u}_j^T =$$

$$\frac{1}{SJ(M_i)} \mathbf{u}_i \begin{pmatrix} \mathbf{D}^{-1}(e^{\mathbf{D}\theta_d} - \mathbf{I}) & 0 \\ 0 & 0 \end{pmatrix} \mathbf{u}_j^T.$$

Example of DSPNs



DSPN of queue with its complete and reduced reachability graphs

Queue M/D/1/3/3 of three consumers: DSPN N.

Queue type: type of incoming process / distribution of service time / number of servers in service station / number of consumers / number of requests.

Symbol 'M': Markov process, and 'D': deterministic distribution.

Queue M/D/1/3/3: Markov incoming process and deterministic service time distribution of 3 consumers with 3 requests (one for each consumer) on 1 server.

Consumers think, then request for service, and are served one by one at service station, if it is free.

Places: $P_N = \{p_1, p_2, p_3, p_4\}$.

Transitions: $T_N = \{t_1, t_2, t_3\}$, where t_2, t_3 are deterministic (t_2 is immediate, deterministic with zero delay), and t_1 is exponential one.

Rates / delays: $\Omega_N(t_1) = m_1\lambda,\ \Omega_N(t_2) = 0,\ \Omega_N(t_3) = \theta$, where m_1 is a number of tokens in place p_1 (rate of t_1 depend on input flow).

Transition names according to their types: exponential t_1 is named e, immediate t_2 is named i, and deterministic t_3 is named d.

Interpretation of places.

p_1: number of thinking consumers (ThinkNo),

p_2: number of consumers that requested service (RequestNo),

p_3: consumer is at service station (Service),

p_4: service station is free (Free).

Interpretation of markings.

$M_1 = (3, 0, 0, 1)$: 3 consumers think about service, and service station is free (3T+F).

$M_2 = (2, 1, 0, 1)$: 2 consumers think about service, 1 consumer requests service, and service station is free (2T+R+F).

$M_3 = (2, 0, 1, 0)$: 2 consumers think about service, and 1 consumer is served (2T+S).

$M_4 = (1, 1, 1, 0)$: 1 consumer thinks about service, 1 consumer requests service, and 1 consumer is served (T+R+S).

$M_5 = (0, 2, 1, 0)$: 2 consumers request service, and 1 consumer is served (2R+S).

$M_6 = (1, 2, 0, 1)$: 1 consumer thinks about service, 2 consumers request service, and service station is free (T+2R+F).

Markings M_2 and M_6 are vanishing: zero sojourn time, corresponds to service immediately after request, if service station is free.

Other markings are tangible.

$RS_T(N) = \{M_1, M_3, M_4, M_5\}$ and $RS_V(N) = \{M_2, M_6\}$.

Eliminating of vanishing markings from complete reachability graph $RG(N)$: reduced reachability graph $RG^*(N)$.

Interpretation of transitions and their rates / delays.

1. When consumer has thought about service, after time that exponentially distributed with parameter $m_1\lambda$, (s)he requests service:

   transition t_1 with rate $m_1\lambda$.

2. hen service has been requested, and service station is free, then immediately, with delay 0, service starts:

   transition t_2 with delay 0.

3. When consumer is at service station, after time θ (s)he is served:

   transition t_3 with delay θ.

The only deterministic transition t_3 cannot be enabled concurrently with other ones (and with itself) in all markings: the analysis applicability condition is fulfilled.

Transition t_3 is *concurrent independent* one.

States of reduced reachability graph $RG^*(N)$: tangible markings $M_1 \in ME(N)$ and $M_3, M_4, M_5 \in MD(N)$.

Reordering: $M_3 \mapsto M_1',\ M_4 \mapsto M_2',\ M_5 \mapsto M_3',\ M_1 \mapsto M_4'$.

Complete and reduced TRMs for CTMC are

$$\mathbf{Q} = \begin{pmatrix} -2\lambda & 2\lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ 3\lambda & 0 & 0 & -3\lambda \end{pmatrix} \quad \mathbf{Q}' = \begin{pmatrix} -2\lambda & 2\lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

TPM resulting by deterministic transition firing is

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

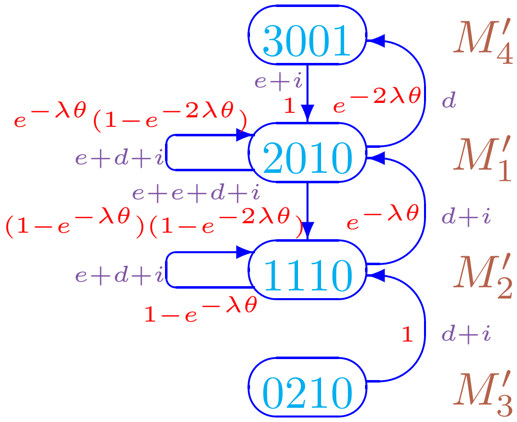
Matrix exponential is

$$e^{\mathbf{Q}'\theta} = \begin{pmatrix} e^{-2\lambda\theta} & e^{-\lambda\theta}(1-e^{-2\lambda\theta}) & (1-e^{-\lambda\theta})(1-e^{-2\lambda\theta}) & 0 \\ 0 & e^{-\lambda\theta} & 1-e^{-\lambda\theta} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrix exponential changed by probabilities of deterministic transition firing is

$$e^{\mathbf{Q}'\theta}\Delta = \begin{pmatrix} e^{-\lambda\theta}(1-e^{-2\lambda\theta}) & (1-e^{-\lambda\theta})(1-e^{-2\lambda\theta}) & 0 & e^{-2\lambda\theta} \\ e^{-\lambda\theta} & 1-e^{-\lambda\theta} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$EDTMC^*(N)$



EDTMC for DSPN of queue

TPM for EDTMC $EDTMC^*(N)$ based on $RG^*(N)$ is

$$\mathbf{P} = \begin{pmatrix} e^{-\lambda\theta}(1-e^{-2\lambda\theta}) & (1-e^{-\lambda\theta})(1-e^{-2\lambda\theta}) & 0 & e^{-2\lambda\theta} \\ e^{-\lambda\theta} & 1-e^{-\lambda\theta} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Steady state PMF of “visit” probabilities for EDTMC is a solution of equation system

$$\begin{cases} (1-e^{-\lambda\theta}+e^{-3\lambda\theta})\psi_1 - e^{-\lambda\theta}\psi_2 = \psi_4 \\ (1-e^{-\lambda\theta})(1-e^{-2\lambda\theta})\psi_1 = e^{-\lambda\theta}\psi_2 \\ \psi_3 = 0 \\ e^{-2\lambda\theta}\psi_1 = \psi_4 \\ \psi_1+\psi_2+\psi_3+\psi_4 = 1 \end{cases}$$

The result is

$$\psi = \frac{1}{1 + e^{-2\lambda\theta} + 2e^{-3\lambda\theta}}(e^{-\lambda\theta}, (1 - e^{-\lambda\theta})(1 - e^{-2\lambda\theta}), 0, e^{-3\lambda\theta}).$$

Since deterministic transition t_3 is independent, $SJ(M_1) = SJ(M_2) = \theta$.

Average sojourn time vector for markings of N is

$$SJ = \left(\theta, \theta, \theta, \frac{1}{3\lambda}\right).$$

Steady state PMF ψ weighted by SJ is

$$\left(\psi_1\theta, \psi_2\theta, 0, \frac{\psi_4}{3\lambda}\right).$$

Sojourn time in M_1 and M_2 must be redistributed between them and M_3.

Let $c_k = \int_0^\theta e^{-k\lambda x}dx = \frac{1-e^{-k\lambda\theta}}{k\lambda\theta}$, $(1 \le k \le 3)$.

Conversion matrix is

$$\mathbf{C} = \begin{pmatrix} c_2 & c_1 - c_3 & 1 - c_1 - c_2 + c_3 & 0 \\ 0 & c_1 & 1 - c_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Weighted steady-state PMF changed by conversion matrix is

$$\left(\psi_1\theta c_2, \psi_1\theta(c_1 - c_3) + \psi_2\theta c_1, \psi_1\theta(1 - c_1 - c_2 + c_3) + \psi_2\theta(1 - c_1), \frac{\psi_4}{3\lambda}\right).$$

The last step: normalization of converted weighted steady-state PMF.

Stable state PMF for DSPN is

$$\varphi = \frac{1}{\psi_1 + \psi_2 + \frac{\psi_4}{3\lambda\theta}} \times$$

$$\left(\psi_1 c_2, \psi_1(c_1 - c_3) + \psi_2 c_1, \psi_1(1 - c_1 - c_2 + c_3) + \psi_2(1 - c_1), \frac{\psi_4}{3\lambda\theta}\right).$$

Summary for DSPNs

DSPNs are standardly unlabeled:

acceptable to model logically different activities:

all transitions of DSPN from queue example;

not acceptable to model logically equal activities.

Transition labeling:

$L_N(t_1) = Require,\ L_N(t_2) = \tau,\ L_N(t_3) = Serve.$

Rates and delays are associated with actions:

$Require$ has rate λ, τ has delay 0, and $Serve$ has delay θ.

Transition interleaving in DSPNs: interleaving semantics for labeled DSPNs.

Definition of DSPN transition labeling: not presenteqd yet.

DSPNs are an extension of GSPNs by arbitrary fixed (deterministic) delays, not zero only, as in GSPNs.

DSPNs have good expressive power, but their analysis is complex: calculation of many matrix exponentials.

Complexity grows very fast: adding new token (consumer) or another deterministic transition in DSPN of queue example.

Elimination of restricting conditions: deterministic DTSPNs (DDTSPNs) [ZCH97].

DDTSPNs are discrete analogue of DSPNs: deterministic and geometric transitions.

Constant distributionof deterministic transitions is a partial case of geometric one: no restriction by number of enabled determinsitic transitions.

Decision complexity of DSPNs: partition by subsystems and numerical methods.

Overview and discussion

The results obtained

Description of four well-known types of SPNs.

Analysis methods and illustrative examples.

Comparison and application areas.

Ways to define transition labeling: behavioural equivalences.

The most perspective model: DTSPNs and their extensions, like DDTSPNs.

Advantages and disadvantages of stochastic Petri nets

Advantages

- Convenient for theoretical reasoning on behaviour of systems with shared resources and for use in development tools.
- Performance can be evaluated from SPN structure, and detailed analysis is accomplished using MC with well-known algorithms.
- Applicable when synchronization is important: analysis of systems with interacting components.

Disadvantages

- High complexity of large system specification because of absence of modularity and intricateness of the corresponding SPNs.
- More abstract SPNs with better expressive power: analytical and structural restrictions or partitioning, simulation and numerical methods.
- Concurrency of the PN underlying an SPN is reflected only partially in the corresponding MC: in the best case, it has step semantics that is not “true concurrent”.

Equivalences for Stochastic Petri Nets and Stochastic Process Algebras[a]

Abstract: Labeled discrete time stochastic Petri nets (LDTSPNs) are proposed.

The visible behavior of LDTSPNs is described by transition labels. The dynamic behavior is defined.

Trace and bisimulation probabilistic equivalences are introduced.

A diagram of their interrelations is presented.

Some of the equivalences are characterized via formulas of probabilistic modal logics.

The equivalences are used to compare stationary behavior of nets.

Stochastic algebra of finite processes $StAFP_0$ is proposed with a net semantics based on a subclass of LDTSPNs.

Keywords: Stochastic Petri nets, step semantics, probabilistic equivalences, bisimulation, modal logics, stationary behavior, stochastic process algebras.

[a]The joint work with Peter Buchholz, Faculty of Computer Science IV, University of Dortmund, Germany.

Contents

Introduction

Previous work

Transition labeling

- CTSPNs [Buc95]
- GSPNs [Buc98]
- DTSPNs [BT00]

Equivalences

- Probabilistic transition systems (PTSs) [BM89,Chr90,LS91,BHe97,KN98]
- SPAs [HR94,Hil94,BGo98]
- Markov process algebras (MPAs) [Buc94,BKe01]
- CTSPNs [Buc95]
- GSPNs [Buc98]
- Stochastic automata (SAs) [Buc99]
- Stochastic event structures (SESs) [MCW03]

Probabilistic modal logics

- Logic PML [LS91]

Process algebras

- AFP_0 [KCh85]
- PBC [BDH92]

Labeled discrete time stochastic Petri nets

Formal model

Definition 86 *A labeled discrete time stochastic Petri net (LDTSPN) is a tuple* $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$*:*

- P_N *and* T_N *are finite sets of places and transitions* *(*$P_N \cup T_N \neq \emptyset,\ P_N \cap T_N = \emptyset$*);*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \rightarrow I\!N$ *is the arc weight function;*
- $\Omega_N : T_N \rightarrow (0; 1)$ *is the transition conditional probability function;*
- $L_N : T_N \rightarrow Act_\tau = Act \cup \{\tau\}$ *is the transition labeling function;*
- $M_N \in I\!N_{fin}^{P_N}$ *is the initial marking.*

Concurrent transition firings at discrete time moments.

LDTSPNs have *step* semantics.

Behavior of the model

Let M be a marking of a LDTSPN $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$. Then $t \in Ena(M)$ fires in the next time moment with probability $\Omega_N(t)$, if no other transition is enabled in M.

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U is ready for firing in M*:

$$PF(U, M) = \prod_{t \in U} \Omega_N(t) \cdot \prod_{u \in Ena(M) \setminus U} (1 - \Omega_N(u)).$$

In the case $U = \emptyset$ we define

$$PF(\emptyset, M) = \begin{cases} \prod_{u \in Ena(M)} (1 - \Omega_N(u)) & Ena(M) \neq \emptyset; \\ 1 & \text{otherwise}. \end{cases}$$

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^\bullet U \subseteq M$ or $U = \emptyset$ and $tang(M)$. The *probability that the set of transitions U fires in M*:

$$PT(U, M) = \frac{PF(U, M)}{\sum_{\{V \mid {}^\bullet V \subseteq M\}} PF(V, M)}.$$

If $U = \emptyset$ then $M = \widetilde{M}$.

Firing of U changes marking M to $\widetilde{M} = M - {}^{\bullet}U + U^{\bullet}$, $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT(U, M)$.

We write $M \xrightarrow{U} \widetilde{M}$ if $\exists \mathcal{P}\ M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$.

For $U = \{t\}$ we write $M \xrightarrow{t}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{t} \widetilde{M}$.

For $A \in I\!N_{fin}^{Act_\tau}$ we define $vis(A) = \sum_{a \in A \cap Act} a$.

Let $A \in I\!N_{fin}^{Act}$. $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ is a step starting in M, performing transitions that are *visibly* labeled by A and ending in $\widetilde{M}$.

The probability $\mathcal{P} = PS(A, M, \widetilde{M})$ is

$$PS(A, M, \widetilde{M}) = \sum_{\{U \subseteq Ena(M) \mid M \xrightarrow{U} \widetilde{M},\ vis(L_N(U)) = A\}} PT(U, M).$$

We write $M \xrightarrow{A} \widetilde{M}$ if $\exists \mathcal{P}\ M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$.

For $A = \{a\}$ we write $M \xrightarrow{a}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{a} \widetilde{M}$.

Definition 87 *For a LDTSPN N we define the following notions.*

- *The* reachability set $RS(N)$ *is the minimal set of markings s.t.*
  - $M_N \in RS(N)$;
  - *if* $M \in RS(N)$ *and* $M \stackrel{A}{\rightarrow} \widetilde{M}$ *then* $\widetilde{M} \in RS(N)$.
- *The* reachability graph $RG(N)$ *is a directed labeled graph with*
  - *the set of nodes* $RS(N)$;
  - *an arc labeled by* $A,\ \mathcal{P}$ *from node* M *to* $\widetilde{M}$ *if* $M \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{M}$.
- *The* underlying Discrete Time Markov Chain (DTMC) $DTMC(N)$ *is a DTMC with*
  - *the state space* $RS(N)$;
  - *a transition* $M \rightarrow_{\mathcal{P}} \widetilde{M}$ *if at least one arc from* M *to* $\widetilde{M}$ *exists in* $RG(N)$. *The probability* $\mathcal{P} = PM(M, \widetilde{M})$ *is*

$$PM(M, \widetilde{M}) = \sum_{A \in \mathbb{N}_{fin}^{Act}} PS(A, M, \widetilde{M});$$

  - *the initial state* $s_1 = M_N$.

An *internal step* $M \stackrel{\emptyset}{\rightarrow}_{\mathcal{P}} \widetilde{M}$ takes place when

- $\widetilde{M}$ is reachable from M by firing a set of internal transitions or
- no transition fires.

The recursive definition for $k \geq 0$ empty steps:

$$PS^k(\emptyset, M, \widetilde{M}) = \begin{cases} \sum_{\overline{M} \in RS(N)} PS^{k-1}(\emptyset, M, \overline{M}) \cdot PS(\emptyset, \overline{M}, \widetilde{M}) & \text{if } k \geq 1; \\ 1 & \text{if } k = 0 \text{ and } M = \widetilde{M}; \\ 0 & \text{otherwise.} \end{cases}$$

The *probability of reaching* $\widetilde{M}$ *from* M *by internal steps, followed by an visible step* A is

$$PS^*(A, M, \widetilde{M}) = PS(A, \overline{M}, \widetilde{M}) \sum_{k=0}^{\infty} PS^k(\emptyset, M, \overline{M}).$$

New transition relation: $M \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PS^*(A, M, \widetilde{M})$ and $A \neq \emptyset$.

We write $M \stackrel{A}{\twoheadrightarrow} \widetilde{M}$ if $\exists \mathcal{P}\ M \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$.

For $A = \{a\}$ we write $M \stackrel{a}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \stackrel{a}{\twoheadrightarrow} \widetilde{M}$.

$RS^*(N)$ and $RG^*(N)$ are the *visible reachability set* and *graph*.

The *visible underlying DTMC* $DTMC^*(N)$ with state space $RS^*(N)$ and transition probabilities

$$PM^*(M, \widetilde{M}) = \sum_{A \in \mathbb{N}_{fin}^{Act} \setminus \emptyset} PS^*(A, M, \widetilde{M}).$$

We write $M \twoheadrightarrow_{\mathcal{P}} \widetilde{M}$ if $\mathcal{P} = PM^*(M, \widetilde{M})$.

A *trap* is a loop of internal transitions starting and ending in some marking M which occurs with probability 1.

For each $\overline{M}$, the sum $\sum_{k=0}^{\infty} PS^k(\emptyset, M, \overline{M})$ is finite as long as no traps exist.

In this case, $PS^*(A, M, \widetilde{M})$ defines a probability distribution:

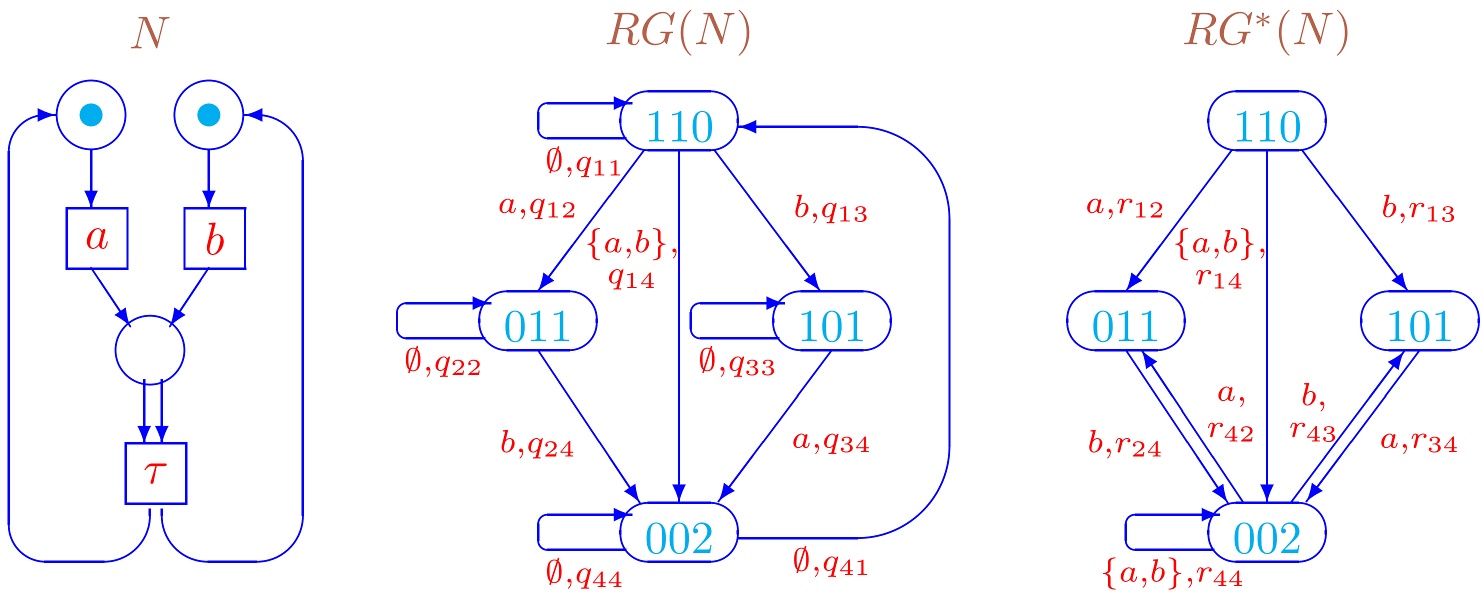
$$\sum_{A \in \mathbb{N}_{fin}^{Act} \setminus \emptyset} \sum_{\widetilde{M} \in RS^*(N)} PS^*(A, M, \widetilde{M}) = 1.$$

Interleaving semantics: the *interleaving transition relation*.

Let N be a LDTSPN, $M, \widetilde{M} \in RS^*(N)$, $a \in Act$ and $M \stackrel{a}{\twoheadrightarrow} \widetilde{M}$. We write $M \stackrel{a}{\rightharpoonup\!\!\!\rightharpoonup}_{\mathcal{P}} \widetilde{M}$, if $\mathcal{P} = PS_i^*(a, M, \widetilde{M})$ and

$$PS_i^*(a, M, \widetilde{M}) = \frac{PS^*(\{a\}, M, \widetilde{M})}{\sum_{\{b \in Act \mid \exists \overline{M}\ M \stackrel{b}{\twoheadrightarrow} \overline{M}\}} PS^*(\{b\}, M, \overline{M})}.$$

Example of LDTSPNs



A LDTSPN and the corresponding reachability graphs

$q_{11} = \overline{\Omega}_N(t_1)\overline{\Omega}_N(t_2)$ $q_{12} = \Omega_N(t_1)\overline{\Omega}_N(t_2)$ $q_{13} = \overline{\Omega}_N(t_1)\Omega_N(t_2)$

$q_{14} = \Omega_N(t_1)\Omega_N(t_2)$ $q_{22} = \overline{\Omega}_N(t_2)$ $q_{24} = \Omega_N(t_2)$

$q_{33} = \overline{\Omega}_N(t_1)$ $q_{34} = \Omega_N(t_1)$ $q_{41} = \Omega_N(t_3)$

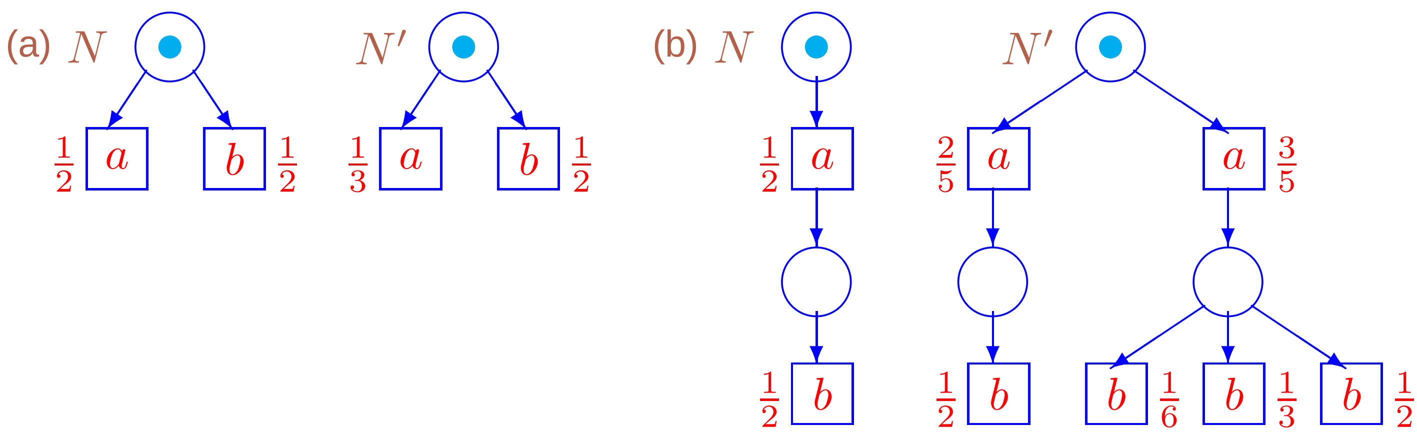
$q_{44} = \overline{\Omega}_N(t_3)$

$r_{12} = r_{42} = \frac{q_{12}}{1-q_{11}}$ $r_{13} = r_{43} = \frac{q_{13}}{1-q_{11}}$ $r_{14} = r_{44} = \frac{q_{14}}{1-q_{11}}$

$r_{24} = 1$ $r_{34} = 1$

Stochastic simulation

Properties of probabilistic relations



PP: Properties of probabilistic equivalences

- In Figure PP(a) LDTSPNs N and N' could not be related by any (even trace) probabilistic equivalence, since only in N' action a has probability $\frac{1}{3}$.
- In Figure PP(b) LDTSPNs N and N' are related by any (even bisimulation) probabilistic equivalence, since in our model probabilities of consequent actions are multiplied, and that of alternative ones are summed.

Probabilistic τ-trace equivalences

Definition 88 *A visible interleaving probabilistic trace of a LDTSPN N is a pair $(\sigma, PT^*(\sigma))$, where $\sigma = a_1 \cdots a_n \in Act^*$ and*

$$PT^*(\sigma) = \sum_{\{M_1,\ldots,M_n \mid M_N \stackrel{a_1}{\twoheadrightarrow}_{\mathcal{P}_1} M_1 \stackrel{a_2}{\twoheadrightarrow}_{\mathcal{P}_2} \cdots \stackrel{a_n}{\twoheadrightarrow}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

The set of all visible interleaving probabilistic traces of a LDTSPN N is $VisIntProbTraces(N)$. LDTSPNs N and N' are interleaving probabilistic τ-trace equivalent, $N \equiv^{\tau}_{ip} N'$, if

$$VisIntProbTraces(N) = VisIntProbTraces(N').$$

Definition 89 *A visible step probabilistic trace of a LDTSPN N is a pair $(\Sigma, PT^*(\Sigma))$, where $\Sigma = A_1 \cdots A_n \in (\mathbb{N}^{Act}_{fin})^*$ and*

$$PT^*(\Sigma) = \sum_{\{M_1,\ldots,M_n \mid M_N \stackrel{A_1}{\twoheadrightarrow}_{\mathcal{P}_1} M_1 \stackrel{A_2}{\twoheadrightarrow}_{\mathcal{P}_2} \cdots \stackrel{A_n}{\twoheadrightarrow}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

The set of all visible step probabilistic traces of a LDTSPN N is $VisStepProbTraces(N)$. LDTSPNs N and N' are step probabilistic τ-trace equivalent, $N \equiv^{\tau}_{sp} N'$, if

$$VisStepProbTraces(N) = VisStepProbTraces(N').$$

Probabilistic τ-bisimulation equivalences

Let for LDTSPN N $\mathcal{L} \subseteq RS^*(N),\ M \in RS^*(N)$ and $A \in I\!N_{fin}^{Act}$.

We write $M \overset{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{L}$ if $\mathcal{P} = PM_A^*(M, \mathcal{L})$ and

$$PM_A^*(M, \mathcal{L}) = \sum_{\{\widetilde{M} \in \mathcal{L} \mid M \overset{A}{\twoheadrightarrow} \widetilde{M}\}} PS^*(A, M, \widetilde{M}).$$

We write $M \overset{A}{\twoheadrightarrow} \mathcal{L}$ if $\exists \mathcal{P}\ M \overset{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{L}$.

For $A = \{a\}$ we write $M \overset{a}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{L}$ and $M \overset{a}{\twoheadrightarrow} \mathcal{L}$.

Similarly, we define $M \overset{a}{\rightharpoondown}_{\mathcal{P}} \mathcal{L}$ based on the interleaving transition relation.

Definition 90 *Let N be a LDTSPN. An equivalence $\mathcal{R} \subseteq RS^*(N)^2$ is a $\star$-probabilistic τ-bisimulation between M_1 and M_2 of N, $\star \in${interleaving, step}, $\mathcal{R} : M_1 \underline{\leftrightarrow}_{\star p}^{\tau} M_2,\ \star \in \{i, s\}$, if $\forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$*

- $\forall x \in Act$ *and* $\hookrightarrow = \rightharpoondown$, *if* $\star = i$;
- $\forall x \in I\!N_{fin}^{Act}$ *and* $\hookrightarrow = \twoheadrightarrow$, *if* $\star = s$;

$$M_1 \overset{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{L} \ \Leftrightarrow\ M_2 \overset{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{L}.$$

M_1 and M_2 are $\star$-probabilistic τ-bisimulation equivalent, $\star \in${interleaving, step}, $M_1 \underline{\leftrightarrow}_{\star p}^{\tau} M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_{\star p}^{\tau} M_2,\ \star \in \{i, s\}$.

Definition 91 *Let N and N' be two LDTSPNs. $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is a $\star$-probabilistic τ-bisimulation between N and N', $\star \in${interleaving, step}, $\mathcal{R} : N \underline{\leftrightarrow}_{\star p}^{\tau} N'$, if $\mathcal{R} : M_N \underline{\leftrightarrow}_{\star p}^{\tau} M_{N'},\ \star \in \{i, s\}$.*

N and N' are $\star$-probabilistic τ-bisimulation equivalent, $\star \in${interleaving, step}, $N \underline{\leftrightarrow}_{\star p}^{\tau} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{\star p}^{\tau} N',\ \star \in \{i, s\}$.

Backward probabilistic τ-bisimulation equivalences

Let for LDTSPN N $\mathcal{L} \subseteq RS^*(N)$, $M \in RS^*(N)$ and $A \in \mathbb{N}_{fin}^{Act}$.

We write $\mathcal{L} \overset{A}{\twoheadrightarrow}_{\mathcal{P}} M$ if $\mathcal{P} = PM_A^*(\mathcal{L}, M)$ and

$$PM_A^*(\mathcal{L}, M) = \sum_{\{\widetilde{M} \in \mathcal{L} \mid \widetilde{M} \overset{A}{\twoheadrightarrow} M\}} PS^*(A, \widetilde{M}, M).$$

We write $\mathcal{L} \overset{A}{\twoheadrightarrow} M$ if $\exists \mathcal{P}\ \mathcal{L} \overset{A}{\twoheadrightarrow}_{\mathcal{P}} M$.

For $A = \{a\}$ we write $\mathcal{L} \overset{a}{\twoheadrightarrow}_{\mathcal{P}} M$ and $\mathcal{L} \overset{a}{\twoheadrightarrow} M$.

Similarly, we define $\mathcal{L} \overset{a}{\rightharpoondown\!\!\!\!\rightharpoondown}_{\mathcal{P}} M$ based on the interleaving transition relation.

Definition 92 *Let N be a LDTSPN. An* equivalence $\mathcal{R} \subseteq RS^*(N)^2$ *is a* $\star$-backward probabilistic τ-bisimulation *between M_1 and M_2 of N,* $\star \in$ {interleaving, step}, $\mathcal{R} : M_1 \underline{\leftrightarrow}^{\tau}_{\star bp} M_2$, $\star \in \{i, s\}$, *if* $\forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$

- $\forall x \in Act$ *and* $\hookrightarrow = \rightharpoondown\!\!\!\!\rightharpoondown$, *if* $\star = i$;
- $\forall x \in \mathbb{N}_{fin}^{Act}$ *and* $\hookrightarrow = \twoheadrightarrow$, *if* $\star = s$;

$$[M_N]_{\mathcal{R}} = \{M_N\},$$

$$M_1 \overset{x}{\hookrightarrow}_{\mathcal{P}} RS^*(N) \ \Leftrightarrow\ M_2 \overset{x}{\hookrightarrow}_{\mathcal{P}} RS^*(N),$$

$$\mathcal{L} \overset{x}{\hookrightarrow}_{\mathcal{P}} M_1 \ \Leftrightarrow\ \mathcal{L} \overset{x}{\hookrightarrow}_{\mathcal{P}} M_2.$$

M_1 *and* M_2 *are* $\star$-backward probabilistic τ-bisimulation equivalent, $\star \in$ {interleaving, step}, $M_1 \underline{\leftrightarrow}^{\tau}_{\star bp} M_2$, *if* $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}^{\tau}_{\star bp} M_2$, $\star \in \{i, s\}$.

The *indicator function* Γ recovers a LDTSPN by a marking belonging to it.

For LDTSPN N and $M \in RS^*(N)$ we define $\Gamma(M) = N$.

Definition 93 *Let N and N' be two LDTSPNs.*
$\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ *is a* $\star$-backward probabilistic τ-bisimulation *between* N *and* N', $\star \in$ {interleaving, step}, $\mathcal{R} : N \underline{\leftrightarrow}_{\star bp}^{\tau} N'$, $\star \in \{i, s\}$, *if* $\forall \mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}\ \forall M_1, M_2 \in \mathcal{L}$

- $\forall x \in Act$ *and* $\hookrightarrow = \twoheadrightarrow$, *if* $\star = i$;
- $\forall x \in \mathbb{N}_{fin}^{Act}$ *and* $\hookrightarrow = \twoheadrightarrow$, *if* $\star = s$;

$$[M_N]_{\mathcal{R}} = \{M_N, M_{N'}\},$$

$$M_1 \stackrel{x}{\hookrightarrow}_{\mathcal{P}} RS^*(\Gamma(M_1)) \ \Leftrightarrow\ M_2 \stackrel{x}{\hookrightarrow}_{\mathcal{P}} RS^*(\Gamma(M_2)),$$

$$\mathcal{K} \stackrel{x}{\hookrightarrow}_{\mathcal{P}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_1))|}{|\mathcal{K}\cap RS^*(\Gamma(M_1))|}} M_1 \ \Leftrightarrow\ \mathcal{K} \stackrel{x}{\hookrightarrow}_{\mathcal{P}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_2))|}{|\mathcal{K}\cap RS^*(\Gamma(M_2))|}} M_2.$$

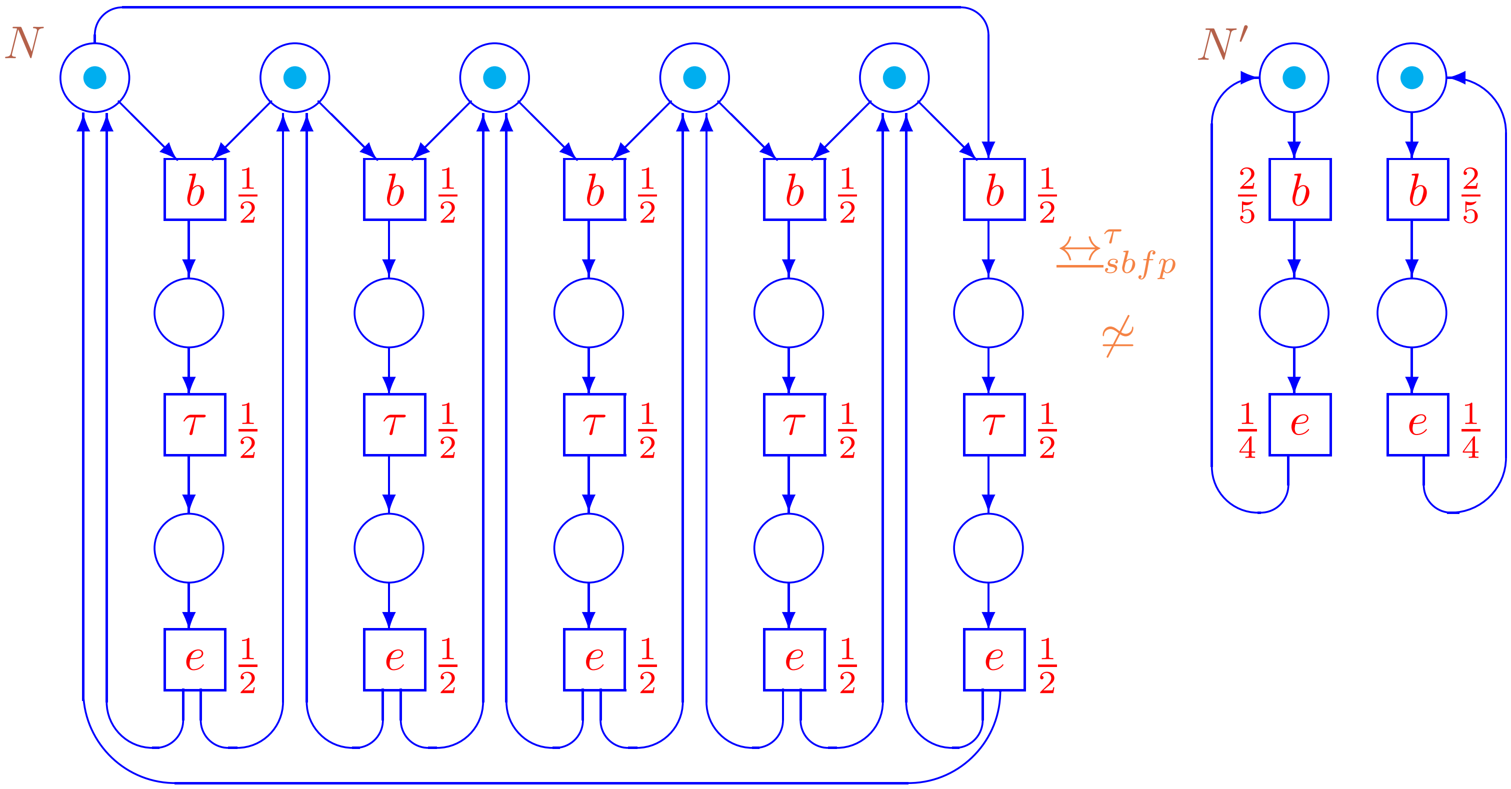
N *and* N' *are* $\star$-backward probabilistic τ-bisimulation equivalent, $\star \in$ {interleaving, step}, $N \underline{\leftrightarrow}_{\star bp}^{\tau} N'$, *if* $\exists \mathcal{R} : N \underline{\leftrightarrow}_{\star bp}^{\tau} N'$, $\star \in \{i, s\}$.

Back and forth probabilistic τ-bisimulation equivalences

Definition 94 *LDTSPNs* N *and* N' *are* $\star$-back and forth probabilistic τ-bisimulation equivalent, $\star \in \{$interleaving, step$\}$, $N \underline{\leftrightarrow}^{\tau}_{\star bfp} N'$, *if*

$$N \underline{\leftrightarrow}^{\tau}_{\star p} N' \text{ and } N \underline{\leftrightarrow}^{\tau}_{\star bp} N', \ \star \in \{i, s\}.$$

Reduction example



The complete and reduced LDTSPNs of the abstract dining philosophers system

b and e correspond to the beginning and the end of eating of some philosopher.

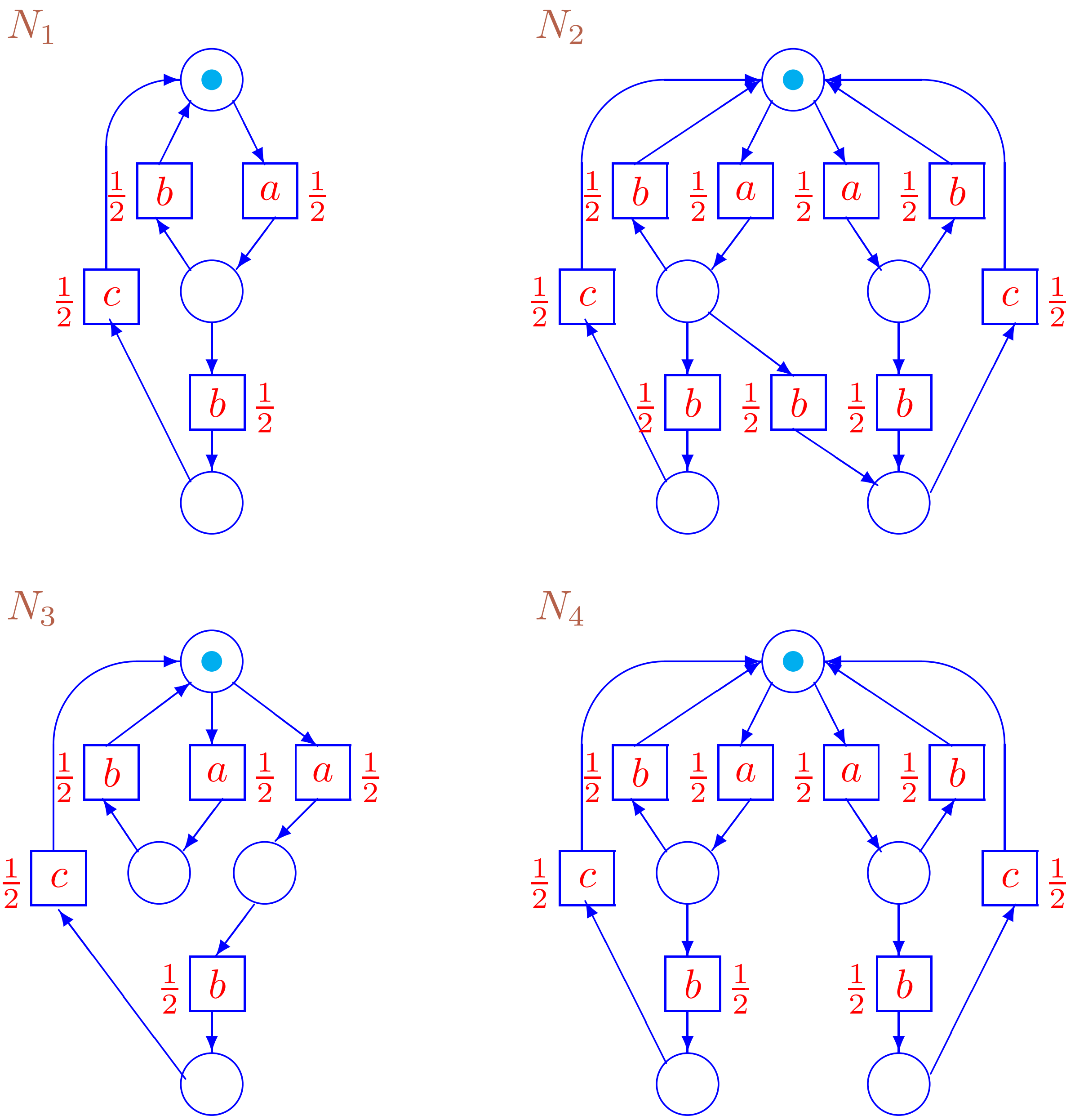
τ corresponds to an activity of some philosopher during the eating.

This activity is not respected in behavioural analysis of the system.

$N \underline{\leftrightarrow}^{\tau}_{sbfp} N'$, hence, N' is a reduction of N w.r.t. $\underline{\leftrightarrow}^{\tau}_{sbfp}$.

$N \not\simeq N'$, since N' is smaller than N.

Examples of the probabilistic relations

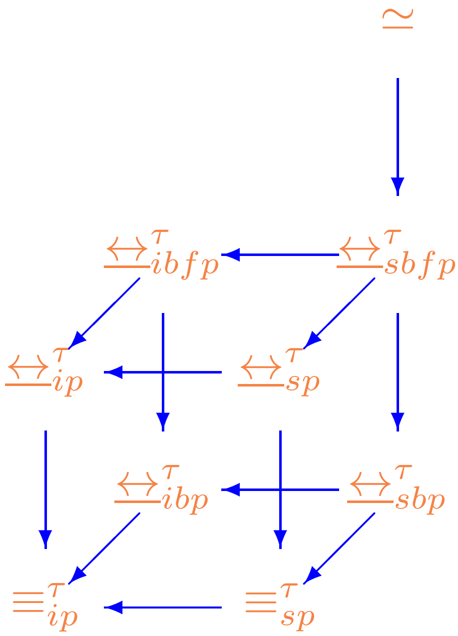


LDTSPNs related via different probabilistic τ-equivalences

$N_1 \equiv^{\tau}_{sp} N_2 \equiv^{\tau}_{sp} N_3 \equiv^{\tau}_{sp} N_4$ $\quad N_1 \underline{\leftrightarrow}^{\tau}_{sp} N_2 \underline{\leftrightarrow}^{\tau}_{sp} N_4$ $\quad N_1 \underline{\leftrightarrow}^{\tau}_{sbp} N_3 \underline{\leftrightarrow}^{\tau}_{sbp} N_4$

$N_1 \underline{\leftrightarrow}^{\tau}_{sbfp} N_4$ $\quad N_2 \not\underline{\leftrightarrow}^{\tau}_{ip} N_3$ $\quad N_2 \not\underline{\leftrightarrow}^{\tau}_{ibp} N_3$

Comparing the probabilistic τ-equivalences



Interrelations of the probabilistic τ-equivalences

Proposition 16 *Let* $\star \in \{i, s\}$. *For LDTSPNs* N *and* N'

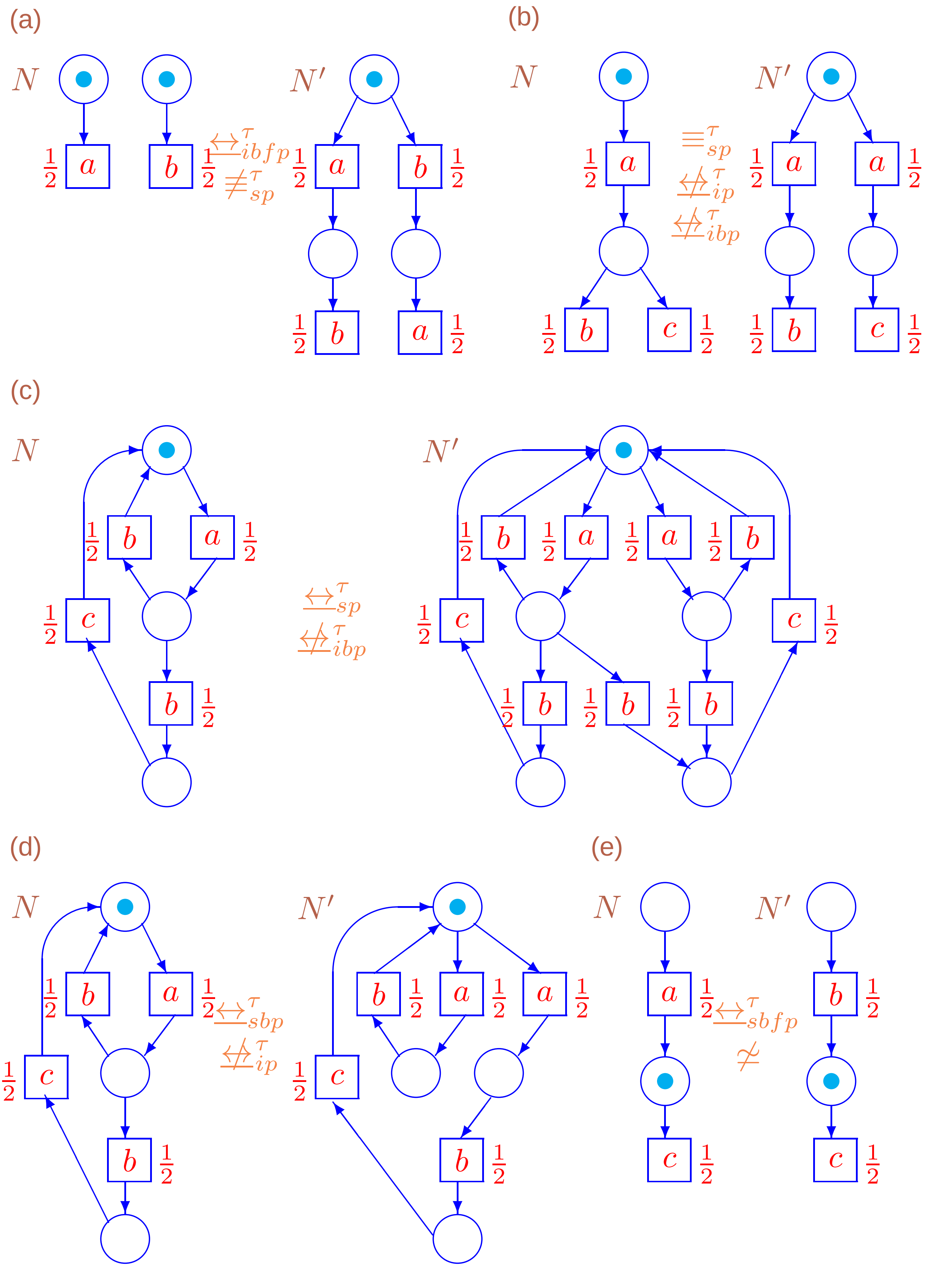
1. $N \underline{\leftrightarrow}^{\tau}_{\star p} N' \;\Rightarrow\; N \equiv^{\tau}_{\star p} N'$;
2. $N \underline{\leftrightarrow}^{\tau}_{\star bp} N' \;\Rightarrow\; N \equiv^{\tau}_{\star p} N'$;
3. $N \underline{\leftrightarrow}^{\tau}_{\star bfp} N' \;\Rightarrow\; N \underline{\leftrightarrow}^{\tau}_{\star p} N'$ *and* $N \underline{\leftrightarrow}^{\tau}_{\star bp} N'$.

Theorem 22 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\equiv^{\tau}, \underline{\leftrightarrow}^{\tau}, \simeq\}$ *and* $\star, \star\star \in \{\_, ip, sp, ibp, sbp, ibfp, sbfp\}$. *For LDTSPNs* N *and* N'

$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \twoheadleftarrow\!\!\twoheadrightarrow_{\star\star} N'$$

iff in the graph in figure above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\twoheadrightarrow_{\star\star}$.

Examples of the probabilistic relations



S: Examples of the probabilistic τ-equivalences

- In Figure S(a), $N \underline{\leftrightarrow}^{\tau}_{ibfp} N'$, but $N \not\equiv^{\tau}_{sp} N'$, since only in the LDTSPN N' actions a and b cannot occur concurrently.
- In Figure S(b), $N \equiv^{\tau}_{sp} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ip} N'$ and $N \underline{\not\leftrightarrow}^{\tau}_{ibp} N'$, since only in the LDTSPN N' an action a can occur so that no action b can occur afterwards.
- In Figure S(c), $N \underline{\leftrightarrow}^{\tau}_{sp} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ibp} N'$, since only in N' there is a place with two input transitions labeled by b. Hence, the probability for a token to go to this place is always more than for that with only one input b-labeled transition.
- In Figure S(d), $N \underline{\leftrightarrow}^{\tau}_{sbp} N'$, but $N \underline{\not\leftrightarrow}^{\tau}_{ip} N'$, since only in the LDTSPN N' an action a can occur so that a sequence of actions bc cannot occur just after it.
- In Figure S(e), $N \underline{\leftrightarrow}^{\tau}_{sbfp} N'$ but $N \not\simeq N'$, since upper transitions of LDTSPNs N and N' are labeled by different actions (a and b).

Logic $IPML$

Definition 95 *$\top$ denotes the truth, $a \in Act$, $\mathcal{P} \in (0;1]$.*

A formula of $IPML$:

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \nabla_a \mid \langle a\rangle_{\mathcal{P}}\Phi$$

IPML is the set of *all formulas* of $IPML$.

Definition 96 *Let N be a LDTSPN and $M \in RS^*(N)$. The satisfaction relation $\models_N \subseteq RS^*(N) \times \mathbf{IPML}$:*

1. *$M \models_N \top$ — always;*
2. *$M \models_N \neg\Phi$, if $M \not\models_N \Phi$;*
3. *$M \models_N \Phi\wedge\Psi$, if $M \models_N \Phi$ and $M \models_N \Psi$;*
4. *$M \models_N \nabla_a$, if not $M \stackrel{a}{\twoheadrightarrow} RS^*(N)$;*
5. *$M \models_N \langle a\rangle_{\mathcal{P}}\Phi$, if $\exists\mathcal{L} \subseteq RS^*(N)\ M \stackrel{a}{\twoheadrightarrow}_{\mathcal{Q}} \mathcal{L},\ \mathcal{Q} \geq \mathcal{P}$ and $\forall\widetilde{M} \in \mathcal{L}\ \widetilde{M} \models_N \Phi$.*

$\langle a\rangle\Phi = \exists\mathcal{P}\ \langle a\rangle_{\mathcal{P}}\Phi$.

$\langle a\rangle_{\mathcal{Q}}\Phi$ implies $\langle a\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

We write $N \models_N \Phi$, if $M_N \models_N \Phi$.

Definition 97 *N and N' are logical equivalent in $IPML$, $N{=_{IPML}}N'$, if $\forall\Phi \in \mathbf{IPML}\ N \models_N \Phi \Leftrightarrow N' \models_{N'} \Phi$.*

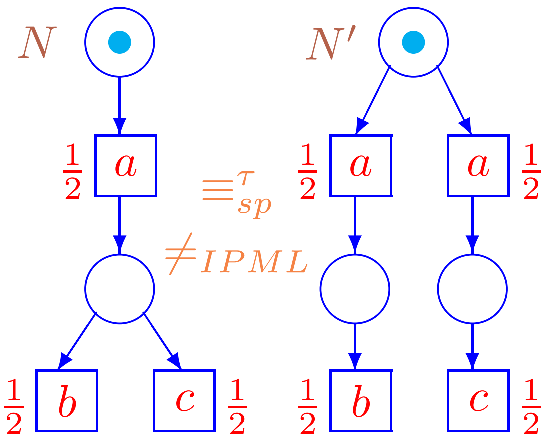
Let for a LDTSPN N $M \in RS^*(N),\ a \in Act$.

The set of *next* to M markings *after occurrence of visible action* a (*visible image set*) is $VisImage(M, a) = \{\widetilde{M} \mid M \stackrel{a}{\twoheadrightarrow} \widetilde{M}\}$.

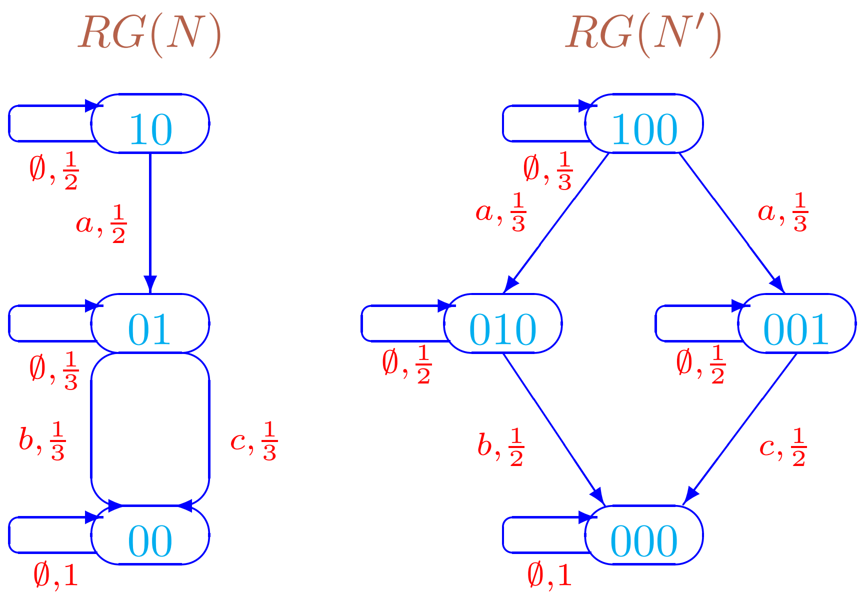
A LDTSPN N is a *image-finite* one, if $\forall M \in RS^*(N)\, \forall a \in Act\ |VisImage(M, a)| < \infty$.

Theorem 23 *For image-finite LDTSPNs N and N'*

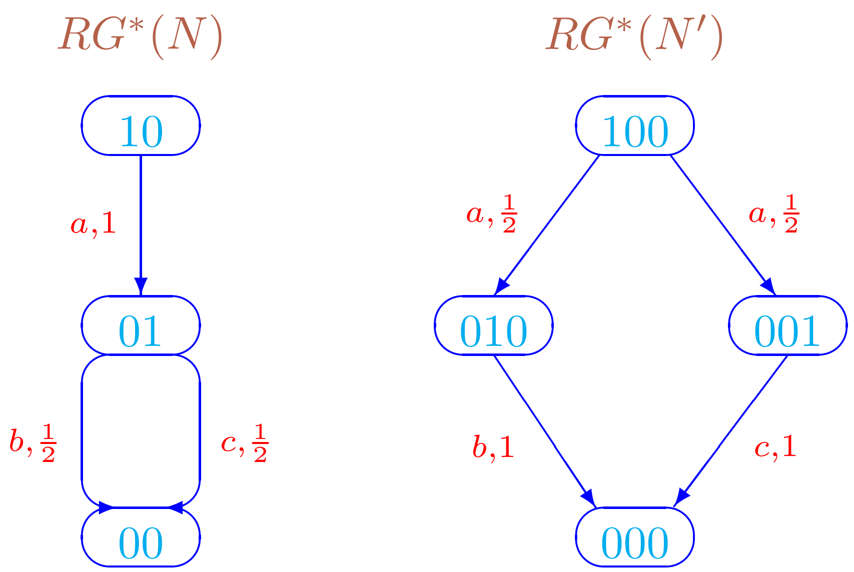
$$N \underline{\leftrightarrow}_{ip}^{\tau} N' \ \Leftrightarrow\ N =_{IPML} N'.$$



Differentiating power of $=_{IPML}$



Reachability graphs of the LDTSPNs above



Visible reachability graphs of the LDTSPNs above

$N \equiv_{sp}^{\tau} N'$, but $N \not\equiv_{IPML} N'$, because for $\Phi = \langle a\rangle_1 \langle b\rangle_{\frac{1}{2}} \top,\ N \models_N \Phi$, but $N' \not\models_{N'} \Phi$, since only in N' an action a can occur so that no action b can occur afterwards.

Logic $SPML$

Definition 98 *$\top$ denotes the truth, $A \in I\!N_{fin}^{Act}$, $\mathcal{P} \in (0;1]$.*

A formula of $SPML$:

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \nabla_A \mid \langle A\rangle_{\mathcal{P}}\Phi$$

SPML is the set of *all formulas* of $SPML$.

Definition 99 *Let N be a LDTSPN and $M \in RS^*(N)$. The satisfaction relation $\models_N \subseteq RS^*(N) \times \mathbf{SPML}$:*

1. $M \models_N \top$ — *always;*
2. $M \models_N \neg\Phi$, *if* $M \not\models_N \Phi$;
3. $M \models_N \Phi\wedge\Psi$, *if* $M \models_N \Phi$ *and* $M \models_N \Psi$;
4. $M \models_N \nabla_A$, *if not* $M \xrightarrow{A} RS^*(N)$;
5. $M \models_N \langle A\rangle_{\mathcal{P}}\Phi$, *if* $\exists\mathcal{L} \subseteq RS^*(N)\ M \xrightarrow{A}_{\mathcal{Q}} \mathcal{L},\ \mathcal{Q} \geq \mathcal{P}$ *and* $\forall\widetilde{M} \in \mathcal{L}\ \widetilde{M} \models_N \Phi$.

$\langle A\rangle\Phi = \exists\mathcal{P}\ \langle A\rangle_{\mathcal{P}}\Phi$.

$\langle A\rangle_{\mathcal{Q}}\Phi$ implies $\langle A\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

We write $N \models_N \Phi$, if $M_N \models_N \Phi$.

Definition 100 *N and N' are logical equivalent in $SPML$, $N =_{SPML} N'$, if $\forall\Phi \in \mathbf{SPML}\ N \models_N \Phi \Leftrightarrow N' \models_{N'} \Phi$.*

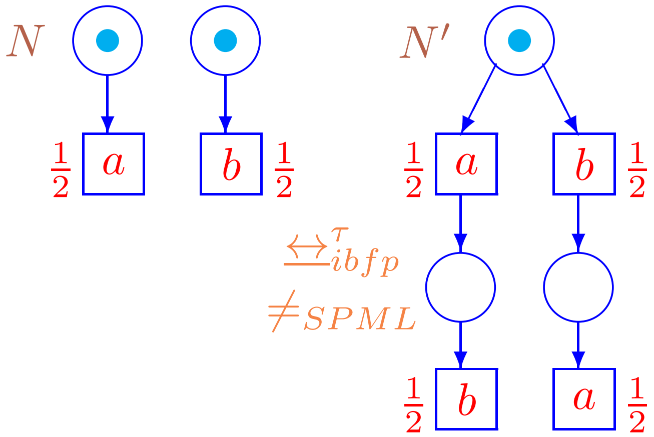
Let for a LDTSPN N $M \in RS^*(N),\ A \in \mathbb{N}_{fin}^{Act}$.

The set of *next* to M markings *after occurrence of multiset of visible actions* A (*visible image set*) is $VisImage(M, A) = \{\widetilde{M} \mid M \stackrel{A}{\twoheadrightarrow} \widetilde{M}\}$.

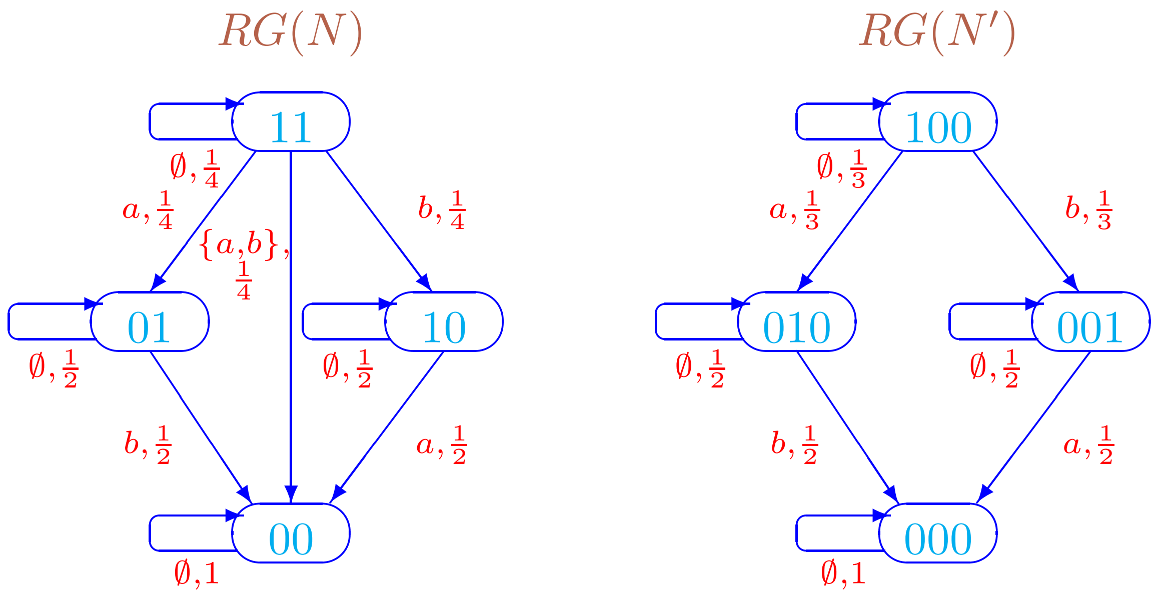
A LDTSPN N is a *image-finite* one, if
$\forall M \in RS^*(N)\ \forall A \in \mathbb{N}_{fin}^{Act}\ |VisImage(M, A)| < \infty$.

Theorem 24 *For image-finite LDTSPNs* N *and* N'

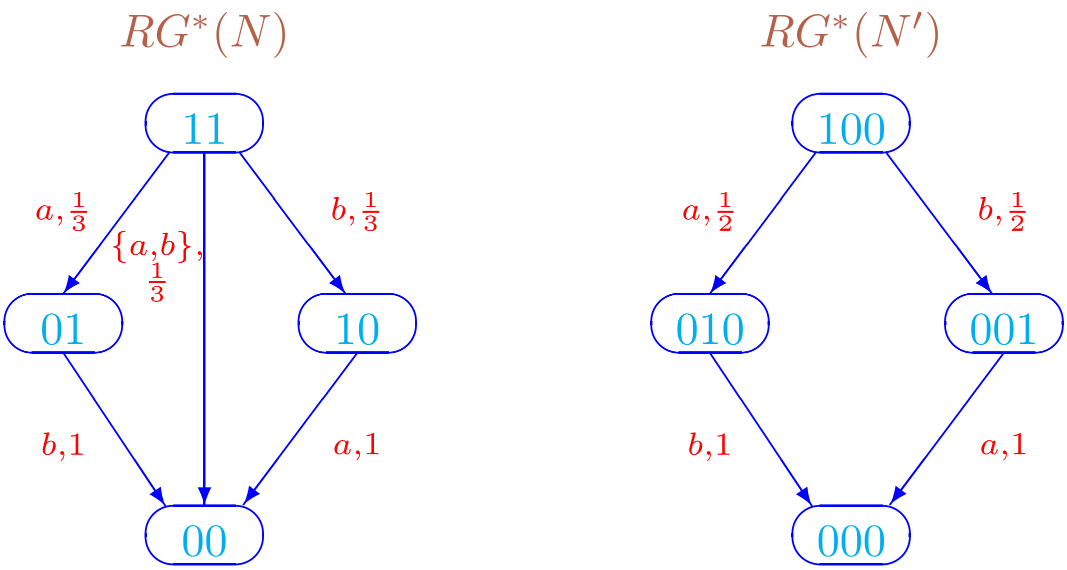
$$N \underline{\leftrightarrow}_{sp}^{\tau} N' \ \Leftrightarrow\ N =_{SPML} N'.$$



Differentiating power of $=_{SPML}$



Reachability graphs of the LDTSPNs above



Visible reachability graphs of the LDTSPNs above

$N \underline{\leftrightarrow}^{\tau}_{ibfp} N'$ but $N \neq_{SPML} N'$, because for $\Phi = \langle \{a, b\} \rangle_{\frac{1}{3}} \top, \ N \models_N \Phi$, but $N' \not\models_{N'} \Phi$, since only in N' actions a and b cannot occur concurrently.

Stationary behaviour

The PMF ψ^* for the *embedded steady-state distribution* after occurrence of a visible action is the unique solution of

$$\begin{cases} \sum_{\widetilde{M}\in RS^*(N)} \psi^*(\widetilde{M}) \cdot PM^*(\widetilde{M}, M) = \psi^*(M) \\ \sum_{M\in RS^*(N)} \psi^*(M) = 1 \end{cases}.$$

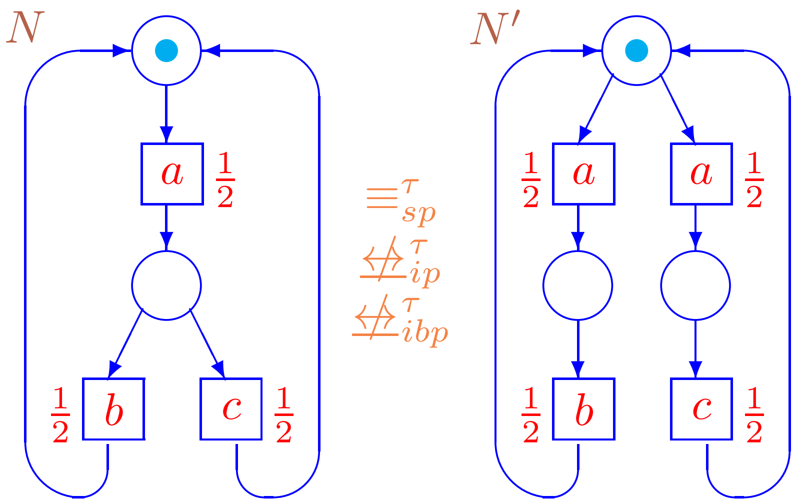
A *visible step trace* of LDTSPN N is a chain $\Sigma = A_1 \cdots A_n \in Act^*$, where $\exists M \in RS^*(N)\ M \stackrel{A_1}{\twoheadrightarrow}_{\mathcal{P}_1} M_1 \stackrel{A_2}{\twoheadrightarrow}_{\mathcal{P}_2} \cdots \stackrel{A_n}{\twoheadrightarrow}_{\mathcal{P}_n} M_n$. The *probability of the step trace Σ to start in the marking M* is

$$PS^*(\Sigma, M) = \sum_{\{M_1,\ldots,M_n \mid M \stackrel{A_1}{\twoheadrightarrow}_{\mathcal{P}_1} M_1 \stackrel{A_2}{\twoheadrightarrow}_{\mathcal{P}_2} \cdots \stackrel{A_n}{\twoheadrightarrow}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

Theorem 25 *Let Σ be a visible step trace of LDTSPNs N and N' and $\mathcal{R} : \underline{\leftrightarrow}^{\tau}_{sp} N'$ or $N\mathcal{R} : \underline{\leftrightarrow}^{\tau}_{sbp} N'$. Then $\forall \mathcal{L} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$\sum_{M\in\mathcal{L}\cap RS^*(N)} \psi^*(M) PS^*(\Sigma, M) = \sum_{M'\in\mathcal{L}\cap RS^*(N')} \psi^*(M') PS^*(\Sigma, M').$$

The trace equivalences do not guarantee the equality from the theorem above.



LDTSPNs for which the equality from the theorem above does not hold

In the figure above, $N \equiv^{\tau}_{sp} N'$, but $N \not\underline{\leftrightarrow}^{\tau}_{ip} N'$ and $N \not\underline{\leftrightarrow}^{\tau}_{ibp} N'$.

The equality from the theorem above does not hold.

For N, the probabilities of being in the possible markings is $\frac{1}{2}, \ \frac{1}{2}$.

For N', the probabilities of being in the possible markings is $\frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{4}$.

Stochastic process algebra $StAFP_0$

Algebra of finite nondeterministic parallel processes AFP_0 [KCh85].

Specification of acyclic nets (A-nets, ANs).

Stochastic algebra of finite processes $StAFP_0$.

Specification of stochastic A-nets (SANs).

Syntax

An *activity* (a, ω):

- $a \in Act$ is the *action* label;
- $\omega \in (0; 1)$ is the *probability* of action a.

AP is the set of *all activities*.

Operations: *concurrency* $\|$, *precedence* ;, *alternative* $\bigtriangledown$.

Definition 101 *Let* $(a, \omega) \in AP$. *A formula of* $StAFP_0$:

$$P ::= \ (a, \omega) \mid P\|P \mid P;P \mid P\bigtriangledown P.$$

$\mathbf{StAFP_0}$ is the set of *all formulas* of $StAFP_0$.

Semantics

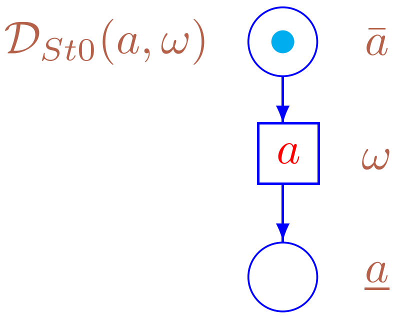
Formulas of $StAFP_0$ specify a subclass of LDTSPNs, *Stochastic A-nets (SANs)*: $T_N \subseteq Act,\ L_N = id_{T_N},\ M_N = {}^\bullet N$.

Thus, a SAN is specified by a quadruple $N = (P_N, T_N, W_N, \Omega_N)$.

The *net representation* of formulas, a mapping $\mathcal{D}_{St0}$ from $\mathbf{StAFP_0}$ to SANs.

Let $(a, \omega) \in AP$. An *atomic net* $\mathcal{D}_{St0}(a, \omega) = (P_N, T_N, W_N, \Omega_N)$, where

- $P_N = \{\bar{a}, \underline{a}\}$;
- $T_N = \{a\}$;
- $W_N = \{(\bar{a}, a), (a, \underline{a})\}$;
- $\Omega_N = \{(a, \omega)\}$.



An atomic net

Let $N = (P_N, T_N, W_N, \Omega_N)$ be a SAN and $Q, R \subseteq P_N$.

A *forming* operation $\otimes$:

$$Q \otimes R = \{q \cup r \mid q \in Q,\ r \in R\}.$$

The *merging* operation μ over a SAN $N = (P_N, T_N, W_N, \Omega_N)$ merges two sets of its places $Q, R \subseteq P$:

$$\mu(N, Q, R) = (\widetilde{P}_N, T_N, \widetilde{W}_N, \Omega_N), \text{ where}$$

- $\widetilde{P}_N = P_N \setminus (Q \cup R) \cup (Q \otimes R)$;
- $\forall t \in T_N\ \widetilde{W}_N(p,t) =$
$$\begin{cases} W_N(p,t), & p \in \widetilde{P}_N \setminus (Q \otimes R); \\ \max\{W_N(r,t), W_N(q,t)\}, & p = (q \cup r) \in Q \otimes R, \\ & q \in Q,\ r \in R. \end{cases}$$
$\forall t \in T_N\ \widetilde{W}_N(t,p) =$
$$\begin{cases} W_N(t,p), & p \in \widetilde{P}_N \setminus (Q \otimes R); \\ \max\{W_N(t,r), W_N(t,q)\}, & p = (q \cup r) \in Q \otimes R, \\ & q \in Q,\ r \in R. \end{cases}$$

Let $N = (P_N, T_N, W_N, \Omega_N)$ and $N' = (P_{N'}, T_{N'}, W_{N'}, \Omega_{N'})$ be two SANs. Net operations:

Concurrency $N \| N' = (P_N \cup P_{N'}, T_N \cup T_{N'}, W_N \cup W_{N'}, \Omega)$, where

$$\Omega(a) = \begin{cases} \Omega_N(a), & a \in T_N \setminus T_{N'}; \\ \Omega_{N'}(a), & a \in T_{N'} \setminus T_N; \\ \Omega_N(a) \cdot \Omega_{N'}(a), & a \in T_N \cap T_{N'}. \end{cases}$$

Precedence $N;N' = \mu(N \| N', N^\bullet, {}^\bullet N')$.

Alternative $N \bigtriangledown N' = \mu(\mu(N \| N', {}^\bullet N, {}^\bullet N'), N^\bullet, N'^\bullet)$.

Nets N and N' combined by $;$ and $\bigtriangledown$ contain no equally named transitions.

Formulas P and P' combined by $;$ and $\bigtriangledown$ contain no identical actions.

Let $P, Q \in \mathbf{StAFP_0}$. The net representation of combined formulas:

1. $\mathcal{D}_{St0}(P\|Q) = \mathcal{D}_{St0}(P)\|\mathcal{D}_{St0}(Q)$;
2. $\mathcal{D}_{St0}(P;Q) = \mathcal{D}_{St0}(P);\mathcal{D}_{St0}(Q)$;
3. $\mathcal{D}_{St0}(P\triangledown Q) = \mathcal{D}_{St0}(P)\triangledown\mathcal{D}_{St0}(Q)$.

Definition 102 *Formulas* P *and* P' *are* semantic equivalent *in* $StAFP_0$, $P =_{St0} P'$, *if* $\mathcal{D}_{St0}(P) \simeq \mathcal{D}_{St0}(P')$.

Axiomatization

Let $P \in \mathbf{StAFP_0}$. The *structure* of P, $\phi_P \in \mathbf{AFP_0}$, specifies the non-stochastic process: replace each activity (a, ω) of P by a.

The *action probability function* Ω_P from actions contained in activities of P to $(0; 1)$. Let $(a, \omega_1), \ldots, (a, \omega_n)$ be *all* activities of P with action a. Then $\Omega_P(a) = \omega_1 \cdots \omega_n$.

The axiom system Θ_{St0}: in accordance with $=_{St0}$. Here $a \in Act$ and $P, Q, G \in \mathbf{StAFP_0}$.

1. Associativity

1.1 $P \| (Q \| R) = (P \| Q) \| R$

1.2 $P; (Q; R) = (P; Q); R$

1.3 $P \bigtriangledown (Q \bigtriangledown R) = (P \bigtriangledown Q) \bigtriangledown R$

2. Commutativity

2.1 $P \| Q = Q \| P$

2.2 $P \bigtriangledown Q = Q \bigtriangledown P$

3. Distributivity

3.1 $P; (Q \| R) = (P_1; Q) \| (P_2; R),\ \phi_P = \phi_{P_1} = \phi_{P_2},\ \Omega_P = \Omega_{P_1} \cdot \Omega_{P_2}$

3.2 $(P \| Q); R = (P; R_1) \| (Q; R_2),\ \phi_R = \phi_{R_1} = \phi_{R_2},\ \Omega_R = \Omega_{R_1} \cdot \Omega_{R_2}$

3.3 $P \bigtriangledown (Q \| R) = (P_1 \bigtriangledown Q) \| (P_2 \bigtriangledown R),\ \phi_P = \phi_{P_1} = \phi_{P_2},\ \Omega_P = \Omega_{P_1} \cdot \Omega_{P_2}$

4. Probability

4.1 $P = P_1 \| P_2,\ \phi_P = \phi_{P_1} = \phi_{P_2},\ \Omega_P = \Omega_{P_1} \cdot \Omega_{P_2}$

The axiom system Θ_{St0} is sound w.r.t. the equivalence $=_{St0}$.

A formula $P \in \mathbf{StAFP_0}$ is a *totally stratified* one iff $P = P_1 \| \cdots \| P_n,\ n \geq 1$ and each $P_i\ (1 \leq i \leq n)$ is a *primitive formula*, does not contain $\|$.

Theorem 26 *Any formula* $P \in \mathbf{StAFP_0}$ *can be transformed (with the use of* Θ_{St0}*) into an equivalent (via* $=_{St0}$*) totally stratified one.*

Overview and open questions

The results obtained

- A new class of stochastic Petri nets with labeled transitions and a step semantics for transition firing (LDTSPNs).
- Equivalences for LDTSPNs which preserve interesting aspects of behavior and thus can be used

  to compare systems and to compute for a given one a minimal equivalent representation [Buc95].
- A diagram of interrelations for the equivalences.
- Logical characterization of the equivalences via probabilistic modal logics.
- An application of the equivalences for comparing stationary behavior of LDTSPNs.
- Stochastic algebra of finite processes $StAFP_0$ for specification of stochastic A-nets (SANs).
- A sound axiomatization of the net equivalence.

Further research

- Other equivalences in interleaving and step semantics:

  interleaving branching bisimulation [PRS92]

  (respecting conflicts with invisible transitions),

  back-forth bisimulations [NMV90,Pin93]

  (moving backward along history of computation).

- True concurrent equivalences:

  partial word and *pomset relations* [PRS92,Vog92,MCW03]

  (partial order models of computation).

- Logical characterization of *back and back-forth* equivalences:

  probabilistic extension of back-forth logic (BFL) [CLP92]

  (probabilistic eventuality operator for back moves).

- More flexible process algebras:

  Petri box calculus (PBC) [BDH92]

  (infinite processes: recursion and iteration).

Equivalences for process algebras: calculus AFP_2

Abstract: A process algebra AFP_2 was proposed by L.A. Cherkasova in 1989. It has a semantics of posets with non-actions and deadlocked actions to respect non-determinism.

Via formulas of AFP_2, one can analyze behavior of A-nets (Acyclic nets). The considered Petri net equivalences are investigated on this net subclass.

Semantic equivalences of formulas AFP_2 (algebraic equivalences) are transferred into A-nets, and their interrelations with the net equivalences are investigated.

A term rewrite system RWS_2 is produced from axiom system Θ_2 for semantic equivalences. Its confluence (in the case of termination) is proved.

A method of automatic check for algebraic equivalences based on RWS_2 was implemented as a program $CANON$ in C programming language.

Keywords: Process algebras, syntax, semantics, semantic (algebraic) equivalences, axiomatization, A-nets, net equivalences, term rewrite systems, implementation.

Contents

Introduction

Process algebras: semantics of concurrency

In process calculi, a process is specified by an algebraic formula.

A verification of its properties is accomplished by means of equivalences, axioms and inference rules.

The calculi below construct a process from atomic *actions* with *precedence, parallelism, non-determinism* and some auxiliary operations.

1. Interleaving semantics.

   CCS [Mil80], CSP [Hoa80], $TCSP$ [Hoa85,Old87a], BPA [BK89].

   Concurrency is interpreted as sequential non-determinism.

2. Step semantics.

   $SCCS$ [Mil83], ACP [BK84], $CCSP$ [Old87b], PBC [BDH92].

   A special operator for simultaneous occurrence of actions.

3. Pomset semantics.

   Algebra of event structures [BCa87].

   A causal dependence relation over actions imposes partial ordering. Two actions are parallel if they are causally independent.

Interleaving calculi are more suitable in technical staff.

Algebras based on step and pomset semantics have more natural specification of concurrency.

Process algebras: specification and analysis

1. *Descriptive* calculi.

   They provide a description of structural properties of systems: specification.

   An example is AFP_0 [Ch89].

2. *Analytical* calculi.

   They combine mechanisms as for specification of processes as for investigation of their behavioral properties: analysis, verification.

   An example is AFP_2 [Ch89].

Calculus AFP_2

Algebra of finite processes AFP_2

AFP_2 has semantics of posets with non-actions and deadlocked actions (to respect non-determinism).

A synchronization is by action name. The only event corresponds to equally named actions.

Syntax

The symbol alphabets.

- $\alpha = \{a, b, \ldots\}$ is an alphabet of *actions*.
- $\bar{\alpha} = \{\bar{a}, \bar{b}, \ldots\}$ is an alphabet of *non-actions*.
- $\Delta_\alpha = \{\delta_a, \delta_b, \ldots\}$ is an alphabet of *deadlocked actions*.

$\hat{\alpha} = \alpha \cup \bar{\alpha} \cup \Delta_\alpha$.

Symbols of $\hat{\alpha}$ are combined into formulas by operations ; (*precedence*), $\bigtriangledown$ (*exclusive or, alternative*), $\|$ (*concurrency*), $\vee$ (*disjunction, union*), $\rceil\!\rceil$ (*"not occur"*), $\tilde{\rceil\!\rceil}$ (*"mistaken not occur"*).

Definition 103 *A formula of AFP_2 is:*

$$P ::= \; a \mid \bar{a} \mid \delta_a \mid \rceil\!\rceil a \mid \tilde{\rceil\!\rceil} P \mid P;Q \mid P\|Q \mid P\bigtriangledown Q \mid P\vee Q.$$

Here $a \in \alpha,\ \bar{a} \in \bar{\alpha},\ \delta_a \in \Delta_\alpha$ are *elementary formulas*.

$\mathbf{AFP_2}$ is the set of *all formulas* of AFP_2.

Definition 104 *Formulas P and P' of AFP_2 are isomorphic, $P \simeq P'$, if they coincide up to associativity w.r.t. $;\,, \|, \vee, \bigtriangledown$ and commutativity w.r.t. $\|, \vee, \bigtriangledown$.*

For example, $(a\|b\|\bar{c}) \vee (c\|\bar{a}\|\bar{b}) \simeq (\bar{a}\|\bar{b}\|c) \vee (b\|a\|\bar{c})$.

Denotational semantics

Let $X \subseteq \hat{\alpha}$. We propose the following notations.

- $X^+ = X \cap \alpha$ is the subset of *actions* of X;
- $X^- = X \cap \bar{\alpha}$ is the subset of *non-actions* of X;
- $\Delta_X = X \cap \Delta_\alpha$ is the subset of *deadlocked actions* of X.

We consider only posets $\rho = (X, \prec)$ over $\hat{\alpha}$ with the following restrictions.

1. a, $\bar{a}$ and δ_a do not occur in X together;
2. $\prec$ is irreflexive;
3. $\forall x, y \in X^- \cup \Delta_X\ (x \not\prec y) \wedge (y \not\prec x)$, all elements of $X^- \cup \Delta_X$ are incomparable;
4. $\forall x \in X^+\ \forall y \in X^- \cup \Delta_X\ (x \not\prec y) \wedge (y \not\prec x)$, all elements of X^+ and $X^- \cup \Delta_X$ are incomparable.

The *modified union* of posets absorbs equal computations and ones which can be continued in another behaviour of nondeterministic process.

$$\rho \tilde{\cup} \rho' = \begin{cases} \rho, & \rho' \trianglelefteq \rho; \\ \rho', & \rho \trianglelefteq \rho'; \\ \{\rho, \rho'\}, & \text{otherwise}. \end{cases}$$

The operations over posets are introduced: ; (*precedence*), $\|$ (*concurrency*), $\bigtriangledown$ (*alternative*), $\rceil\!\rceil$ (*not occur*) and $\tilde{\rceil\!\rceil}$ (*mistaken not occur*).

If a constructed poset ρ does not satisfy the conditions 1-4, we “correct” it with *regularization* operation $Regul$.

It singles out the maximal prefix of ρ “before” some contradictions arise. All the actions occuring “after” that contradictions are announced as the deadlocked ones.

- $D_1 = \{\delta_a \mid (a \in X) \wedge (a \prec a)\} \cup \{\delta_a \mid (a \in X) \wedge (\bar{a} \in X)\} \cup \{\delta_a \mid (a \in X) \wedge (\delta_a \in X)\} \cup \{\delta_a \mid (\bar{a} \in X) \wedge (\delta_a \in X)\} \cup \Delta_X$;
- $D_2 = \{\delta_a \mid (a \in X) \wedge (\delta_b \in D_1) \wedge (\delta_b \prec a)\}$;
- $D_3 = \{\delta_a \mid \bar{a} \in X\}$.

$$D = \begin{cases} \emptyset, & D_1 = \emptyset; \\ D_1 \cup D_2 \cup D_3, & \text{otherwise}. \end{cases}$$

Then $Regul(\rho) = (D, \emptyset) \cup (Y, \prec \cap (Y \times Y))$, where $Y = X \setminus \hat{\alpha}(D)$. If ρ satisfies the conditions 1-4, then $Regul(\rho) = \rho$.

Let $\rho = (X, \prec),\ \rho' = (X, \prec')$. We define poset operations.

Not occur $\rceil\!\rceil \rho = (\bar{\alpha}(X), \emptyset)$.

Mistaken not occur $\tilde{\rceil\!\rceil} \rho = (\Delta_\alpha(X), \emptyset)$.

Precedence

$\rho;\rho' = Regul(X \cup X', \prec \cup \prec' \cup (X^+ \times (X')^+) \cup (\Delta_X \times (X')^+))$.

Concurrency $\rho \| \rho' = Regul(X \cup X', (\prec \cup \prec')^*)$, where $(\prec \cup \prec')^*$ is a transitive closure of $\prec \cup \prec'$.

Alternative

$\rho \bigtriangledown \rho' = Regul(X \cup \bar{\alpha}(X'), \prec, l \cup l') \tilde{\cup} Regul(\bar{\alpha}(X) \cup X', \prec')$ ($\rho \bigtriangledown \rho'$ is not a poset, but a set of two posets describing alternative behaviours).

We extend the operations above to sets of posets. Let $\mathcal{P} = \cup_{i=1}^n \rho_i$ and $\mathcal{P}' = \cup_{j=1}^m \rho'_j$.

Then $\neg\mathcal{P} = \tilde{\cup}_{i=1}^n \neg\rho_i$, where $\neg \in \{\rceil\!\rceil, \tilde{\rceil\!\rceil}\}$ and $\mathcal{P} \circ \mathcal{P}' = \tilde{\cup}_{i=1}^n (\tilde{\cup}_{j=1}^m \rho_i \circ \rho'_j)$, where $\circ \in \{;, \|, \bigtriangledown\}$.

Definition 105 *A* denotational semantics *of* AFP_2 *is a mapping* $\mathcal{D}_2$ *from* $\mathbf{AFP_2}$ *into set of posets.*

1. $\mathcal{D}_2(a) = (\{a\}, \emptyset)$, $\mathcal{D}_2(\bar{a}) = (\{\bar{a}\}, \emptyset)$, $\mathcal{D}_2(\delta_a) = (\{\delta_a\}, \emptyset)$;
2. $\mathcal{D}_2(\neg P) = \neg\mathcal{D}_2(P)$, $\neg \in \{\rceil\!\rceil, \tilde{\rceil\!\rceil}\}$;
3. $\mathcal{D}_2(P \circ Q) = \mathcal{D}_2(P) \circ \mathcal{D}_2(Q)$, $\circ \in \{;, \|, \bigtriangledown\}$;
4. $\mathcal{D}_2(P \vee Q) = \mathcal{D}_2(P) \tilde{\cup} \mathcal{D}_2(Q)$.

Definition 106 *Formulas* P *and* P' *are* semantic equivalent *in* AFP_2, $P =_2 P'$, *iff* $\mathcal{D}_2(P) = \mathcal{D}_2(P')$.

If $\rho = (X, \prec)$ is a poset, then $\rho^+ = (X^+, \prec)$ is the *visible* part of ρ over α.

For any formula P of AFP_2, $\mathcal{D}_2(P) = \cup_{i=1}^n \rho_i$ is a set of posets, which characterize a nondeterministic process specified by P.

An *visible* part of this set is defined as $\mathcal{D}_2^+(P]) = \cup_{i=1}^n \rho_i^+$.

Definition 107 *Formulas* P *and* P' *are* observation semantic equivalent *in* AFP_2, $P =_2^+ P'$, *iff* $\mathcal{D}_2^+(P) = \mathcal{D}_2^+(P')$.

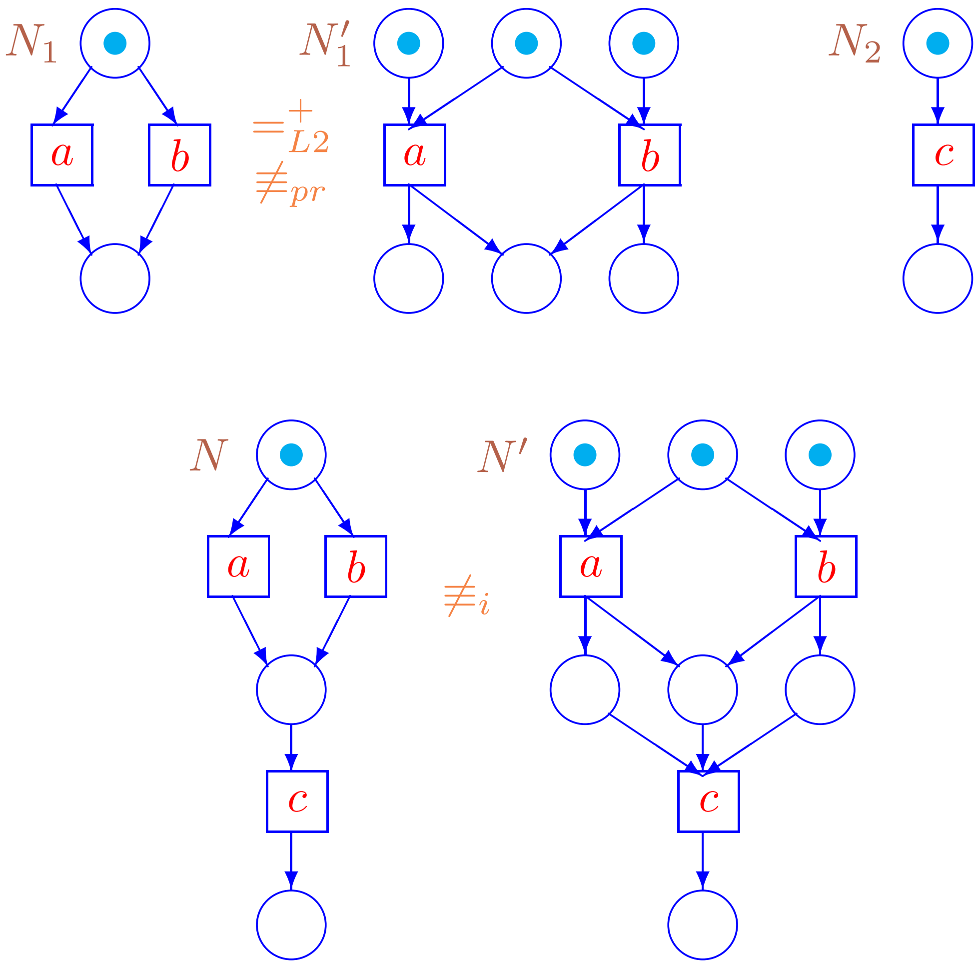
A *context* $\mathcal{C}$ is a formula of AFP_2 with zero or more subformulas replaced by "holes" to be filled by other formulas.

$\mathcal{C}(P)$ means putting of the formula P in each such "hole".

Proposition 17 [Ch89] *For any formulas* P *and* P' *of* AFP_2

$$P =_2 P' \;\Leftrightarrow\; \forall \mathcal{C}\; \mathcal{C}(P) =_2 \mathcal{C}(P').$$

Example of semantic equivalence of AFP_2



A-nets from example of congruence

Thus, $=_2$ is a congruence w.r.t. operations of AFP_2.

But $=_2^+$ is not a congruence.

Let $P_1 = a \bigtriangledown b$, $P_1' = (a \bigtriangledown b)\|a\|b$ and $P_2 = c$.

Then $\mathcal{D}_2^+(P_1) = \mathcal{D}_2^+(P_1') = \{(\{a\}, \emptyset), (\{b\}, \emptyset)$ and $P_1 =_2^+ P_1'$.

But $\mathcal{D}_2^+(P_1;P_2]) = \{(\{a, b\}, \prec_1), (\{b, c\}, \prec_2)\}$, whereas $\mathcal{D}_2^+(P_1';P_2) = \{(\{a\}, \emptyset), (\{b\}, \emptyset)\}$, and $P_1;P_2 \neq_2^+ P_1';P_2$.

Axiomatization

An axiom system Θ_2 is in accordance to the equivalence $=_2$.

Here $a \in \alpha,\ \bar{a} \in \bar{\alpha},\ \delta_a \in \Delta_\alpha,\ P, Q, R \in \mathbf{AFP_2}$.

1. Associativity
   - 1.1 $P \| (Q \| R) = (P \| Q) \| R$
   - 1.2 $P \bigtriangledown (Q \bigtriangledown R) = (P \bigtriangledown Q) \bigtriangledown R$
   - 1.3 $P \vee (Q \vee R) = (P \vee Q) \vee R$
   - 1.4 $P; (Q; R) = (P; Q); R$
2. Commutativity
   - 2.1 $P \| Q = Q \| P$
   - 2.2 $P \bigtriangledown Q = Q \bigtriangledown P$
   - 2.3 $P \vee Q = Q \vee P$
3. Distributivity
   - 3.1 $(P \| Q); R = (P; R) \| (Q; R)$
   - 3.2 $P; (Q \| R) = (P; Q) \| (P; R)$
   - 3.3 $(P \vee Q); R = (P; R) \vee (Q; R)$
   - 3.4 $P; (Q \vee R) = (P; Q) \vee (P; R)$
   - 3.5 $(P \vee Q) \| R = (P \| R) \vee (Q \| R)$
   - 3.6 $P \bigtriangledown (Q \| R) = (P \bigtriangledown Q) \| (P \bigtriangledown R)$

4. Axioms for $\triangledown$ and $\rceil\!\rceil$

4.1 $P \triangledown Q = (P \| (\rceil\!\rceil Q)) \vee ((\rceil\!\rceil P) \| Q)$

4.2 $\rceil\!\rceil (P \| Q) = (\rceil\!\rceil P) \| (\rceil\!\rceil Q)$

4.3 $\rceil\!\rceil (P \vee Q) = (\rceil\!\rceil P) \vee (\rceil\!\rceil Q)$

4.4 $\rceil\!\rceil (P; Q) = (\rceil\!\rceil P) \| (\rceil\!\rceil Q)$

4.5 $\rceil\!\rceil a = \bar{a}$

4.6 $\rceil\!\rceil \bar{a} = \bar{a}$

4.7 $\rceil\!\rceil \delta_a = \bar{a}$

5. Structural properties

5.1 $\bar{a}; P = \bar{a} \| P$

5.2 $P; \bar{a} = P \| \bar{a}$

5.3 $P \| (P; Q) = (P; Q)$

5.4 $Q \| (P; Q) = (P; Q)$

5.5 $P; Q; R = (P; Q) \| (Q; R)$

5.6 $(P; Q) \| (Q; R) = (P; Q) \| (Q; R) \| (P; R)$

5.7 $P \| P = P$

5.8 $P \vee P = P$

5.9 $P \vee Q = P$ or $Q \lhd P$ ($\lhd$ is a strict prefix of formulas, defined later)

6. Axioms for deadlocked actions and $\widetilde{\rceil\!\rceil}$

6.1 $a\|\bar{a} = \delta_a$

6.2 $a; a = \delta_a$

6.3 $a\|\delta_a = \delta_a$

6.4 $\delta_a; P = \delta_a\|(\widetilde{\rceil\!\rceil} P)$

6.5 $P; \delta_a = P\|\delta_a$

6.6 $\delta_a\|(\rceil\!\rceil P) = \delta_a\|(\widetilde{\rceil\!\rceil} P)$

6.7 $\widetilde{\rceil\!\rceil} a = \delta_a$

6.8 $\widetilde{\rceil\!\rceil} \bar{a} = \delta_a$

6.9 $\widetilde{\rceil\!\rceil} \delta_a = \delta_a$

6.10 $\widetilde{\rceil\!\rceil}(P\|Q) = (\widetilde{\rceil\!\rceil} P)\|(\widetilde{\rceil\!\rceil} Q)$

6.11 $\widetilde{\rceil\!\rceil}(P; Q) = (\widetilde{\rceil\!\rceil} P)\|(\widetilde{\rceil\!\rceil} Q)$

6.12 $\widetilde{\rceil\!\rceil}(P \vee Q) = (\widetilde{\rceil\!\rceil} P) \vee (\widetilde{\rceil\!\rceil} Q)$

The axiom system Θ_2 is *sound* for $=_2$: if $P = P'$ is an axiom of Θ_2, then $P =_2 P'$.

To prove that Θ_2 is *complete* for $=_2$, we introduce a canonical form of formulas.

Canonical form of formulas

A canonical form of formulas of AFP_2 is a disjunctive normal form.

Elementary members: symbols from $\hat{\alpha}$ or elementary precedences (of two actions).

Conjunction: $\|$, disjunction: $\vee$.

Let P be a formula of AFP_2. *Alphabet* $\alpha(P)$ of P is:

1. $\alpha(a) = \alpha(\bar{a}) = \alpha(\delta_a) = a$;
2. $\alpha(\neg P) = \alpha(P),\ \neg \in \{\rceil, \tilde{\rceil}\}$;
3. $\alpha(P \circ Q) = \alpha(P) \cup \alpha(Q),\ \circ \in \{;, \|, \bigtriangledown, \vee\}$.

- $\bar{\alpha}(P) = \{\bar{a} \mid a \in \alpha(P)\}$;
- $\Delta_\alpha(P) = \{\delta_a \mid a \in \alpha(P)\}$;
- $\hat{\alpha}(P) = \alpha(P) \cup \bar{\alpha}(P) \cup \Delta_\alpha(P)$.

Contents of P, $cont(P)$, is:

1. $cont(a) = a,\ cont(\bar{a}) = \bar{a},\ cont(\delta_a) = \delta_a$;
2. $cont(\neg P) = cont(P),\ \neg \in \{\rceil, \tilde{\rceil}\}$;
3. $cont(P \circ Q) = cont(P) \cup cont(Q),\ \circ \in \{;, \|, \bigtriangledown, \vee\}$.

- $cont^+(P) = cont(P) \cap \alpha$ is the set of *actions* of P;
- $cont^-(P) = cont(P) \cap \bar{\alpha}$ is the set of *non-actions* of P;
- $\Delta_{cont}(P) = cont(P) \cap \Delta_\alpha$ is the set of *deadlocked actions* of P.

Precedence is a formula $P_1; \dots ;P_n = ;_{i=1}^n P_i$, where $P_i \in \hat{\alpha}\ (1 \le i \le n)$;

Conjunction is a formula $P_1 \| \dots \| P_n = \|_{i=1}^n P_i$, where P_i are precedences $(1 \le i \le n)$.

Disjunction is a formula $P = P_1 \vee \dots \vee P_n = \vee_{i=1}^n P_i$, where $P_i\ (1 \le i \le n)$ are conjunctions.

Normal conjunction is a conjunction $P = \|_{i=1}^{n} P_i$ s.t.:

1. Every formula P_i $(1 \leq i \leq n)$ has one of the forms:
   (a) elementary formula a $(a \in \alpha)$, $\bar{a}$ $(\bar{a} \in \bar{\alpha})$, δ_a $(\delta_a \in \Delta_\alpha)$;
   (b) *elementary precedence* $(a; b)$, where $a, b \in \alpha$ and $a \neq b$;
2. If there is a formula P_i $(1 \leq i \leq n)$ s.t. $P_i = \delta_a$ $(\delta_a \in \Delta_\alpha)$, then there is no another one P_j $(1 \leq j \leq n)$ s.t. $P_j = \bar{b}$ $(\bar{b} \in \bar{\alpha})$;
3. For any formulas P_i and P_j $(1 \leq i \neq j \leq n)$ s.t. $\alpha(P_i) \cap \alpha(P_j) \neq \emptyset$, P_i and P_j have a form of different elementary precedences;
4. For any pair $P_i = (a; b)$ and $P_j = (b; c)$ $(1 \leq i \neq j \leq n)$ there exists a formula $P_k = (a; c)$ $(1 \leq k \leq n)$ describing the transitive closure of the precedence relation for actions a, b and c.

1 (*2,3,4*)*-conjunction* is a conjunction that satisfy the condition 1 (2,3,4) from the definition above.

For example, 1,2,3,4-conjunction is a normal one.

Let P and Q be normal conjunctions. A formula P is a *strict prefix* of Q, $P \lhd Q$, if:

1. $cont^+(P) \subset cont^+(Q)$;
2. elementary precedence $(a; b)$ is a conjunctive member of Q and $b \in cont^+(P)$ iff $(a; b)$ is a conjunctive member of P.

A formula P is a *prefix* of Q, $P \unlhd Q$, if $P \lhd Q$ or $P \simeq Q$.

For example, in the formula $(a\|c\|\bar{b}\|\bar{d}\|\bar{e}) \vee (c\|\delta_a\|\delta_b\|\delta_d\|\delta_e) \vee (a\|\delta_b\|\delta_c\|\delta_d\|\delta_e) \vee ((b; d)\|(b; e)\|\bar{a}\|\bar{c})$, the second and third conjunctions are strict prefixes of the first one.

Definition 108 *A formula P is in canonical form if it is a disjunction $P = \vee_{i=1}^{n} P_i$ with the following properties.*

1. *P_i $(1 \leq i \leq n)$ is a normal conjunction;*
2. *for any P_i and P_j $(1 \leq i \neq j \leq n)$ $P_i \not\simeq P_j$;*
3. *for any P_i and P_j $(1 \leq i \neq j \leq n)$ $\neg(P_i \triangleleft P_j \vee P_j \triangleleft P_i)$.*

As for conjunction, we define *1* (*2,3*)*-disjunction*.

For example, 1,2,3-disjunction is a canonical form.

Each disjunctive member of canonical form characterizes one of alternative behaviours of the nondeterministic process specified by the formula.

It has a form practically coinciding with a poset corresponding to this behaviour.

For example, the formula $(a\|c\|\bar{b}\|\bar{d}\|\bar{e}) \vee ((b;d)\|(b;e)\|\bar{a}\|\bar{c})$ is in canonical form.

A conjunction (disjunction) is *maximal* if there is no longer one containing it as a conjunctive (disjunctive) member.

Theorem 27 *[Ch89] Any formula of AFP_2 can be reduced to the unique (up to isomorphism) canonical form.*

The set of *all canonical forms* of a formula P is $canon(P)$.

Definition 109 *$P =_{\Theta_2} P'$ means that the equality of P and P' can be proved using Θ_2.*

Theorem 28 *[Ch89] For any formulas P and P' of AFP_2*

$$P =_2 P' \Leftrightarrow P =_{\Theta_2} P'.$$

To check equivalence of formulas P and P' of AFP_2, one can reduce them to canonical forms Q and Q' and compare the latter up to isomorphism.

Net and algebraic simulation

Equivalences on A-nets

A descriptive algebra AFP_0 with semantics based on finite A-nets [KCh85].

Any finite A-net can be specified by a formula of the algebra using “regularization” algorithm [Kot78].

A mapping Ξ from the set of all formulas of AFP_0 into that of AFP_2 s.t. the set of posets of the net specified by a formula P of AFP_0, coincide with the set of posets of nondeterministic process specified by the formula $\Xi(P)$ of AFP_2 [Ch89].

Given an A-net specified by the formula P of AFP_0, one can analyze its behavior by means of the same formula P of AFP_2.

Definition 110 *An A-net (Acyclic net) is an acyclic ordinary strictly labeled net* $N = (P_N, T_N, W_N, L_N, M_N)$*:*

1. $\forall p \in P_N\ ({}^\bullet p \neq \emptyset) \vee (p^\bullet \neq \emptyset)$, *there are no isolated places;*
2. $\forall p, q \in P_N\ ({}^\bullet p = {}^\bullet q) \wedge (p^\bullet = q^\bullet)\ \Rightarrow\ p = q$, *there are no “superfluous” places;*
3. $\forall t \in T_N\ ({}^\bullet t \neq \emptyset) \wedge (t^\bullet \neq \emptyset)$, *all transitions have input and output places;*
4. $\forall x \in P_N \cup T_N\ |\downarrow x| < \infty$, *the set of causes is finite;*
5. $\forall p \in P_N\ \forall t, u \in T_N\ t, u \in {}^\bullet p\ \Rightarrow\ t\ \mathbf{al}\ u$, *transitions with common output place are alternative;*
6. $M_N = {}^\bullet N$, *an initial marking is a set of input places of the net.*

The *alternative* relation, $\mathbf{al}$, is defined as follows. Let $t, u \in T_N$ for A-net N. $t \; \mathbf{al} \; u$ if the following is valid.

1. $(t \not\prec_N u) \wedge (u \not\prec_N t)$;
2. $({}^\bullet t \cap {}^\bullet u \neq \emptyset) \vee (\exists p \in {}^\bullet t \, \forall t' \in {}^\bullet p \; t' \, \mathbf{al} \, u) \vee (\exists q \in {}^\bullet u \, \forall u' \in {}^\bullet q \; t \, \mathbf{al} \, u') \vee (t = u)$.

Since we consider nets only with finite processes, item 4 may be ignored.

Items 5 and 6 guarantee a safeness of A-nets.

A mapping $\Xi : \mathbf{AFP_0} \to \mathbf{AFP_2}$ is defined as:

1. $\Xi(a) = a$;
2. $\Xi(P;_0Q) = P;_2Q$;
3. $\Xi(P\|_0Q) = P\|_2Q$;
4. $\Xi(P\nabla_0Q) = P\nabla_2Q$.

The number 0 (2) marks the operations of AFP_0 (AFP_2).

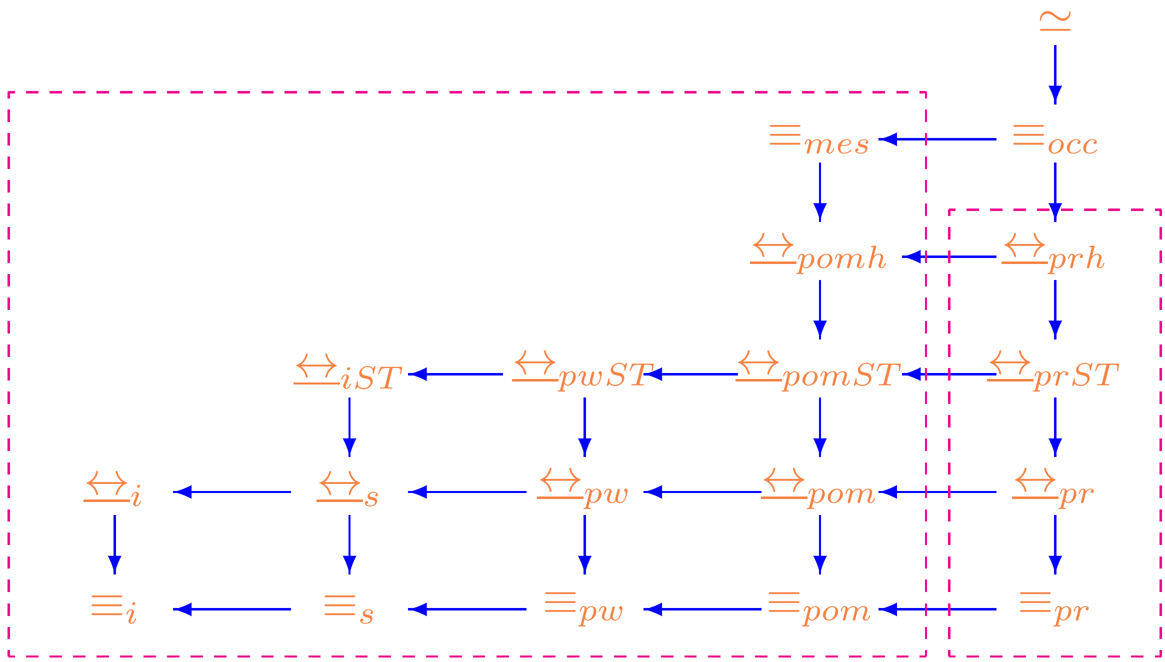
Denotational semantics of AFP_0 is a mapping $\mathcal{D}_0$, which associates with every formula P a set of maximal C-subnets of finite A-net N, specified by the formula.

Theorem 29 *[Ch89] Let P be a formula of AFP_0 and Q be a formula of AFP_2 s.t. $Q = \Xi(P)$. Then*

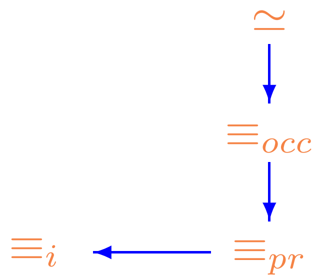
$$\{\rho_C \mid C \in \mathcal{D}_0(P)\} = \mathcal{D}_2^+(Q).$$

Proposition **18** [Tar97] *For A-nets N and N'*

1. $N \equiv_i N' \Leftrightarrow N \equiv_{mes} N'$;
2. $N \equiv_{pr} N' \Leftrightarrow N \underline{\leftrightarrow}_{prh} N'$.



Merging of the basic equivalences on A-nets

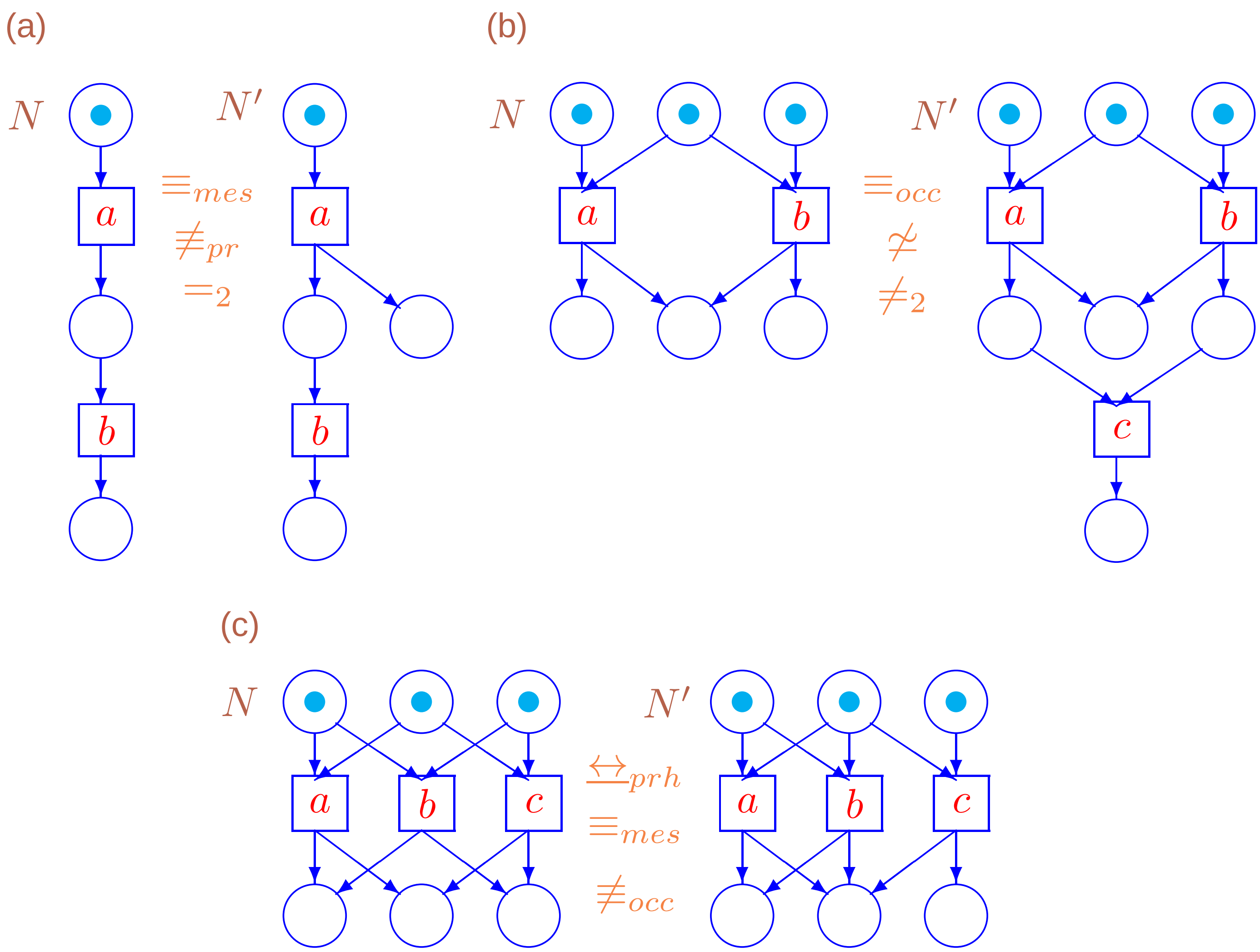


Interrelations of the basic equivalences on A-nets

Theorem 30 *Let* $\leftrightarrow, \twoheadleftrightarrow \in \{\equiv, \simeq\}$, $\star, \star\star \in \{\_, i, pr, occ\}$. *For A-nets* N *and* N'

$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \twoheadleftrightarrow_{\star\star} N'$$

iff there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftrightarrow_{\star\star}$ *in the graph above.*



AN: Examples of the basic equivalences on A-nets

- In Figure AN(a), $N \equiv_i N'$, but $N \not\equiv_{pr} N'$, since a causal net of process of N' with action a not isomorphic to any causal net of process of N.

  $P = a; b, \ P' = (a; b) \| a$.

  $B = \overline{a; b}, \ B' = \overline{[x : ((\{a, x\}; b) \| \hat{x})]}$.

- In Figure AN(c), $N \equiv_{pr} N'$, but $N \not\equiv_{occ} N'$, since only in the unfolding of N' there is a place which is an input one for three transitions.

  $P = (a \bigtriangledown b) \| (b \bigtriangledown c) \| (a \bigtriangledown c), \ P' = (a \bigtriangledown b \bigtriangledown c) \| (a \bigtriangledown b) \| c$.

  $B = \overline{[\{x, y\} : ((\{a, x\} [] \{b, y\}) \| \hat{x} \| \hat{y})]}$,
  $B' = \overline{[\{x, y, z\} : ((\{a, x\} [] \{b, y\}) \| (\hat{x}; \{c, z\}) \| (\hat{y}; \hat{z}))]}$.

- In Figure AN(b), $N \equiv_{occ} N'$, but $N \not\simeq N'$, since only in the net N' there is a transition labeled by c (which never fires).

  $P = (a \bigtriangledown b) \| a \| b, \ P' = (a \bigtriangledown b) \| (a; c) \| (b; c)$.

  $B = \overline{[\{x, y, z\} : ((\{a, x\} [] \{b, y\}) \| (\{b, \hat{y}\} [] \{c, z\}) \| (\{a, \hat{x}\} [] \{c, \hat{z}\}))]}$,
  $B' = \overline{[\{x, y, z\} : ((\{a, x\} [] \{b, y\} [] \{c, z\}) \| (\{a, \hat{x}\} [] \{b, \hat{y}\}) \| \{c, \hat{z}\})]}$.

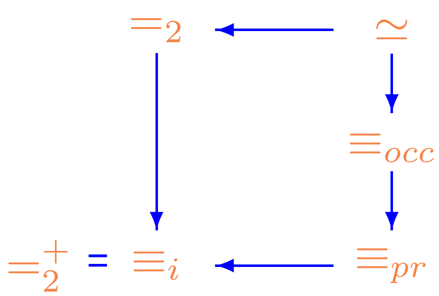
Comparing the net and algebraic equivalences

Definition 111 *Let $\leftrightarrow$ be a formula equivalence of AFP_2, and the formulas P and P' correspond to the finite A-nets N and N' (as described before).*

N and N' are equivalent (w.r.t. $\leftrightarrow$), notation $N \leftrightarrow N'$, iff the formulas corresponding them are also equivalent, $P \leftrightarrow P'$.

Proposition 19 *[Tar97] For A-nets N and N'*

$$N \equiv_i N' \;\Leftrightarrow\; N =_2^+ N'.$$



Interrelations of the basic net and algebraic equivalences

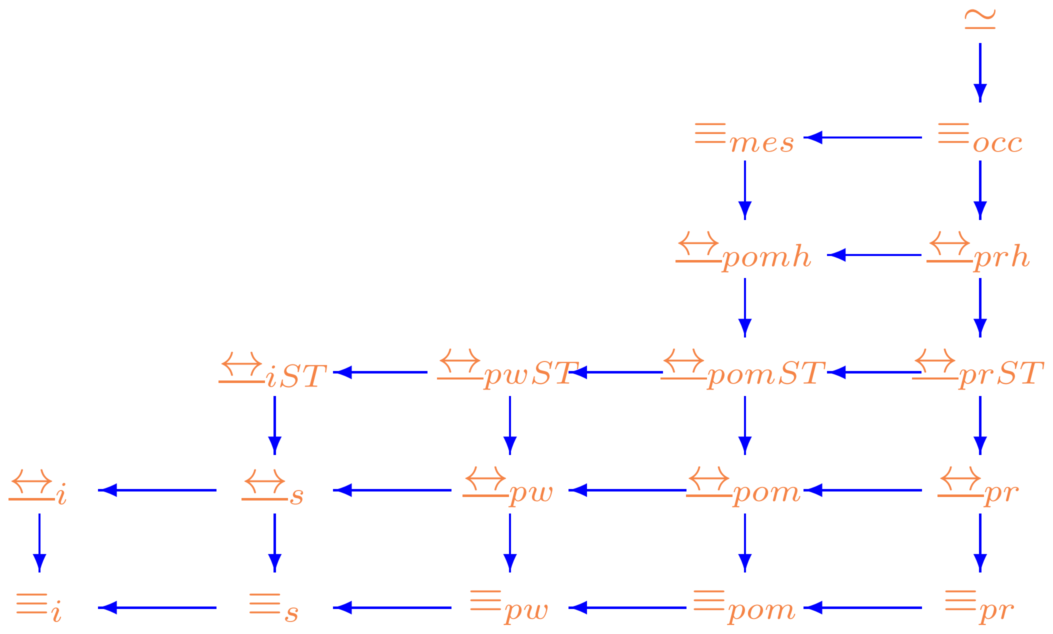
Theorem 31 *Let $\leftrightarrow, \twoheadleftrightarrow \in \{\equiv, \simeq, =\}$, $\star, \star\star \in \{\_, i, pr, occ, \mathcal{D}_2^+, \mathcal{D}_2\}$. For A-nets N and N'*

$$N \leftrightarrow_\star N' \;\Rightarrow\; N \leftrightarrow_{\star\star} N'$$

iff there exists a directed path from $\leftrightarrow_\star$ to $\twoheadleftrightarrow_{\star\star}$ in the graph above.

Equivalences on weakly labeled A-nets

Definition 112 *A* weakly labeled A-net *is an net with all properties of A-net with exception of strict labeling.*

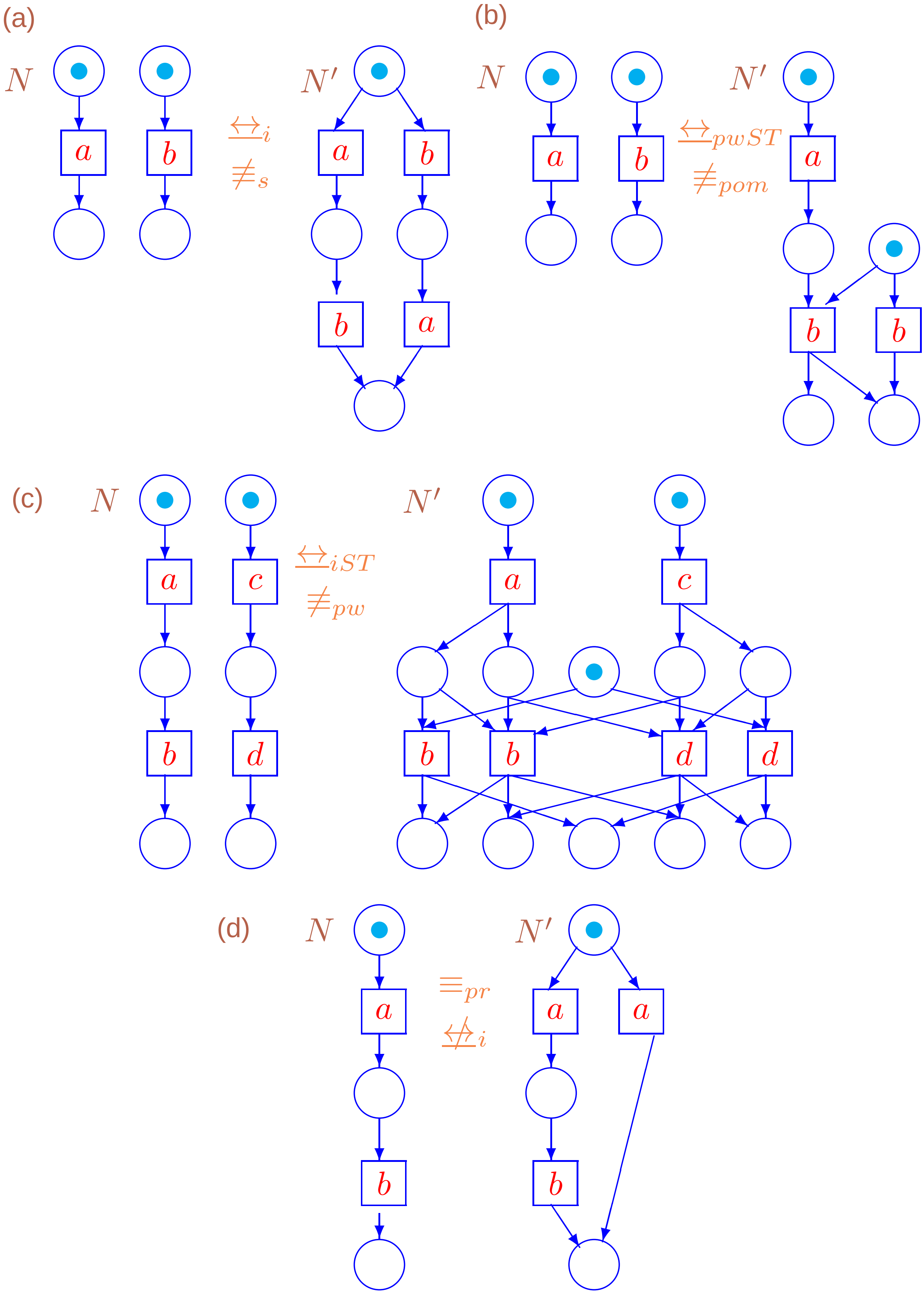


Interrelations of the basic equivalences on weakly labeled A-nets

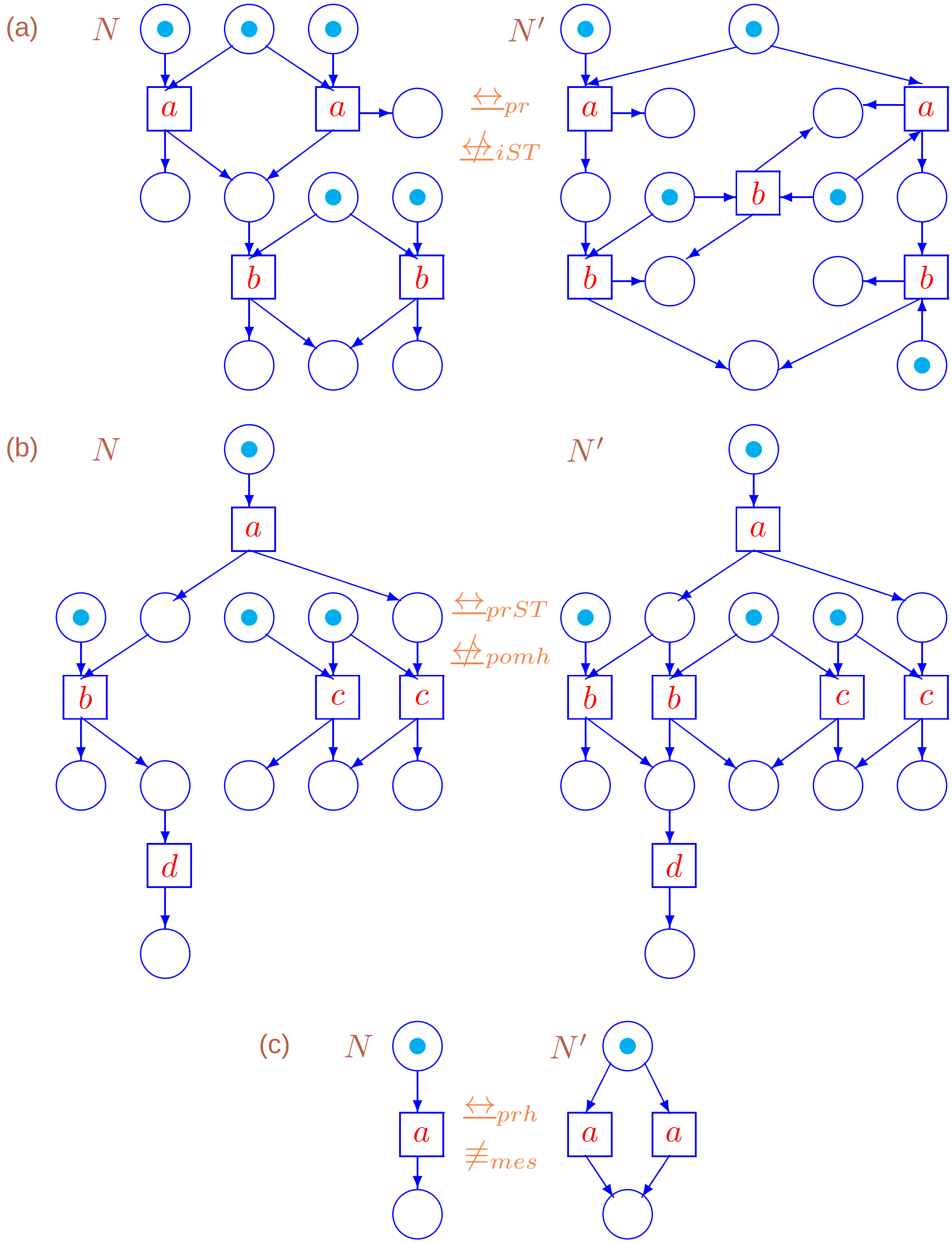
Theorem 32 *Let* $\leftrightarrow, \leftrightarrow\!\!\!\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}, \simeq\}$, $\star, \star\star \in \{\_, i, s, pwpom, pr, iST, pwST, pomST, prST, pomh, prh, mes, occ\}$*. For weakly labeled A-nets* N *and* N'

$$N \leftrightarrow_{\star} N' \ \Rightarrow \ N \leftrightarrow\!\!\!\leftrightarrow_{\star\star} N'$$

iff there exists a directed path from $\leftrightarrow_{\star}$ *to* $\leftrightarrow\!\!\!\leftrightarrow_{\star\star}$ *in the graph above.*



LAN: Examples of weakly labeled A-nets



LAN1: Examples of weakly labeled A-nets (continued)

In the following examples, $E,\ E'$ are formulas of $AFLP_2$ [Tar96] and $B,\ B'$ are that of PBC [BDH92].

- In Figure LAN(a), $N\underline{\leftrightarrow}_i N'$, but $N\not\equiv_s N'$, since only in N' actions a and b can occur concurrently.

  $E = e\|f,\ E' = (e_1;f_1) \bigtriangledown (e_2;f_2)$.

  $B = \overline{a\|b},\ B' = \overline{(a;b)[](b;a)}$.

- In Figure LAN(c), $N\underline{\leftrightarrow}_{iST} N'$, but $N\not\equiv_{pw} N'$, since N is associated with the pomset s.t. even less sequential one cannot be executed in N'.

  $E = (e;f)\|(g;h),\ E' = (e;(f_1 \bigtriangledown f_2))\|(e;(f_2 \bigtriangledown h_1))\|$
  $(g;(f_2 \bigtriangledown h_1))\|(g;(h_1 \bigtriangledown h_2))\|(f_1 \bigtriangledown h_2)$.

  $B = \overline{(a;b)\|(c;d)}$,
  $B' = \overline{[\{x,y_1,y_2,y_2',z,v_1,v_1',v_2\} : ((\{a,x\};(\{b,y_1\}[]\{b,y_2\}))\|}$
  $\overline{(\{a,\hat{x}\};(\{b,\widehat{y_2},y_2'\}[]\{d,v_1\}))\|(\{c,z\};(\{b,\widehat{y_2'}\}[]\{d,\widehat{v_1},v_1'\}))\|}$
  $\overline{(\{c,\hat{z}\};(\{d,\widehat{v_1'}\}[]\{d,v_2\}))\|(\{b,\widehat{y_1}\}[]\{d,\widehat{v_2}\}))]}$.

- In Figure LAN(b), $N\underline{\leftrightarrow}_{pwST} N'$, but $N\not\equiv_{pom} N'$, since only in N' action b can depend on a.

  $E = e\|f,\ E' = (e;f_1)\|(f_1 \bigtriangledown f_2)$.

  $B = \overline{a\|b},\ B' = \overline{[x : ((a;\{b,x\})\|(b[]\hat{x}))]}$.

- In Figure AN(a), $N\equiv_{mes} N'$, but $N\not\equiv_{pr} N'$.

  $E = e;f,\ E' = (e;f)\|e$.

  $B = \overline{a;b},\ B' = \overline{[x : ((\{a,x\};b)\|\hat{x})]}$.

- In Figure LAN(d), $N\equiv_{pr} N'$, but $N\not\underline{\leftrightarrow}_i N'$, since only in N' action a can occur so that b cannot occur after it.

  $E = e;f,\ E' = (e_1;f) \bigtriangledown e_2$.

  $B = \overline{a;b},\ B' = \overline{(a;b)[]a}$.

- In Figure LAN1(a), $N \underline{\leftrightarrow}_{pr} N'$, but $N \underline{\not\leftrightarrow}_{iST} N'$, since only in N' action a can begin working so that no b can start unless a finishes.

  $E = ((e_1 \bigtriangledown e_2); f_1) \| (f_1 \bigtriangledown f_2) \| e_1 \| e_2 \| f_2,$
  $E' = ((e_1; f_1) \bigtriangledown (e_2; f_3)) \| (f_1 \bigtriangledown f_2) \| (e_2 \bigtriangledown f_2) \| e_1 \| f_3.$
  $B = \overline{[\{x_1, x_2, y_1, y_2\} : (((\{a, x_1\}[]\{a, x_2\}); \{b, y_1\}) \| (\widehat{y_1}[]\{b, y_2\}) \| \widehat{x_1} \| \widehat{x_2} \| \widehat{y_2})]},$
  $B' = \overline{[\{x_1, x_2, y_1, y_2, y_3\} : ((\{a, x_1\}; \{b, y_1\})[](\{a, x_2\}; \{b, y_3\})) \| (\widehat{y_1}[]\{b, y_2\}) \| (\widehat{x_2}[]\widehat{y_2}) \| \widehat{x_1} \| \widehat{y_3})]}.$

- In Figure LAN1(b), $N \underline{\leftrightarrow}_{prST} N'$, but $N \underline{\not\leftrightarrow}_{pomh} N'$, since only in N' actions a and b can occur so that the next action, c, must depend on a.

  $E = (e; f; h) \| (e; g_2) \| (g_1 \bigtriangledown g_2) \| f \| g_1,$
  $E' = (e; (f_1 \bigtriangledown f_2); h) \| (e; g_2) \| (f_2 \bigtriangledown g_1) \| (g_1 \bigtriangledown g_2) \| f_1.$
  $B = \overline{[\{x, y, z_1, z_2\} : ((\{a, x\}; \{b, y\}; d) \| (\hat{x}; \{c, z_2\}) \| (\{c, z_1\}[]\widehat{z_2}) \| \hat{y} \| \widehat{z_1})]},$
  $B' = \overline{[\{x, y_1, y_2, z_1, z_2\} : (\{a, x\}; (\{b, y_1\}[]\{b, y_2\}); d) \| (\hat{x}; \{c, z_2\}) \| (\widehat{y_2}[]\{c, z_1\}) \| (\widehat{z_1}[]\widehat{z_2}) \| \widehat{y_1})]}.$

- In Figure LAN1(c), $N \underline{\leftrightarrow}_{prh} N'$, but $N \not\equiv_{mes} N'$, since only MES that corresponding to N' has two conflict actions a.

  $E = e,\ E' = e_1 \bigtriangledown e_2.$
  $B = \overline{a},\ B' = \overline{a[]a}.$

- In Figure AN(b), $N \equiv_{occ} N'$, but $N \not\simeq N'$.

  $E = (e \bigtriangledown f) \| e \| f,\ E' = (e \bigtriangledown f) \| (e; g) \| (f; g).$
  $B = \overline{[\{x, y\} : ((\{a, x\}[]\{b, y\}) \| \hat{x} \| \hat{y})},$
  $B' = \overline{\{x, y, z\} : ((\{a, x\}[]\{b, y\}) \| (\hat{x}; \{c, z\}) \| (\hat{y}; \hat{z}))]}.$

Term rewriting

Term rewrite system RWS_2

Let $P = P_1 \circ \ldots \circ P_{i-1} \circ P_i \circ P_{i+1} \circ \ldots \circ P_n,\ \circ \in \{;, \|, \nabla, \vee\}$.

A *substitution* $[P]^{P_i}_Q$ of subformula P_i by subformula Q in P is $P_1 \circ \ldots P_{i-1} \circ Q \circ P_{i+1} \circ \ldots \circ P_n$.

In the rules of $RWS_2,\ P, Q, R$ denote formulas of AFP_2 and $a, b, c \in \alpha,\ \bar{a}, \bar{b} \in \bar{\alpha},\ \delta_a, \delta_b \in \Delta_\alpha,\ \diamondsuit \in \{^-, \delta\}$.

The numbers in parentheses are the that of equalities of Θ_2 used to produce the corresponding transition rules.

1.1 $\circ \in \{;, \|, \vee\} \ \Rightarrow$

$P \circ (Q \circ R) \to (P \circ Q) \circ R$

(1.1, 1.3, 1.4)

2.1 $(\bullet, \circ) \in \{(\|, ;), (\vee, ;), (\vee, \|)\} \ \Rightarrow$

$(P \circ Q) \bullet R \to (P \bullet R) \circ (Q \bullet R)$

(3.1, 3.3, 3.5)

2.2 $(\bullet, \circ) \in \{(\|, ;), (\vee, ;), (\vee, \|)\} \ \Rightarrow$

$P \bullet (Q \circ R) \to (P \bullet Q) \circ (P \bullet R)$

(2.1, 3.2, 3.4, 3.5)

3.1 $P \nabla Q \to (P \| (\rceil Q)) \vee ((\rceil P) \| Q)$

(4.1)

4.1 $\circ \in \{\|, ;\}, \neg \in \{\rceil, \tilde{\rceil}\} \ \Rightarrow$

$\neg(P \circ Q) \to (\neg P) \| (\neg Q)$

(4.2, 4.4, 6.10, 6.11)

4.2 $\neg \in \{\rceil, \tilde{\rceil}\} \ \Rightarrow$

$\neg(P \vee Q) \to (\neg P) \vee (\neg Q)$

(4.3, 6.12)

4.3 $P = a$ or $P = \Diamond a \;\Rightarrow$

$\rceil\!|P \to \bar{a}$

(4.5, 4.6, 4.7)

4.4 $P = a$ or $P = \Diamond a \;\Rightarrow$

$\tilde{\rceil}\!|P \to \delta_a$

(6.7, 6.8, 6.9)

5.1 $P, Q, R \in \hat{\alpha} \;\Rightarrow$

$(P;Q);R \to ((P;Q)\|(Q;R))\|(P;R)$

(5.5, 5.6)

5.2 $Q \in \hat{\alpha} \;\Rightarrow$

$\bar{a};Q \to \bar{a}\|Q$

(5.1)

5.3 $P \in \hat{\alpha} \;\Rightarrow$

$P;\bar{a} \to P\|\bar{a}$

(5.2)

5.4 $a;a \to \delta_a$

(6.2)

5.5 $Q = b$ or $Q = \Diamond b \;\Rightarrow$

$\delta_a;Q \to \delta_a\|\delta_b$

(6.4, 6.7, 6.8, 6.9)

5.6 $P \in \hat{\alpha} \;\Rightarrow$

$P;\delta_a \to P\|\delta_a$

(6.5)

6.1 P is 1-conjunction, $P' = \delta_a$ is a conjunctive member of $P \Rightarrow$

$P\|\bar{b} \to P\|\delta_b$

(1.1, 2.1, 4.5, 6.6, 6.7)

6.2 P is 1-conjunction, $P' = \bar{b}$ is a conjunctive member of $P \Rightarrow$

$P\|\delta_a \to [P]^{P'}_{\delta_b}\|\delta_a$

(1.1, 2.1, 4.5, 6.6, 6.7)

7.1 P is 1,2-conjunction, P' is a conjunctive member of P, $P' = a$ or $P' = b \Rightarrow$

$P\|(a;b) \to [P]^{P'}_{(a;b)}$

(1.1, 2.1, 5.3, 5.4)

7.2 P is 1,2-conjunction, P' is a conjunctive member of P, $P' = (a;b)$ or $P' = (b;a) \Rightarrow$

$P\|a \to P$

(1.1, 2.1, 5.3, 5.4)

7.3 P is 1,2-conjunction, $P' = a$ is a conjunctive member of $P \Rightarrow$

$P\|\Diamond a \to [P]^{P'}_{\delta_a}$

(1.1, 2.1, 6.1, 6.3)

7.4 P is 1,2-conjunction, P' is a conjunctive member of P, $P' = \Diamond a \Rightarrow$

$P\|a \to [P]^{P'}_{\delta_a}$

(1.1, 2.1, 6.1, 6.3)

7.5 P is 1,2-conjunction, $P' = (a;b)$ is a conjunctive member of $P \Rightarrow$

$P\|\Diamond a \to [P]^{P'}_{\delta_b}\|\delta_a$

(1.1, 1.4, 2.1, 5.1, 6.1, 6.3, 6.4, 6.7)

7.6 P is 1,2-conjunction, $P' = (b; a)$ is a conjunctive member of $P \ \Rightarrow$

$P \| \Diamond a \rightarrow [P]_b^{P'} \| \delta_a$

(1.1, 2.1, 5.2, 6.1, 6.3, 6.5)

7.7 P is 1,2-conjunction, P' is a conjunctive member of P, $P' = \Diamond a \ \Rightarrow$

$P \| (a; b) \rightarrow [P]_{\delta_a}^{P'} \| \delta_b$

(1.1, 1.4, 2.1, 5.1, 6.1, 6.3, 6.4, 6.7)

7.8 P is 1,2-conjunction, P' is a conjunctive member of P, $P' = \Diamond a \ \Rightarrow$

$P \| (b; a) \rightarrow [P]_{\delta_a}^{P'} \| b$

(1.1, 2.1, 5.2, 6.1, 6.3, 6.5)

7.9 P is 1,2-conjunction, $P' = Q$ is a conjunctive member of $P \ \Rightarrow$

$P \| Q \rightarrow P$

(1.1, 2.1, 5.7)

8.1 P is 1,2,3-conjunction, $P' = (a; b)$ is a conjunctive member of P, in the maximal 1,2,3-conjunction containing P as a conjunctive member, there is no conjunctive member $P'' = (a; c) \ \Rightarrow$

$P \| (b; c) \rightarrow (P \| (b; c)) \| (a; c)$

(1.1, 2.1, 5.6)

8.2 P is 1,2,3-conjunction, $P' = (c; a)$ is a conjunctive member of P, in the maximal 1,2,3-conjunction containing P as a conjunctive member there is no conjunctive member $P'' = (b; a) \ \Rightarrow$

$P \| (b; c) \rightarrow (P \| (b; c)) \| (b; a)$

(1.1, 2.1, 5.6)

9.1 P is 1-disjunction, P' is a disjunctive member of $P,\ P' \simeq Q\ \Rightarrow$

$P \vee Q \to P$

(1.1, 1.3, 2.1, 2.3, 5.8)

10.1 P is 1,2-disjunction, Q is a normal conjunction, P' is a disjunctive member of $P,\ Q \lhd P'\ \Rightarrow$

$P \vee Q \to P$

(1.3, 2.3, 5.9)

10.2 P is 1,2-disjunction, Q is a normal conjunction, P' is a disjunctive member of $P,\ P' \lhd Q\ \Rightarrow$

$P \vee Q \to [P]_Q^{P'}$

(1.3, 2.3, 5.9)

Notices on RWS_2

- Rule 1.1 (left associativity): to avoid infinite chains
  $P \circ (Q \circ R) \to (P \circ Q) \circ R \to P \circ (Q \circ R) \to \cdots, \circ \in \{;, \|, \vee\}$.

  No commutativity rules: to avoid infinite chains
  $P \circ Q \to Q \circ P \to P \circ Q \to \cdots, \circ \in \{\|, \vee\}$.

  Symmetrical rules are required.

- Rules 2.1-2.2 (symmetrical distributivity): to obtain disjunction of conjunctions with precedences or elementary formulas as conjunctive members.

- Rule 3.1: to remove $\bigtriangledown$.

- Rules 4.1–4.4: to remove $\rceil\!\rceil$ and $\tilde{\rceil\!\rceil}$.

- Rules 5.1–5.6: to transform precedences into elementary ones (property 1 of normal conjunction).

  Conjunctive (disjunctive) members we want to transform in a pair are not always adjacent: search in conjunction (disjunction) is required.

- Rules 6.1–6.2: to avoid conjunction of non-actions and deadlocked actions (property 2 of normal conjunction).

- Rules 7.1–7.9: to avoid common alphabet symbols in conjunctive members, with exception of that in two different elementary precedences (property 3 of normal conjunction).

- Rules 8.1–8.2: to add a “transitive closure” elementary precedence to the pair of ones with common action (property 4 of normal conjunction).

  Search in a maximal conjunction: to avoid infinite chains
  $(a; b)\|(b; c) \to ((a; b)\|(b; c))\|(a; c) \to$
  $(((a; b)\|(b; c))\|(a; c))\|(a; c) \to \cdots$.

- Rule 9.1: to remove isomorphic disjunctive members (property 1 of normal disjunction).
- Rules 10.1–10.2: to remove prefixed disjunctive members (property 2 of normal disjunction).

Rules 6.1–6.2 and 7.5–7.8 are based on the following derived axioms. Numbers over equality signs are that of axioms of Θ_2. Symbol $*$ marks reverse axiom application. Numbers in parentheses are that of previous derived axioms.

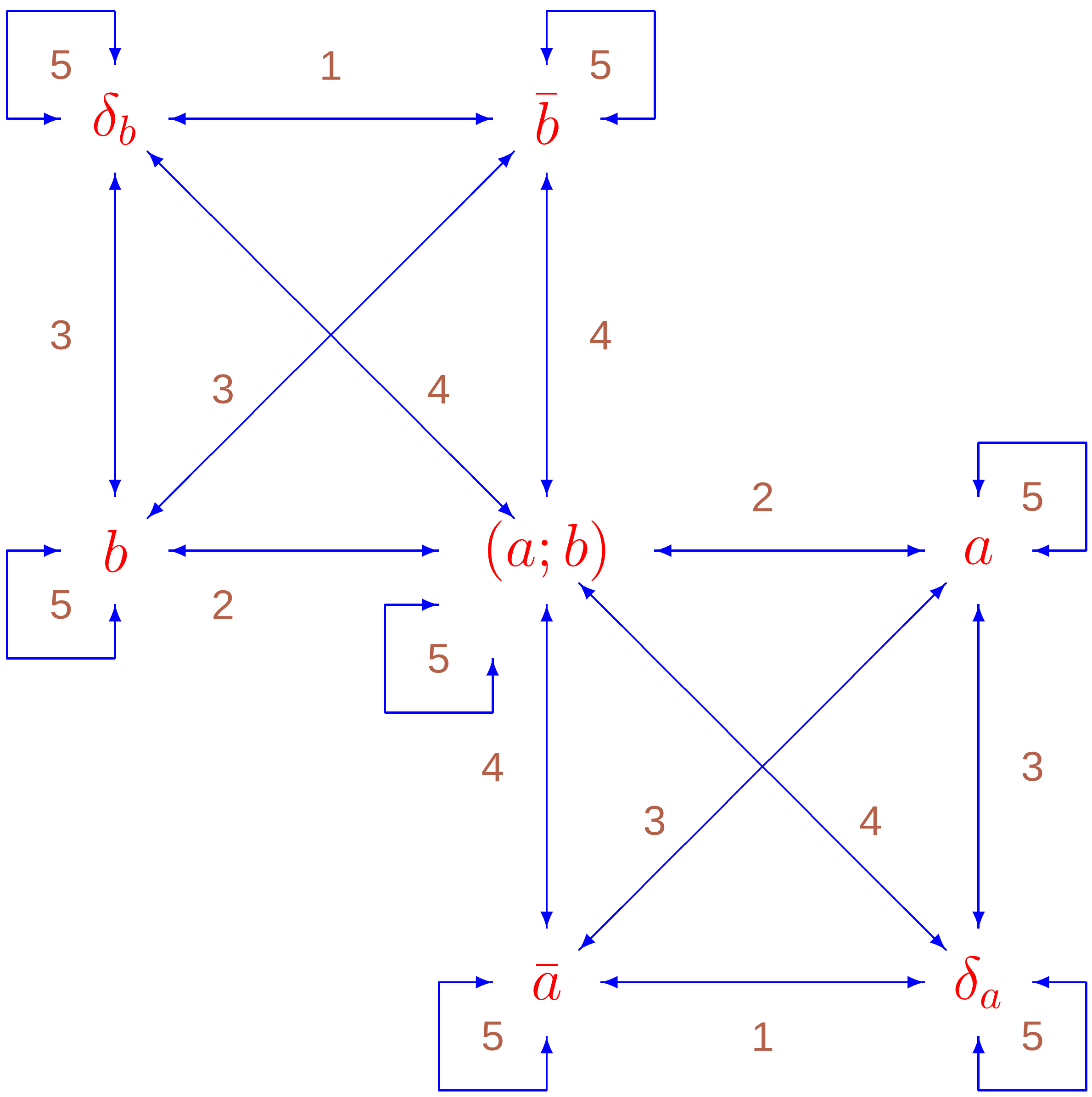
1. $\bar{a}\|(a;b) \stackrel{5.1*}{=} \bar{a};(a;b) \stackrel{1.4}{=} (\bar{a};a);b \stackrel{5.1}{=} (\bar{a}\|a);b \stackrel{2.1}{=} (a\|\bar{a});b \stackrel{6.1}{=} \delta_a;b \stackrel{6.4}{=} \delta_a\|(\tilde{\prod}b) \stackrel{6.7}{=} \delta_a\|\delta_b$;
2. $\delta_a\|(a;b) \stackrel{6.1*}{=} (a\|\bar{a})\|(a;b) \stackrel{1.1*}{=} a\|(\bar{a}\|(a;b)) \stackrel{(1)}{=} a\|(\delta_a\|\delta_b) \stackrel{1.1}{=} (a\|\delta_a)\|\delta_b \stackrel{6.3}{=} \delta_a\|\delta_b$;
3. $\bar{a}\|(b;a) \stackrel{2.1}{=} (b;a)\|\bar{a} \stackrel{5.2*}{=} (b;a);\bar{a} \stackrel{1.1*}{=} b;(a;\bar{a}) \stackrel{5.2}{=} b;(a\|\bar{a}) \stackrel{6.1}{=} b;\delta_a \stackrel{6.5}{=} b\|\delta_a \stackrel{2.1}{=} \delta_a\|b$;
4. $\delta_a\|(b;a) \stackrel{6.1*}{=} (a\|\bar{a})\|(b;a) \stackrel{1.1*}{=} a\|(\bar{a}\|(b;a)) \stackrel{(3)}{=} a\|(\delta_a\|b) \stackrel{1.1}{=} (a\|\delta_a)\|b \stackrel{6.3}{=} \delta_a\|b$;
5. $\delta_a\|\bar{b} \stackrel{4.5*}{=} \delta_a\|(\prod b) \stackrel{6.6}{=} \delta_a\|(\tilde{\prod}b) \stackrel{6.7}{=} \delta_a\|\delta_b$.

Confluence of RWS_2

Proposition 20 *[Tar97] No rule from the groups 1–5 can be applied to a formula of AFP_2 iff it is a disjunction of 1-conjunctions.*

Proposition 21 *[Tar97] No rule from the groups 1–6 can be applied to a formula of AFP_2 iff it is a disjunction of 1,2-conjunctions.*

Proposition 22 *[Tar97] No rule from the groups 1–7 can be applied to a formula of AFP_2 iff it is a disjunction of 1,2,3-conjunctions.*



Conjunctive members with intersecting alphabets

Proposition 23 *[Tar97] No rule from the groups 1–8 can be applied to a formula of AFP_2 iff it is a 1-disjunction.*

Proposition 24 *[Tar97] No rule from the groups 1–9 can be applied to a formula of AFP_2 iff it is a 1,2-disjunction.*

Theorem 33 *[Tar97] No rule from RWS_2 can be applied to a formula of AFP_2 iff it is in the canonical form.*

Hence, to check semantic equivalence of two formulas of AFP_2, it is enough to transform them to the canonical forms with the use of RWS_2 and then check these canonical forms for isomorphism.

Implementation

Program $CANON$

A program $CANON$ in C programming language (more than 2000 lines) based on the previous results. It transforms any formula of AFP_2 into canonical form.

A structure of function main.

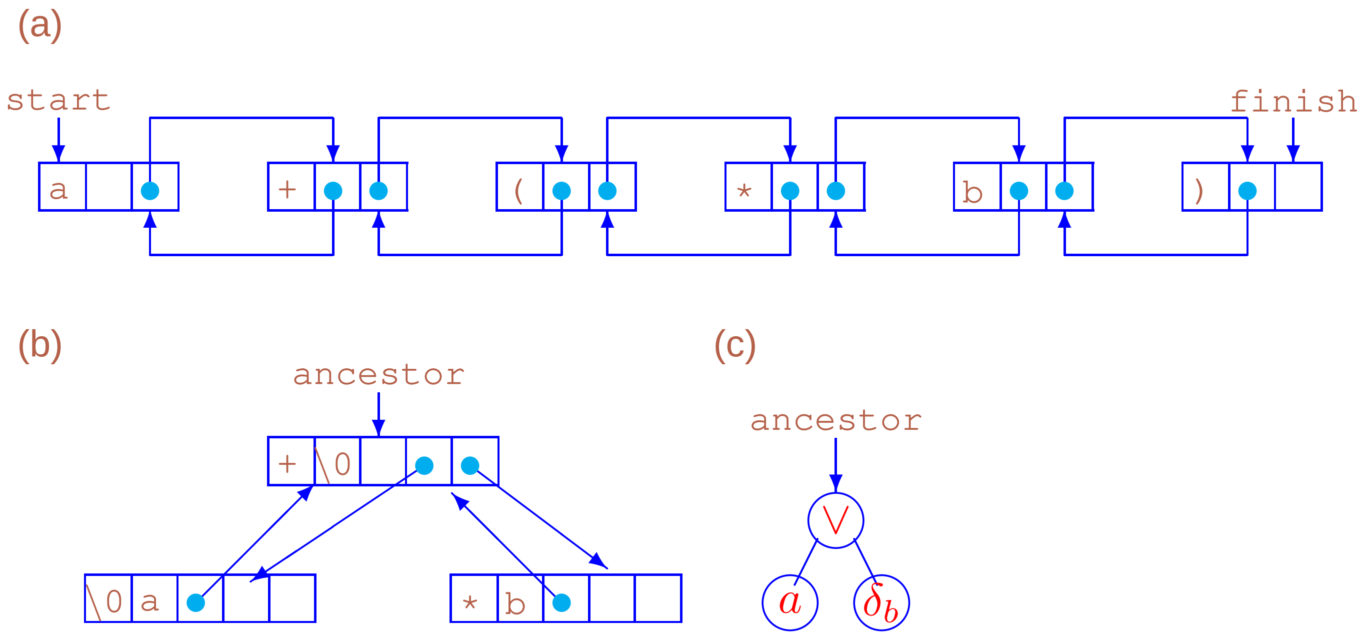
```
Print information about program
and format of input formula;

Print "Formula has been read";

Transform list into tree; Dispose list;

Print formula;

step=1; /*step number*/

do
{
 Print step;

 nar=0; /*the initial number of rule
          applications at the present step*/

 Apply rules; Print nar;

 step++; /*next step*/
}
while(nar!=0);

Print canonical form.
```

Special symbol representation in $CANON$

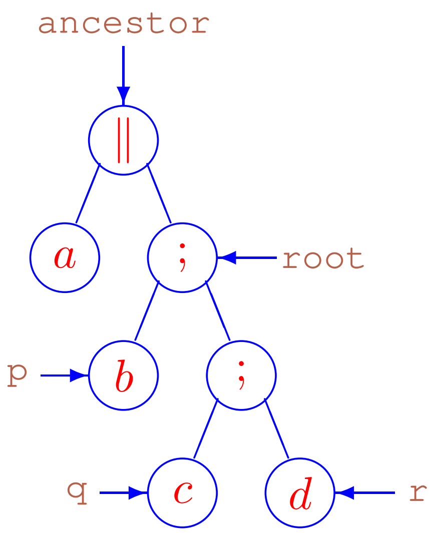
Initial symbol	$-$	δ	$\rceil\rceil$	$\tilde{\rceil\rceil}$	;	$\|$	$\bigtriangledown$	$\vee$
Symbolic constant	NOT	DLT	NOC	MNO	PRC	CNC	ALT	DSJ
ASCII-symbol	-	*	`	~	;	|	#	+

A structure of formulas.

1. a;
2. -a , *a ;
3. `a , ~a ;
4. `(P) , ~(P) ;
5. a#b , a+b , a|b , a;b ;
6. a#(P) , a+(P) , a|(P) , a;(P) ;
7. (P)#a , (P)+a , (P)|a , (P);a ;
8. (P)#(Q) , (P)+(Q) , (P)|(Q) , (P);(Q) .



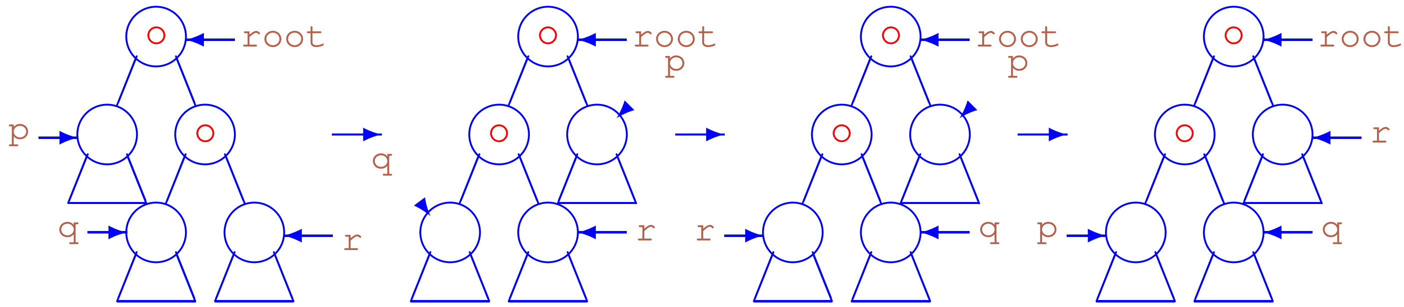
List and tree representations of the formula $a \vee \delta_b$



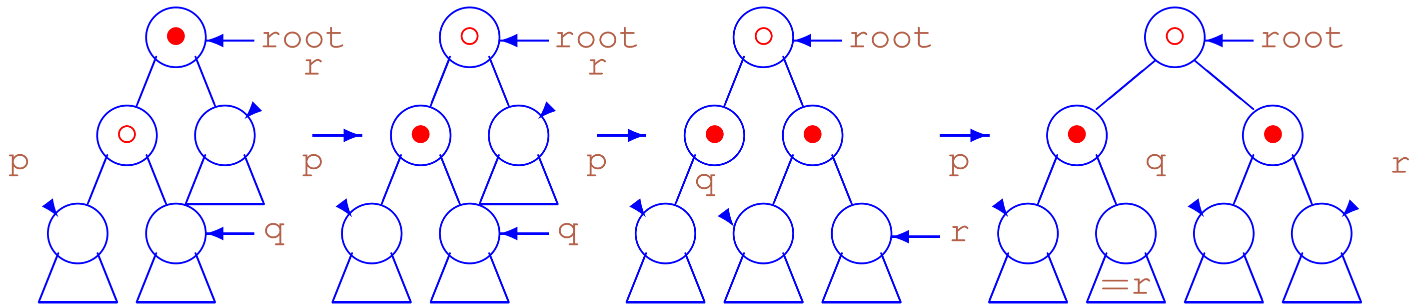
A tree to which the rule 1.1 can be applied

A structure of rules.

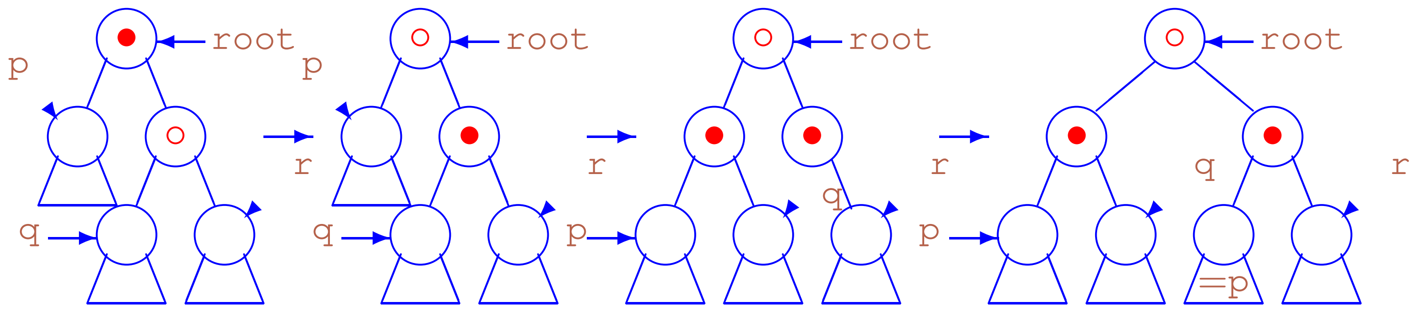
```
if(root!=NULL)
{
 if(the rule is directly applicable
    to the tree with pointer root)
 {
  Set pointers to subtrees corresponding
  to subformulas in the rule;

  Print rule number and subformulas;

  Transform tree in accordance to the rule;

 (*addrnar)++; /*increase counter of rules
                 applied at the present step*/

 Print new formula;
 }
 else
 {
  Apply rule to the left subtree;

  Apply rule to the right subtree;
 }
}
```



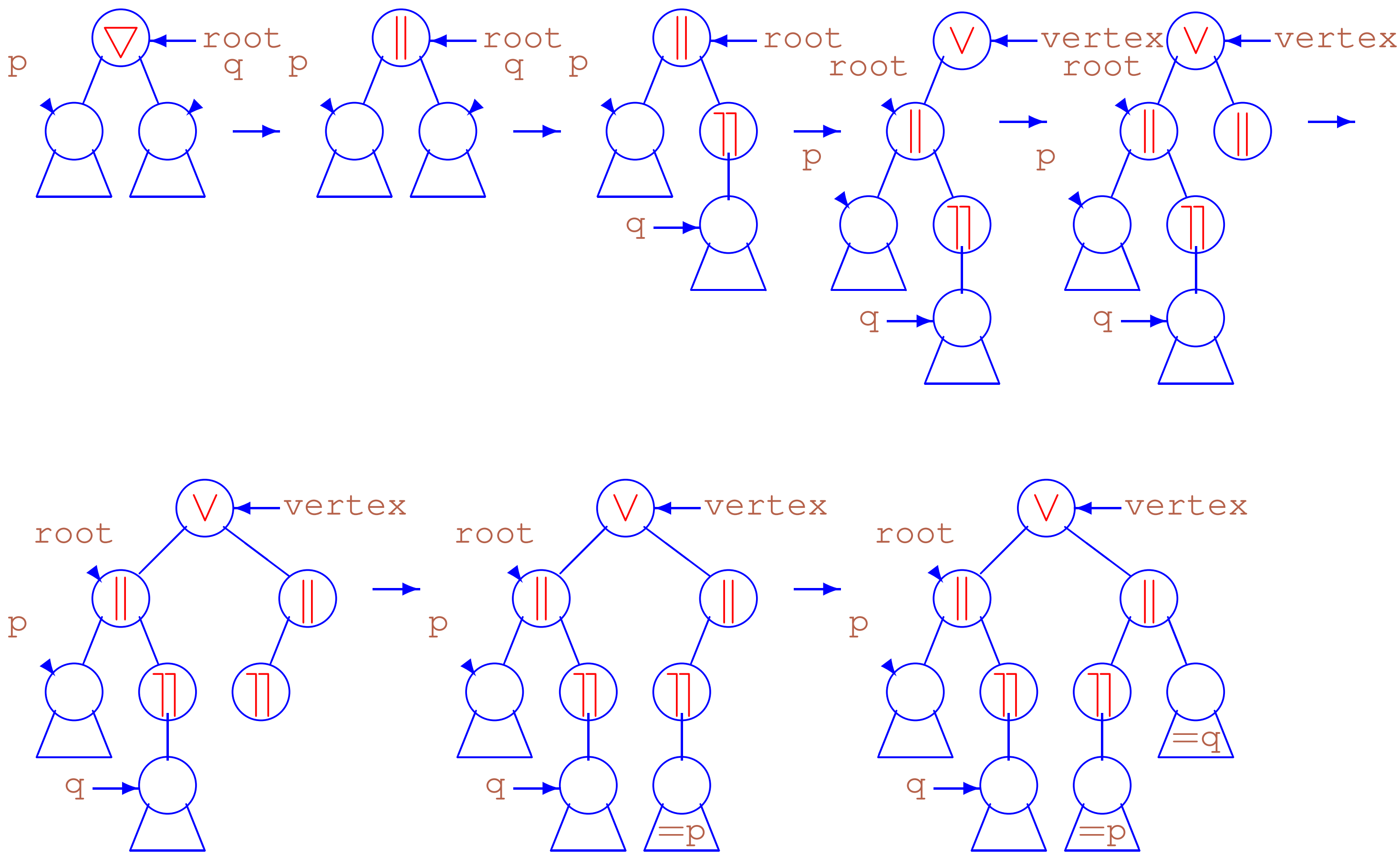
Tree transformations with rule 1.1



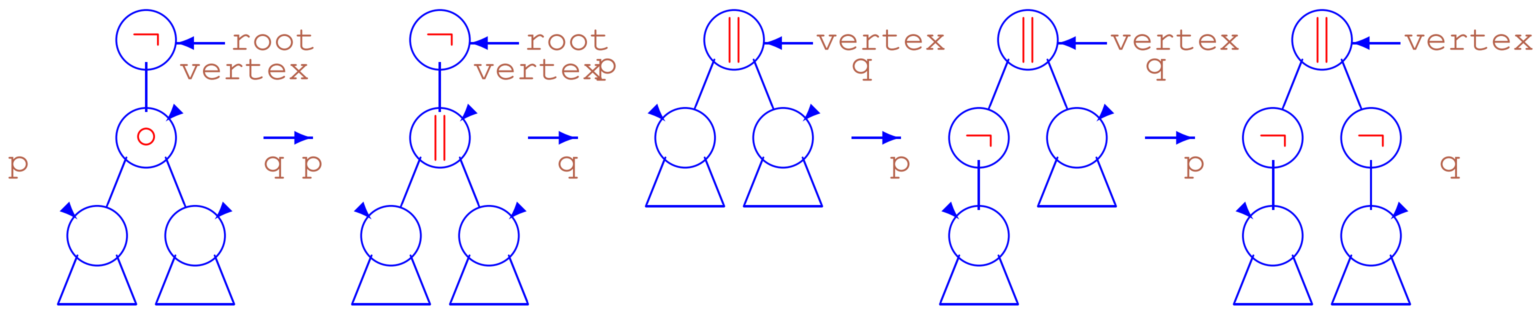
Tree transformations with rule 2.1



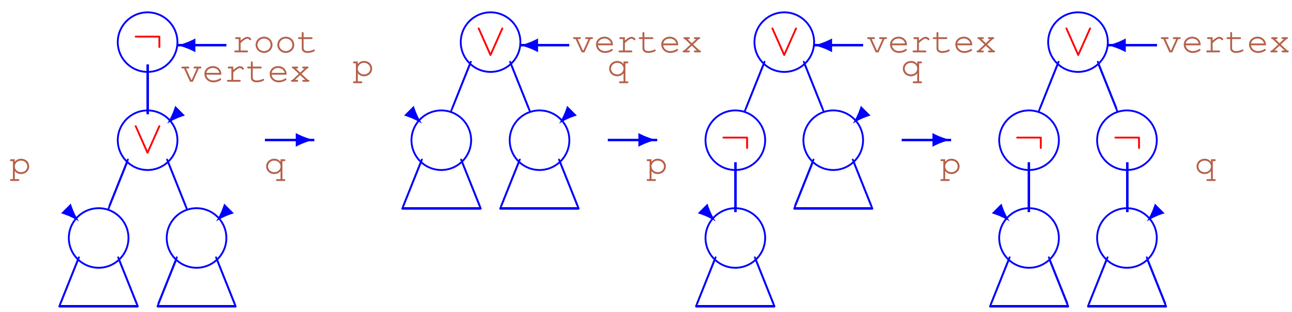
Tree transformations with rule 2.2



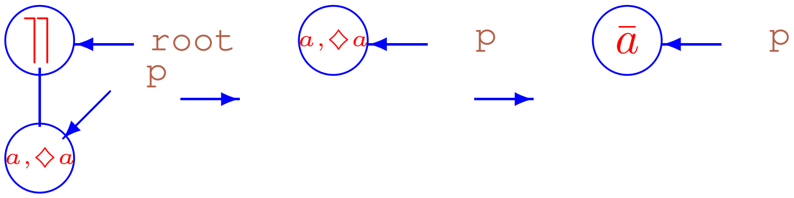
Tree transformations with rule 3.1



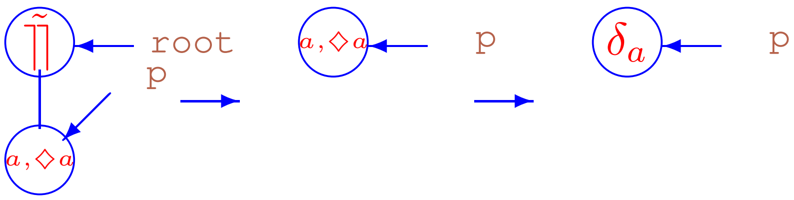
Tree transformations with rule 4.1



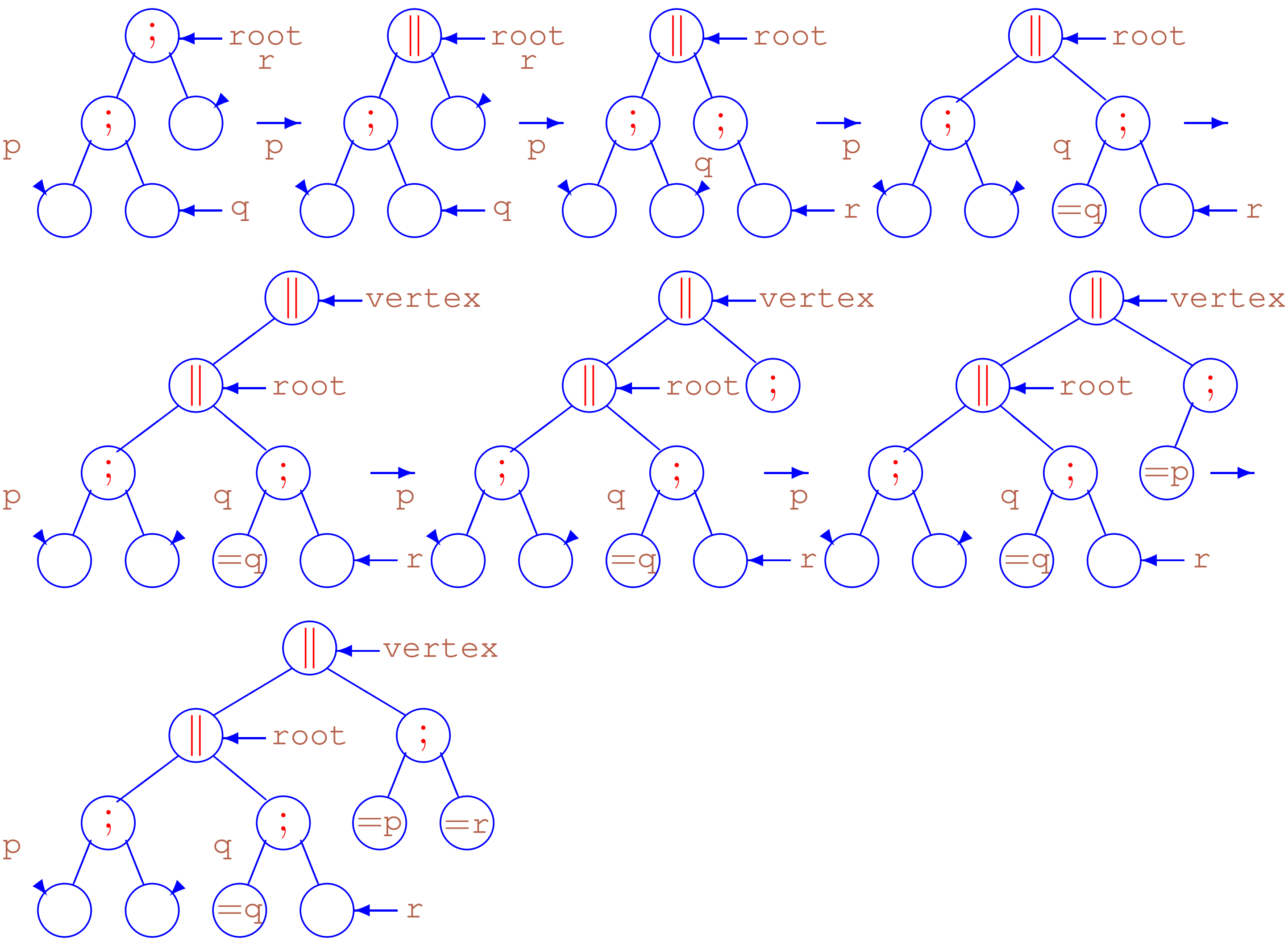
Tree transformations with rule 4.2



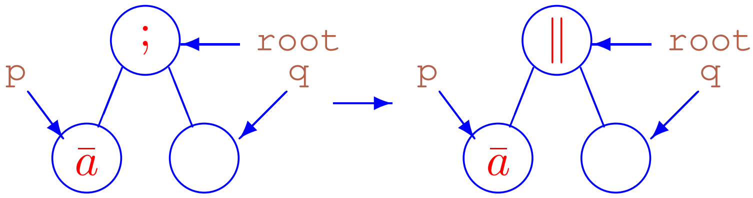
Tree transformations with rule 4.3



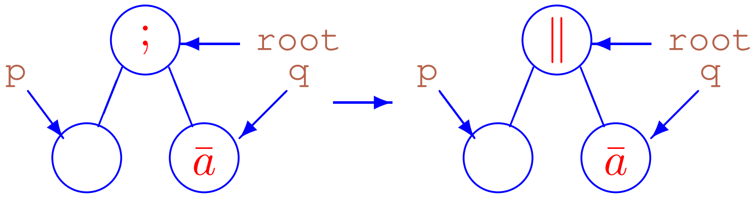
Tree transformations with rule 4.4



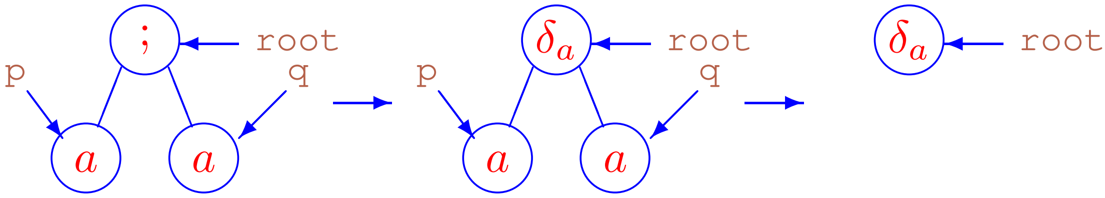
Tree transformations with rule 5.1



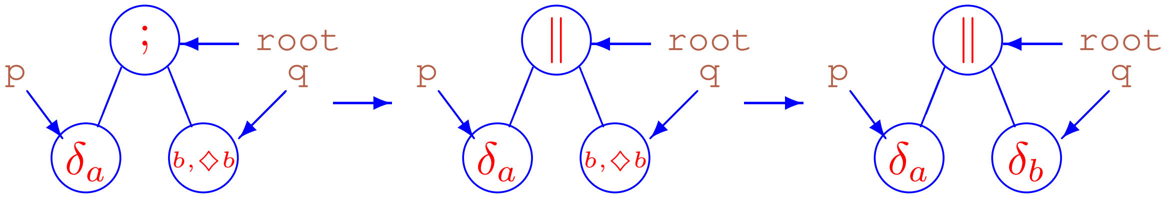
Tree transformations with rule 5.2



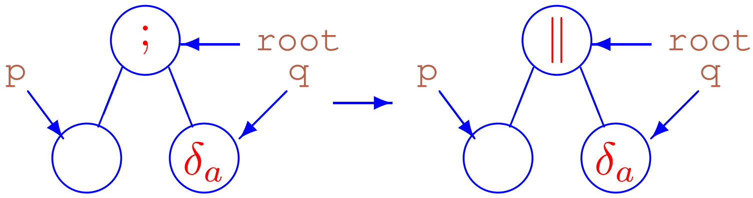
Tree transformations with rule 5.3



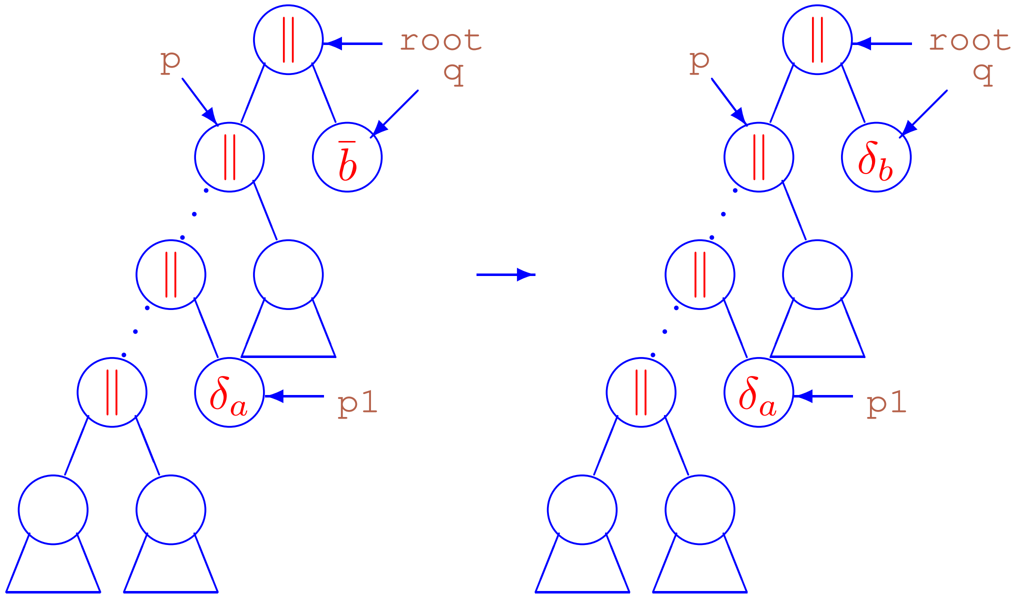
Tree transformations with rule 5.4



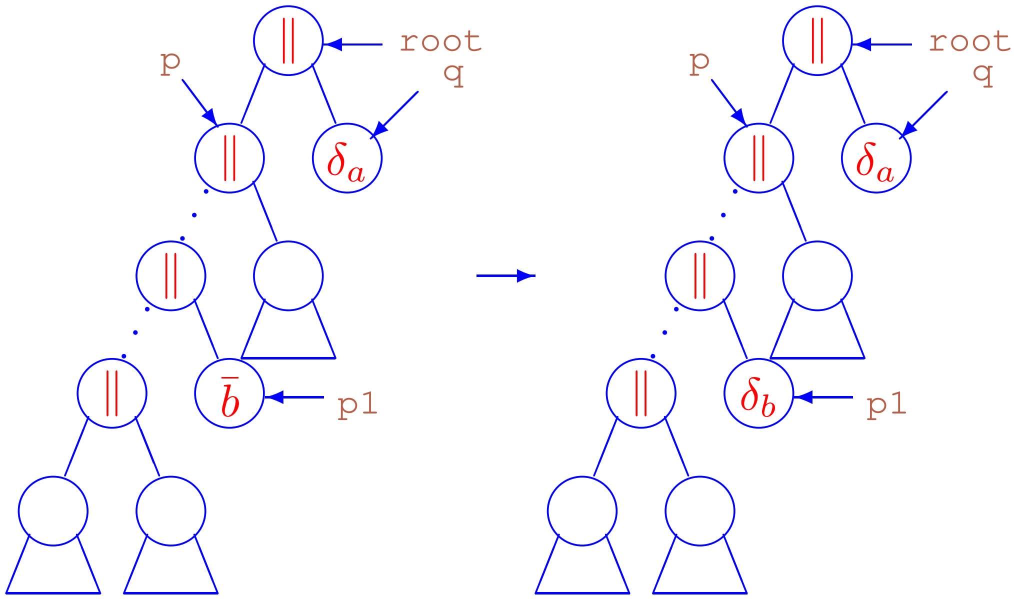
Tree transformations with rule 5.5



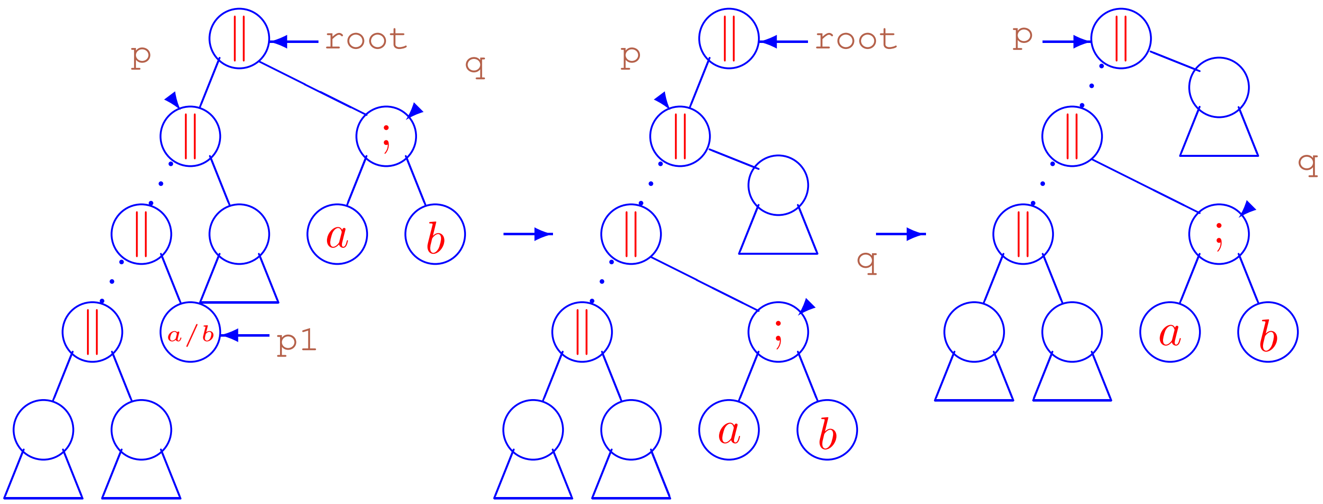
Tree transformations with rule 5.6



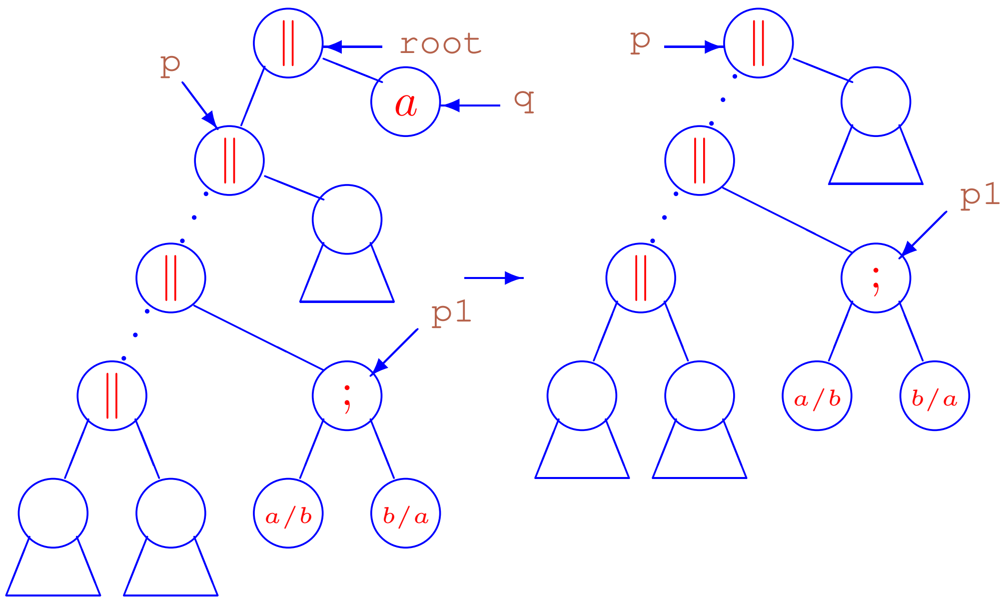
Tree transformations with rule 6.1



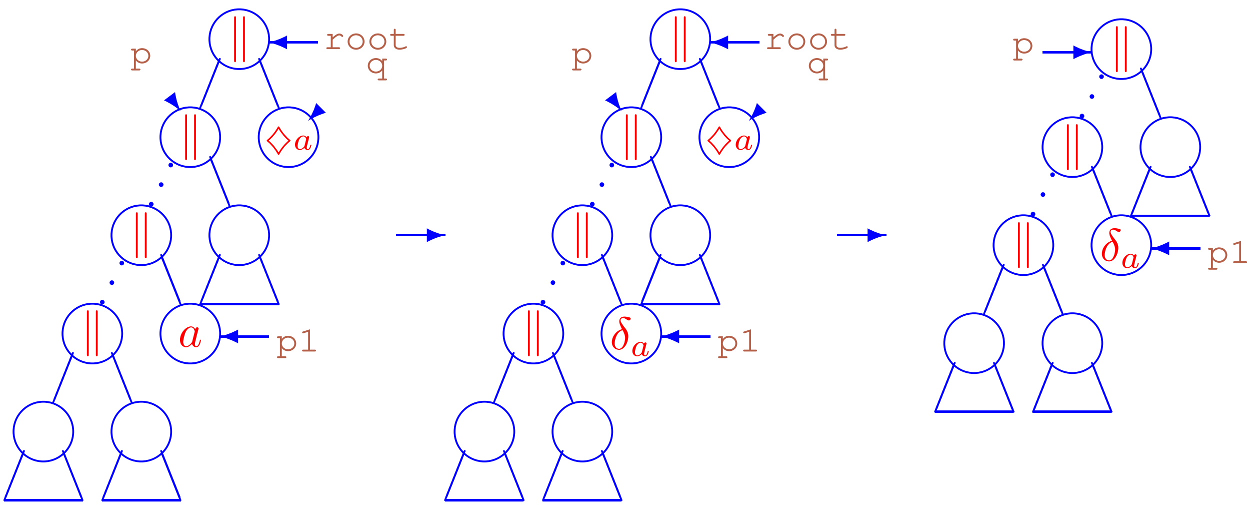
Tree transformations with rule 6.2



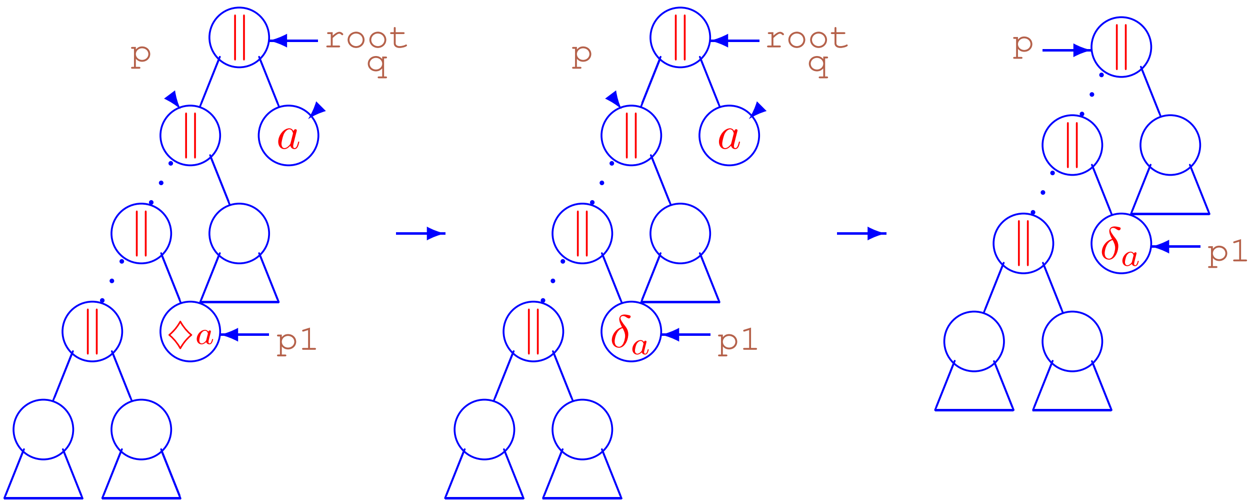
Tree transformations with rule 7.1



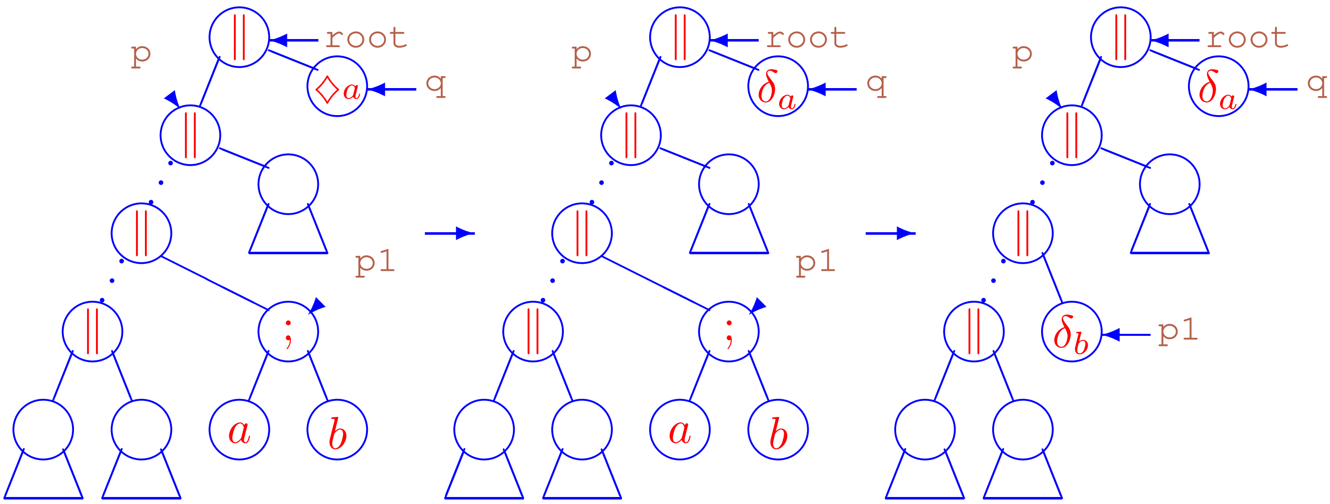
Tree transformations with rule 7.2



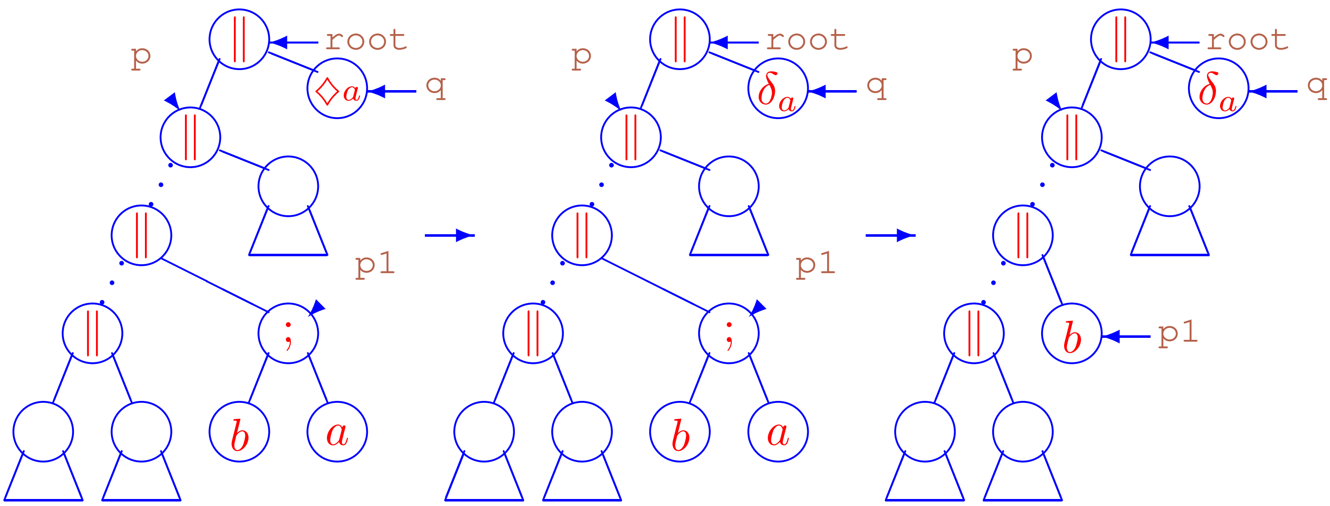
Tree transformations with rule 7.3



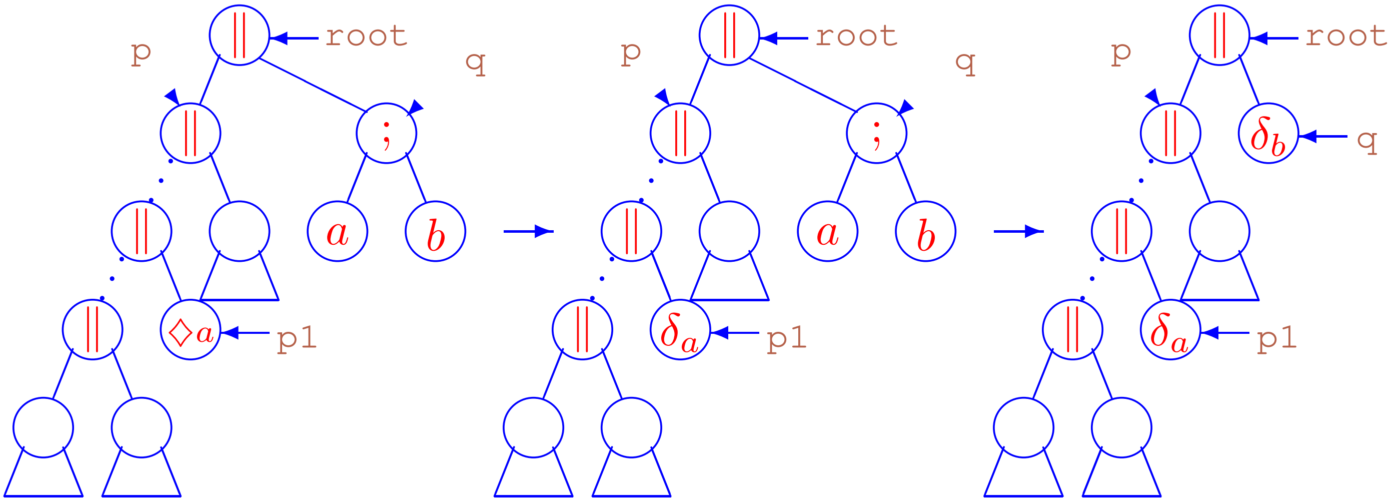
Tree transformations with rule 7.4



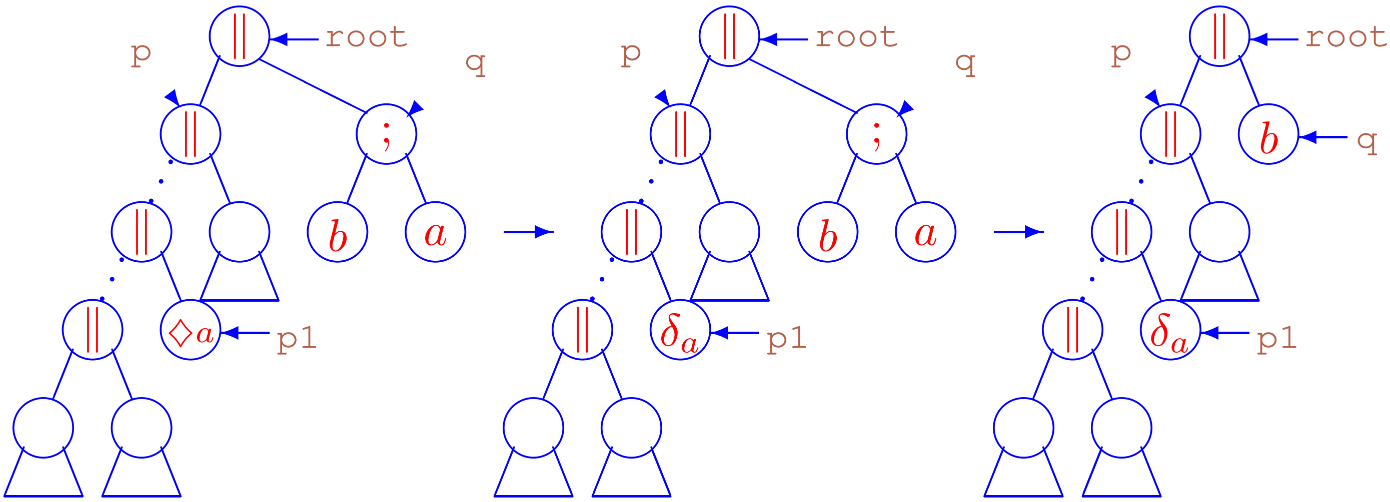
Tree transformations with rule 7.5



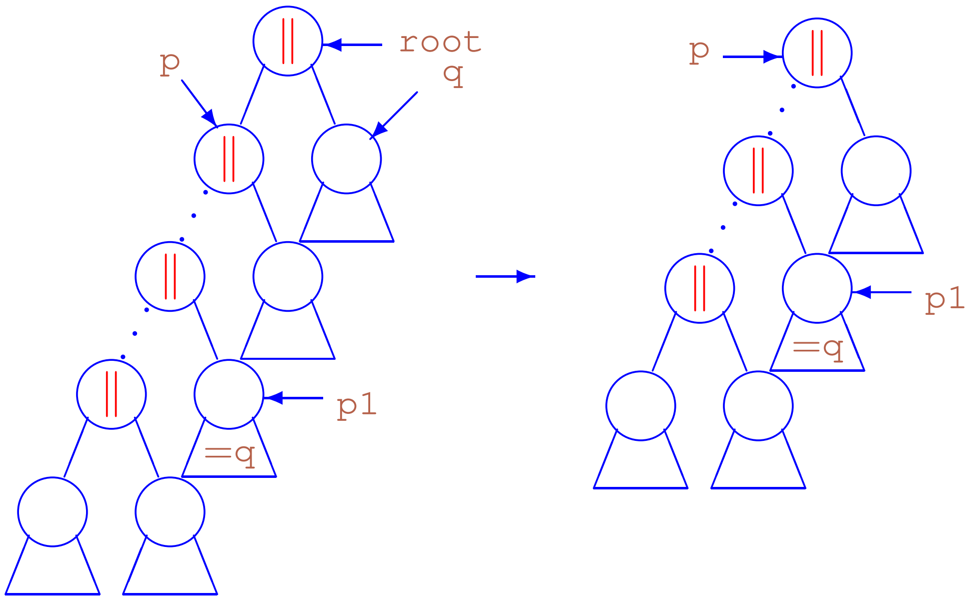
Tree transformations with rule 7.6



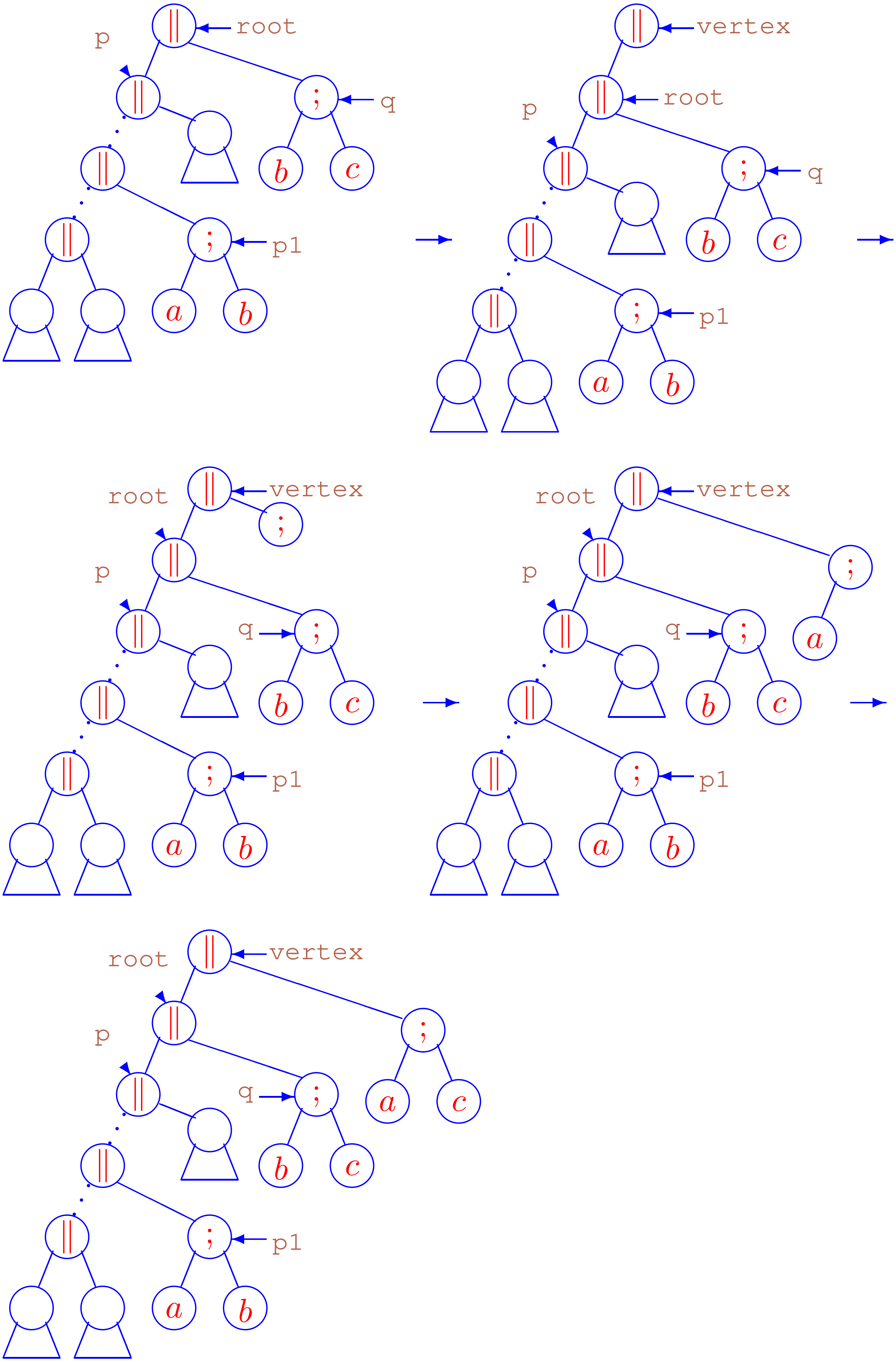
Tree transformations with rule 7.7



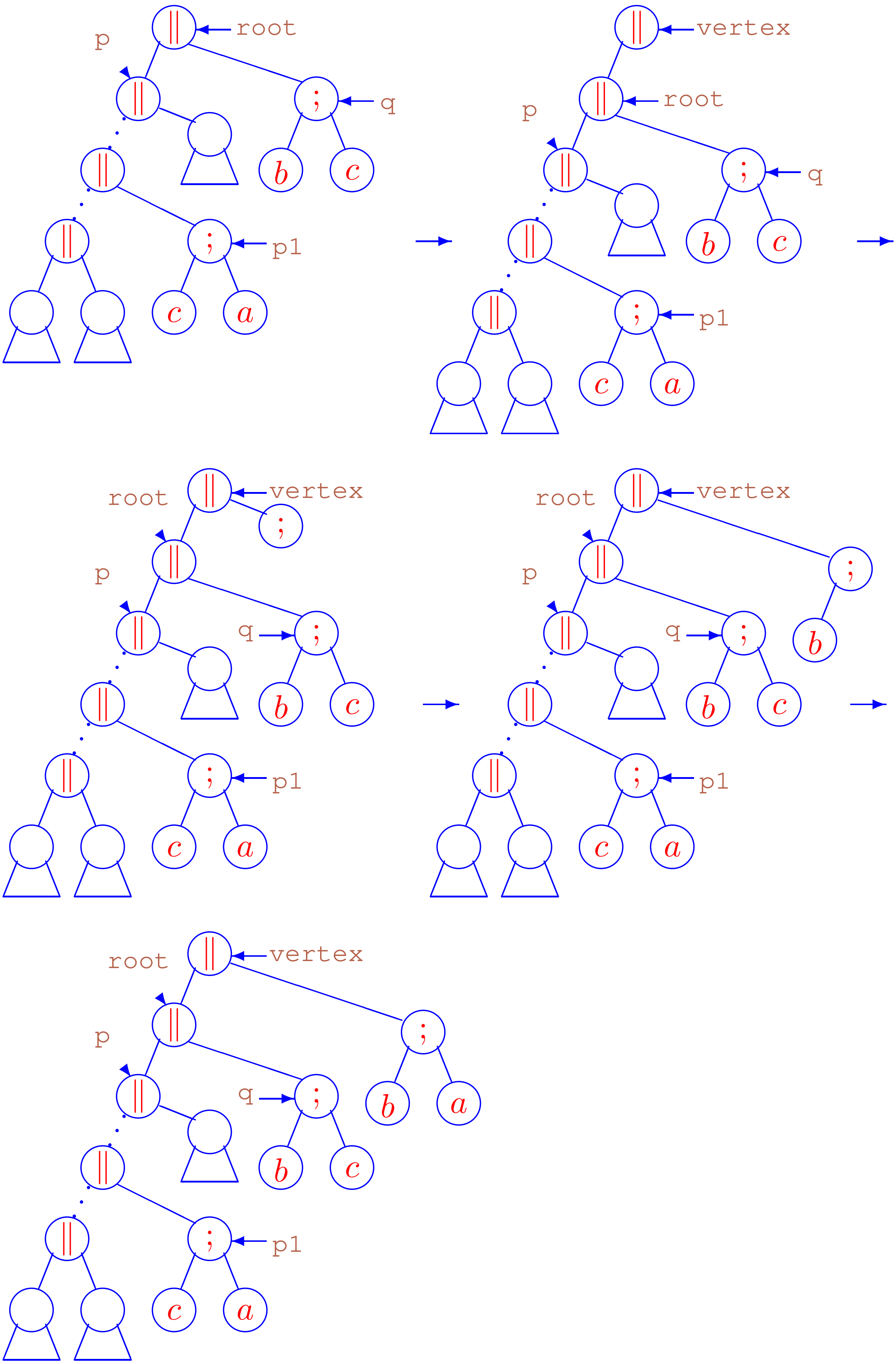
Tree transformations with rule 7.8



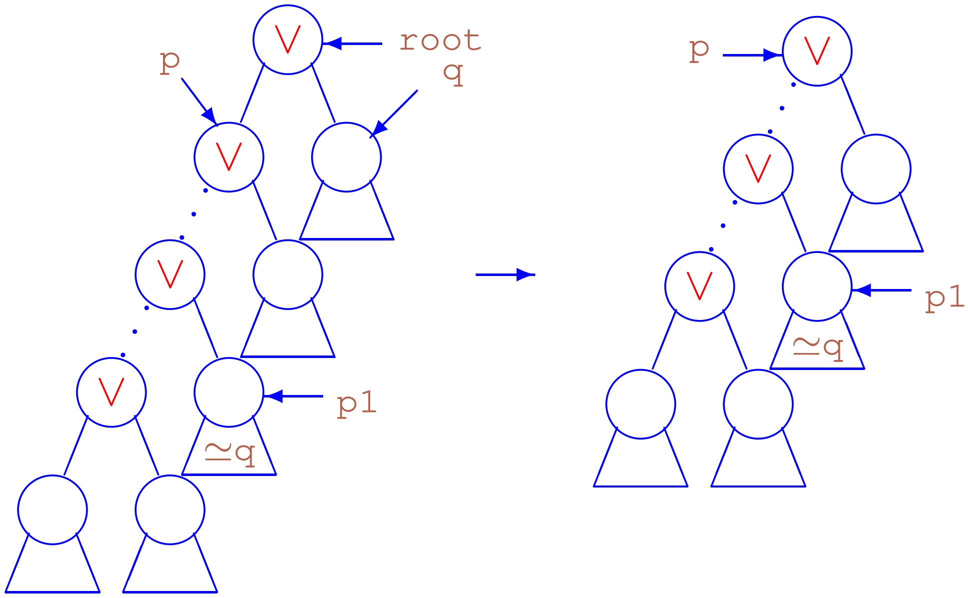
Tree transformations with rule 7.9



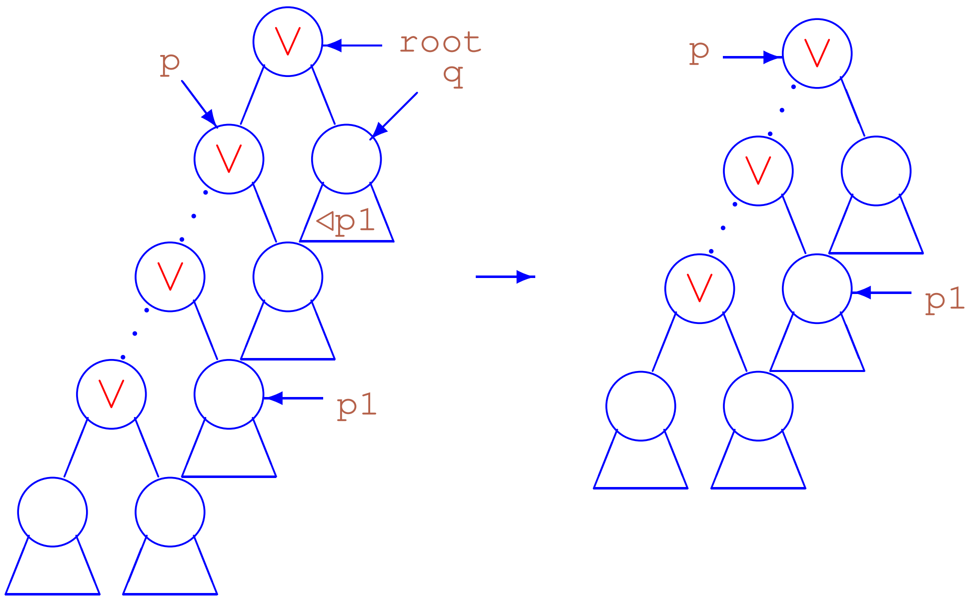
Tree transformations with rule 8.1



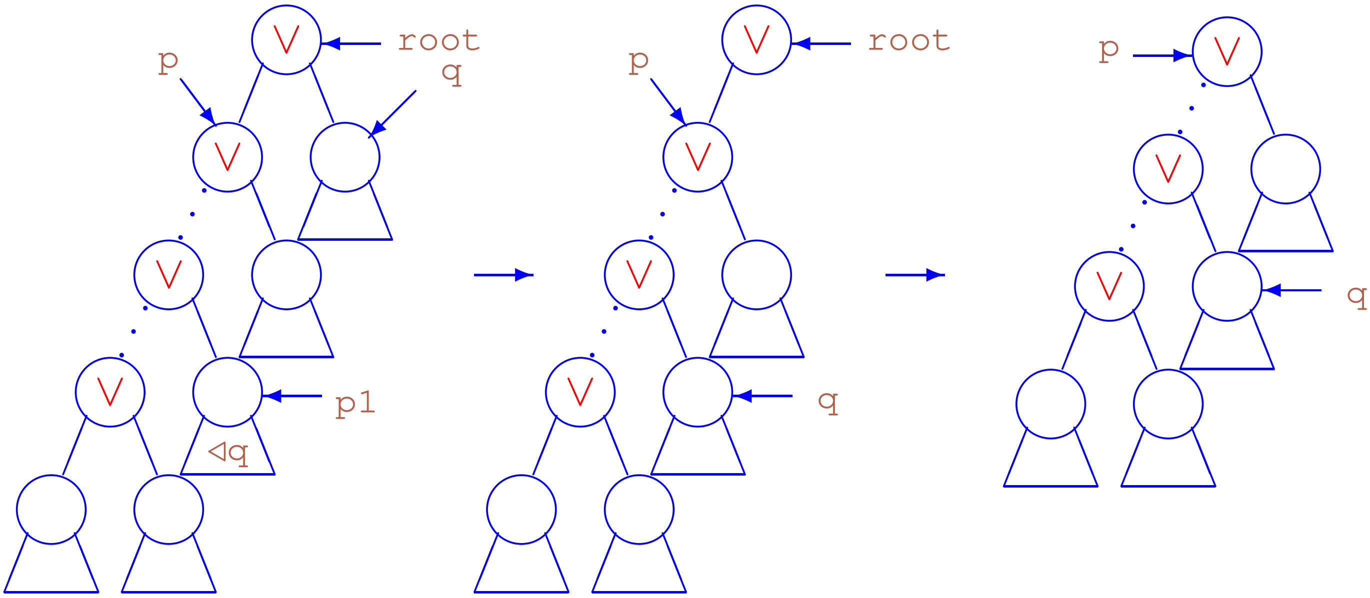
Tree transformations with rule 8.2



Tree transformations with rule 9.1



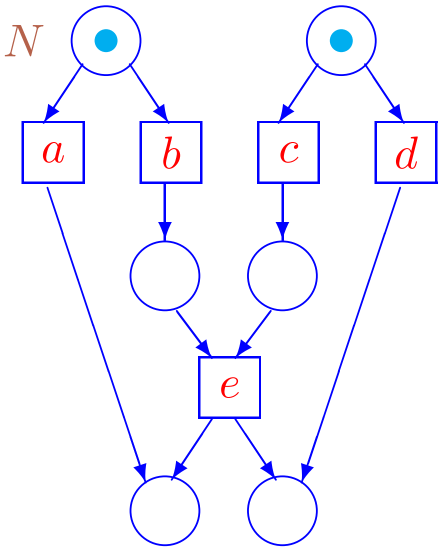
Tree transformations with rule 10.1



Tree transformations with rule 10.2

Example of formula transformation with $CANON$

The initial formula: $(a \bigtriangledown (b;e))\|(d \bigtriangledown (c;e))$.



A-net for the formula $(a \bigtriangledown (b;e))\|(d \bigtriangledown (c;e))$

```
The author of this program is I.V. Tarasyuk
Program CANON transforms formulas of algebras AFP_2, AFLP_2
into canonical form
Input formula should be in one of the following forms:

1. a
2. -a  *a
3. `a  ~a
4. `(P)  ~(P)
5. a;b  a|b  a#b  a+b
6. a;(P)  a|(P)  a#(P)  a+(P)
7. (P);a  (P)|a  (P)#a  (P)+a
8. (P);(Q)  (P)|(Q)  (P)#(Q)  (P)+(Q)

where a and b are symbols of elementary actions,
P and Q are formulas types 2-8

Input formula
Sign of end is EOF

Formula has been read

Your formula is:
(a#(b;e))|(d#(c;e))

Step 1

Rule 3.1 is applied
P=a
Q=(b;e)
New formula is:
((a|(`(b;e)))+((`a)|(b;e)))|(d#(c;e))

Rule 3.1 is applied
P=d
Q=(c;e)
New formula is:
((a|(`(b;e)))+((`a)|(b;e)))|((d|(`(c;e)))+((`d)|(c;e)))

Rule 4.1 is applied
P=b
Q=e
New formula is:
((a|((`b)|(`e)))+((`a)|(b;e)))|((d|(`(c;e)))+((`d)|(c;e)))
```

```
Rule 4.1 is applied
P=c
Q=e
New formula is:
((a|((`b)|(`e)))+((`a)|(b;e)))|((d|((`c)|(`e)))+((`d)|(c;e)))

Rule 4.3 is applied
P=b
New formula is:
((a|((-b)|(`e)))+((`a)|(b;e)))|((d|((`c)|(`e)))+((`d)|(c;e)))

Rule 4.3 is applied
P=e
New formula is:
((a|((-b)|(-e)))+((`a)|(b;e)))|((d|((`c)|(`e)))+((`d)|(c;e)))

Rule 4.3 is applied
P=a
New formula is:
((a|((-b)|(-e)))+((-a)|(b;e)))|((d|((`c)|(`e)))+((`d)|(c;e)))

Rule 4.3 is applied
P=c
New formula is:
((a|((-b)|(-e)))+((-a)|(b;e)))|((d|((-c)|(`e)))+((`d)|(c;e)))

Rule 4.3 is applied
P=e
New formula is:
((a|((-b)|(-e)))+((-a)|(b;e)))|((d|((-c)|(-e)))+((`d)|(c;e)))

Rule 4.3 is applied
P=d
New formula is:
((a|((-b)|(-e)))+((-a)|(b;e)))|((d|((-c)|(-e)))+((-d)|(c;e)))

Number of applied rules in step 1 is 10

Step 2

Rule 1.1 is applied
P=a
Q=(-b)
R=(-e)
New formula is:
(((a|(-b))|(-e))+((-a)|(b;e)))|((d|((-c)|(-e)))+((-d)|(c;e)))
```

```
Rule 1.1 is applied
P=d
Q=(-c)
R=(-e)
New formula is:
(((a|(-b))|(-e))+((-a)|(b;e)))|(((d|(-c))|(-e))+((-d)|(c;e)))

Rule 2.1 is applied
P=((a|(-b))|(-e))
Q=((-a)|(b;e))
R=(((d|(-c))|(-e))+((-d)|(c;e)))
New formula is:
(((a|(-b))|(-e))|(((d|(-c))|(-e))+((-d)|(c;e))))+
(((-a)|(b;e))|(((d|(-c))|(-e))+((-d)|(c;e))))

Rule 2.2 is applied
P=((a|(-b))|(-e))
Q=((d|(-c))|(-e))
R=((-d)|(c;e))
New formula is:
((((a|(-b))|(-e))|((d|(-c))|(-e)))+(((a|(-b))|(-e))|((-d)|(c;e))))+
(((-a)|(b;e))|(((d|(-c))|(-e))+((-d)|(c;e))))

Rule 2.2 is applied
P=((-a)|(b;e))
Q=((d|(-c))|(-e))
R=((-d)|(c;e))
New formula is:
((((a|(-b))|(-e))|((d|(-c))|(-e)))+(((a|(-b))|(-e))|((-d)|(c;e))))+
((((-a)|(b;e))|((d|(-c))|(-e)))+(((-a)|(b;e))|((-d)|(c;e))))

Number of applied rules in step 2 is 5

Step 3

Rule 1.1 is applied
P=((((a|(-b))|(-e))|((d|(-c))|(-e)))+(((a|(-b))|(-e))|((-d)|(c;e))))
Q=(((-a)|(b;e))|((d|(-c))|(-e)))
R=(((-a)|(b;e))|((-d)|(c;e)))
New formula is:
(((((a|(-b))|(-e))|((d|(-c))|(-e)))+(((a|(-b))|(-e))|((-d)|(c;e))))+
(((-a)|(b;e))|((d|(-c))|(-e))))+(((-a)|(b;e))|((-d)|(c;e)))

Number of applied rules in step 3 is 1
```

```
Step 4

Rule 1.1 is applied
P=((a|(-b))|(-e))
Q=(d|(-c))
R=(-e)
New formula is:
((((((a|(-b))|(-e))|(d|(-c)))|(-e))+((((a|(-b))|(-e))|((-d)|(c;e))))+
(((-a)|(b;e))|((d|(-c))|(-e))))+(((-a)|(b;e))|((-d)|(c;e)))

Rule 1.1 is applied
P=((a|(-b))|(-e))
Q=(-d)
R=(c;e)
New formula is:
((((((a|(-b))|(-e))|(d|(-c)))|(-e))+((((a|(-b))|(-e))|(-d))|(c;e)))+
(((-a)|(b;e))|((d|(-c))|(-e))))+(((-a)|(b;e))|((-d)|(c;e)))

Rule 1.1 is applied
P=((-a)|(b;e))
Q=(d|(-c))
R=(-e)
New formula is:
((((((a|(-b))|(-e))|(d|(-c)))|(-e))+((((a|(-b))|(-e))|(-d))|(c;e)))+
((((-a)|(b;e))|(d|(-c)))|(-e)))+(((-a)|(b;e))|((-d)|(c;e)))

Rule 1.1 is applied
P=((-a)|(b;e))
Q=(-d)
R=(c;e)
New formula is:
((((((a|(-b))|(-e))|(d|(-c)))|(-e))+((((a|(-b))|(-e))|(-d))|(c;e)))+
((((-a)|(b;e))|(d|(-c)))|(-e)))+((((-a)|(b;e))|(-d))|(c;e))

Number of applied rules in step 4 is 4

Step 5

Rule 1.1 is applied
P=((a|(-b))|(-e))
Q=d
R=(-c)
New formula is:
(((((((a|(-b))|(-e))|d)|(-c))|(-e))+((((a|(-b))|(-e))|(-d))|(c;e)))+
((((-a)|(b;e))|(d|(-c)))|(-e)))+((((-a)|(b;e))|(-d))|(c;e))
```

```
Rule 1.1 is applied
P=((-a)|(b;e))
Q=d
R=(-c)
New formula is:
((((((a|(-b))|(-e))|d)|(-c))|(-e))+((((a|(-b))|(-e))|(-d))|(c;e)))+
(((((-a)|(b;e))|d)|(-c))|(-e)))+((((-a)|(b;e))|(-d))|(c;e))

Number of applied rules in step 5 is 2

Step 6

Rule 7.6 is applied
P=((((-a)|(b;e))|d)|(-c))
P'=(b;e)
Q=(-e)
New formula is:
((((((a|(-b))|(-e))|d)|(-c))|(-e))+((((a|(-b))|(-e))|(-d))|(c;e)))+
(((((-a)|b)|d)|(-c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))

Rule 7.8 is applied
P=(((a|(-b))|(-e))|(-d))
P'=(-e)
Q=(c;e)
New formula is:
((((((a|(-b))|(-e))|d)|(-c))|(-e))+((((a|(-b))|(*e))|(-d))|c))+
(((((-a)|b)|d)|(-c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))

Rule 7.9 is applied
P=((((a|(-b))|(-e))|d)|(-c))
P'=(-e)
Q=(-e)
New formula is:
(((((a|(-b))|(-e))|d)|(-c))+((((a|(-b))|(*e))|(-d))|c))+
(((((-a)|b)|d)|(-c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))

Number of applied rules in step 6 is 3

Step 7

Rule 6.1 is applied
P=((a|(-b))|(*e))
P'=(*e)
Q=(-d)
New formula is:
(((((a|(-b))|(-e))|d)|(-c))+((((a|(-b))|(*e))|(*d))|c))+
(((((-a)|b)|d)|(-c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))
```

```
Rule 6.2 is applied
P=((a|(-b))|(*e))
P'=(-b)
Q=(*d)
New formula is:
((((((a|(-b))|(-e))|d)|(-c))+((((a|(*b))|(*e))|(*d))|c))+
(((((-a)|b)|d)|(-c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))

Rule 6.2 is applied
P=((((-a)|b)|d)|(-c))
P'=(-c)
Q=(*e)
New formula is:
((((((a|(-b))|(-e))|d)|(-c))+((((a|(*b))|(*e))|(*d))|c))+
(((((-a)|b)|d)|(*c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))

Number of applied rules in step 7 is 3

Step 8

Rule 6.2 is applied
P=((((-a)|b)|d)|(*c))
P'=(-a)
Q=(*e)
New formula is:
((((((a|(-b))|(-e))|d)|(-c))+((((a|(*b))|(*e))|(*d))|c))+
(((((*a)|b)|d)|(*c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))

Number of applied rules in step 8 is 1

Step 9

Number of applied rules in step 9 is 0

Canonical form is:
((((((a|(-b))|(-e))|d)|(-c))+((((a|(*b))|(*e))|(*d))|c))+
(((((*a)|b)|d)|(*c))|(*e)))+((((-a)|(b;e))|(-d))|(c;e))
```

Canonical form is: $(a\|d\|\bar{b}\|\bar{c}\|\bar{e}) \vee (a\|c\|\delta_b\|\delta_d\|\delta_e) \vee (b\|d\|\delta_a\|\delta_c\|\delta_e) \vee ((b;e)\|(c;e)\|\bar{a}\|\bar{d})$.

Discrete time stochastic Petri box calculus

Abstract: In [MVF01], a continuous time stochastic extension $sPBC$ of finite PBC was proposed.

In [MVCC03], iteration operator was added to $sPBC$.

Algebra $sPBC$ has interleaving semantics, but PBC has step one.

We constructed a discrete time stochastic extension $dtsPBC$ of finite PBC [Tar05] and enriched it with iteration [Tar06].

Step operational semantics is defined in terms of labeled probabilistic transition systems.

Denotational semantics is defined in terms of a subclass of labeled DTSPNs (LDTSPNs) called discrete time stochastic Petri boxes (dts-boxes).

We propose a variety of stochastic equivalences.

The interrelations of all the introduced equivalences are investigated.

It is explained how to use the equivalences for transition systems and discrete time Markov chains reduction.

A logical characterization of the equivalences is presented via probabilistic modal logics.

We demonstrate how to apply the equivalences to compare stationary behaviour.

A congruence relation is defined.

The case studies of performance evaluation are presented.

Keywords: Stochastic Petri nets, stochastic process algebras, Petri box calculus, iteration, discrete time, transition systems, operational semantics, dts-boxes, denotational semantics, empty loops, stochastic equivalences, reduction, modal logics, stationary behaviour, congruence, performance evaluation.

Contents

- **Stationary behaviour**
  - **Theoretical background**
  - **Steady state and equivalences**
  - **Simplification of performance analysis**
- **Preservation by algebraic operations**
- **Case studies**
  - **Shared memory system**
  - **Dining philosophers system**
- **Overview and open questions**
  - **The results obtained**
  - **Further research**

Introduction

Previous work

- Continuous time (subsets of $\mathbb{R}_+$): interleaving semantics
  - *Continuous time stochastic Petri nets (CTSPNs)* [Mol82,FN85]:
    exponential transition firing delays,
    Continuous time Markov chain (CTMC).
  - *Generalized stochastic Petri nets (GSPNs)* [MCB84,CMBC93]:
    exponential and zero transition firing delays,
    Semi-Markov chain (SMC).
- Discrete time (subsets of $\mathbb{N}$): interleaving and step semantics
  - *Discrete time stochastic Petri nets (DTSPNs)* [Mol85,ZG94]:
    geometric transition firing delays,
    Discrete time Markov chain (DTMC).
  - *Discrete time deterministic and stochastic Petri nets (DTDSPNs)* [ZFH01]:
    geometric and fixed transition firing delays,
    Semi-Markov chain (SMC).
  - *Discrete deterministic and stochastic Petri nets (DDSPNs)* [ZCH97]:
    phase and fixed transition firing delays,
    Semi-Markov chain (SMC).

Stochastic process algebras

- $MTIPP$ [HR94]
- $GSPA$ [BKLL95]
- $PEPA$ [Hil96]
- $S\pi$ [Pri96]
- $EMPA$ [BGo98]
- $GSMPA$ [BBGo98]
- $sACP$ [AHR00]
- TCP^{dst} [MVi08]

More stochastic process calculi

- $TIPP$ [GHR93]
- $TPCCS$ [Han94]
- $PM-TIPP$ [Ret95]
- PPA [NFL95]
- $prBPA$, ACP_π^+ [And99]
- $StAFP_0$ [BT01]
- $SM-PEPA$ [Brad05]
- $iPEPA$ [HBC13]

Algebra PBC and its extensions

- *Petri box calculus* PBC [BDH92]
- *Time Petri box calculus* $tPBC$ [Kou00]
- *Timed Petri box calculus* $TPBC$ [MF00]
- *Stochastic Petri box calculus* $sPBC$ [MVF01,MVCC03]
- *Ambient Petri box calculus* $APBC$ [FM03]
- *Arc time Petri box calculus* $atPBC$ [Nia05]
- *Generalized stochastic Petri box calculus* $gsPBC$ [MVCR08]
- *Discrete time stochastic Petri box calculus* $dtsPBC$ [Tar05,Tar06]
- *Discrete time stochastic and immediate Petri box calculus* $dtsiPBC$ [TMV10,TMV13]

Classification of stochastic process algebras

Time	Interleaving semantics	Non-interleaving semantics
Continuous	$MTIPP$ (CTMC), $PEPA$ (CTMP), $EMPA$ (SMC, CTMC), $sPBC$ (CTMC), $gsPBC$ (SMC)	$GSPA$ (GSMP), $S\pi$, $GSMPA$ (GSMP)
Discrete	TCP^{dst} (DTMRC)	$sACP$, $dtsPBC$ (DTMC), $dtsiPBC$ (SMC, DTMC)

The SPNs-based denotational semantics: orange SPA names.

The underlying stochastic process: in parentheses near the SPA names.

Transition labeling

- CTSPNs [Buc95]
- GSPNs [Buc98]
- DTSPNs [BT00]

Stochastic equivalences

- Probabilistic transition systems (PTSs) [BM89,Chr90,LS91,BHe97,KN98]
- SPAs [HR94,Hil94,BGo98]
- Markov process algebras (MPAs) [Buc94,BKe01]
- CTSPNs [Buc95]
- GSPNs [Buc98]
- Stochastic automata (SAs) [Buc99]
- Stochastic event structures (SESs) [MCW03]

Syntax

The *set of all finite multisets* over X is $\mathbb{N}_{fin}^X$.

$Act = \{a, b, \ldots\}$ is the set of *elementary actions*.

$\widehat{Act} = \{\hat{a}, \hat{b}, \ldots\}$ is the set of *conjugated actions (conjugates)* s.t. $\hat{a} \neq a$ and $\hat{\hat{a}} = a$.

$\mathcal{A} = Act \cup \widehat{Act}$ is the set of *all actions*.

$\mathcal{L} = \mathbb{N}_{fin}^{\mathcal{A}}$ is the set of *all multiactions*.

The *alphabet* of $\alpha \in \mathcal{L}$ is $\mathcal{A}(\alpha) = \{x \in \mathcal{A} \mid \alpha(x) > 0\}$.

An *activity (stochastic multiaction)* is a pair (α, ρ), where $\alpha \in \mathcal{L}$ and $\rho \in (0; 1)$ is the probability of multiaction α.

$\mathcal{SL}$ is the set of *all activities*.

The *alphabet* of $(\alpha, \rho) \in \mathcal{SL}$ is $\mathcal{A}(\alpha, \rho) = \mathcal{A}(\alpha)$.

The *alphabet* of $\Gamma \in \mathbb{N}_{fin}^{\mathcal{SL}}$ is $\mathcal{A}(\Gamma) = \cup_{(\alpha,\rho)\in\Gamma}\mathcal{A}(\alpha)$.

For $(\alpha, \rho) \in \mathcal{SL}$, its *multiaction part* is $\mathcal{L}(\alpha, \rho) = \alpha$ and its *probability part* is $\Omega(\alpha, \rho) = \rho$.

The *multiaction part* of $\Gamma \in \mathbb{N}_{fin}^{\mathcal{SL}}$ is $\mathcal{L}(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma} \alpha$.

The operations: *sequential execution* ;, *choice* $[]$, *parallelism* $\|$, *relabeling* $[f]$, *restriction* rs, *synchronization* sy and *iteration* $[**]$.

Sequential execution and choice have the standard interpretation.

Parallelism does not include synchronization unlike that in standard process algebras.

Relabeling functions $f : \mathcal{A} \to \mathcal{A}$ are bijections preserving conjugates: $\forall x \in \mathcal{A}\ f(\hat{x}) = \widehat{f(x)}$.

For $\alpha \in \mathcal{L}$, let $f(\alpha) = \sum_{x\in\alpha} f(x)$.

For $\Gamma \in I\!N_{fin}^{\mathcal{SL}}$, let $f(\Gamma) = \sum_{(\alpha,\rho)\in\Gamma}(f(\alpha), \rho)$.

Restriction over $a \in Act$: any process behaviour containing a or its conjugate $\hat{a}$ is not allowed.

Let $\alpha, \beta \in \mathcal{L}$ be two multiactions s.t. for $a \in Act$ we have $a \in \alpha$ and $\hat{a} \in \beta$, or $\hat{a} \in \alpha$ and $a \in \beta$. Then synchronization of α and β by a is $\alpha \oplus_a \beta = \gamma$:

$$\gamma(x) = \begin{cases} \alpha(x) + \beta(x) - 1, & x = a \text{ or } x = \hat{a}; \\ \alpha(x) + \beta(x), & \text{otherwise}. \end{cases}$$

In the iteration, the initialization subprocess is executed first,
then the body one is performed zero or more times,
finally, the termination one is executed.

Static expressions specify the structure of processes.

Definition 113 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$*. A static expression of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}E{*}E].$$

$StatExpr$ is the set of *all static expressions* of $dtsPBC$.

Definition 114 *Let* $(\alpha, \rho) \in \mathcal{SL}$ *and* $a \in Act$*. A regular static expression of* $dtsPBC$ *is*

$$E ::= \ (\alpha, \rho) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}D{*}E],$$

where $D ::= \ (\alpha, \rho) \mid D;E \mid D[]D \mid D[f] \mid D \text{ rs } a \mid D \text{ sy } a \mid [D{*}D{*}E].$

$RegStatExpr$ is the set of *all regular static expressions* of $dtsPBC$.

Dynamic expressions specify the states of processes.

Dynamic expressions are obtained from static ones annotated with upper or lower bars.

The *underlying static expression* of a dynamic one: removing all upper and lower bars.

Definition 115 *Let $E \in StatExpr$ and $a \in Act$. A* dynamic expression *of $dtsPBC$ is*

$$G ::= \overline{E} \mid \underline{E} \mid G;E \mid E;G \mid G[]E \mid E[]G \mid G\|G \mid G[f] \mid G \text{ rs } a \mid G \text{ sy } a \mid [G{*}E{*}E] \mid [E{*}G{*}E] \mid [E{*}E{*}G].$$

$DynExpr$ is the set of *all dynamic expressions* of $dtsPBC$.

Definition 116 *A dynamic expression is* regular *if its underlying static expression is regular.*

$RegDynExpr$ is the set of *all regular dynamic expressions* of $dtsPBC$.

Operational semantics

Inaction rules

Inaction rules: instantaneous structural transformations.

Let $E, F, K \in RegStatExpr$ and $a \in Act$.

Inaction rules for overlined and underlined regular static expressions

$\overline{E;F} \Rightarrow \overline{E};F$	$\underline{E};F \Rightarrow E;\overline{F}$	$E;\underline{F} \Rightarrow \underline{E;F}$
$\overline{E[]F} \Rightarrow \overline{E}[]F$	$\overline{E[]F} \Rightarrow E[]\overline{F}$	$\underline{E}[]F \Rightarrow \underline{E[]F}$
$E[]\underline{F} \Rightarrow \underline{E[]F}$	$\overline{E\|F} \Rightarrow \overline{E}\|\overline{F}$	$\underline{E}\|\underline{F} \Rightarrow \underline{E\|F}$
$\overline{E[f]} \Rightarrow \overline{E}[f]$	$\underline{E}[f] \Rightarrow \underline{E[f]}$	$\overline{E \text{ rs } a} \Rightarrow \overline{E} \text{ rs } a$
$\underline{E} \text{ rs } a \Rightarrow \underline{E \text{ rs } a}$	$\overline{E \text{ sy } a} \Rightarrow \overline{E} \text{ sy } a$	$\underline{E} \text{ sy } a \Rightarrow \underline{E \text{ sy } a}$
$\overline{[E{*}F{*}K]} \Rightarrow [\overline{E}{*}F{*}K]$	$[\underline{E}{*}F{*}K] \Rightarrow [E{*}\overline{F}{*}K]$	$[E{*}\underline{F}{*}K] \Rightarrow [E{*}\overline{F}{*}K]$
$[E{*}\underline{F}{*}K] \Rightarrow [E{*}F{*}\overline{K}]$	$[E{*}F{*}\underline{K}] \Rightarrow \underline{[E{*}F{*}K]}$	

Let $E, F \in RegStatExpr,\ G, H, \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

Inaction rules for arbitrary regular dynamic expressions

$$\frac{G \Rightarrow \widetilde{G},\ \circ \in \{;, []\}}{G \circ E \Rightarrow \widetilde{G} \circ E} \qquad \frac{G \Rightarrow \widetilde{G},\ \circ \in \{;, []\}}{E \circ G \Rightarrow E \circ \widetilde{G}} \qquad \frac{G \Rightarrow \widetilde{G}}{G \| H \Rightarrow \widetilde{G} \| H} \qquad \frac{H \Rightarrow \widetilde{H}}{G \| H \Rightarrow G \| \widetilde{H}}$$

$$\frac{G \Rightarrow \widetilde{G}}{G[f] \Rightarrow \widetilde{G}[f]} \qquad \frac{G \Rightarrow \widetilde{G},\ \circ \in \{\text{rs}, \text{sy}\}}{G \circ a \Rightarrow \widetilde{G} \circ a} \qquad \frac{G \Rightarrow \widetilde{G}}{[G * E * F] \Rightarrow [\widetilde{G} * E * F]} \qquad \frac{G \Rightarrow \widetilde{G}}{[E * G * F] \Rightarrow [E * \widetilde{G} * F]}$$

$$\frac{G \Rightarrow \widetilde{G}}{[E * F * G] \Rightarrow [E * F * \widetilde{G}]}$$

Definition 117 *A regular dynamic expression is operative if no inaction rule can be applied to it.*

$OpRegDynExpr$ is the set of *all operative regular dynamic expressions* of $dtsPBC$.

We shall consider regular expressions only and omit the word “regular”.

Definition 118 $\approx\ = (\Rightarrow \cup \Leftarrow)^*$ *is the structural equivalence of dynamic expressions in* $dtsPBC$.

G *and* G' *are structurally equivalent,* $G \approx G'$*, if they can be reached each from other by applying inaction rules in a forward or backward direction.*

Action and empty loop rules

Action rules: execution of non-empty multisets of activities at a time step.

Empty loop rule: execution of the empty multiset of activities at a time step.

Let $(\alpha,\rho),(\beta,\chi)\in\mathcal{SL},\ E,F\in RegStatExpr$,
$G,H\in OpRegDynExpr,\ \widetilde{G},\widetilde{H}\in RegDynExpr$,
$a\in Act$ and $\Gamma,\Delta\in I\!N_{fin}^{\mathcal{SL}}\setminus\{\emptyset\},\ \Gamma'\in I\!N_{fin}^{\mathcal{SL}}$.

Action and empty loop rules

$$\textbf{El}\ G\stackrel{\emptyset}{\rightarrow}G \qquad \textbf{B}\ \overline{(\alpha,\rho)}\stackrel{\{(\alpha,\rho)\}}{\longrightarrow}\underline{(\alpha,\rho)} \qquad \textbf{SC1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{G\circ E\stackrel{\Gamma}{\rightarrow}\widetilde{G}\circ E}$$

$$\textbf{SC2}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ \circ\in\{;,[]\}}{E\circ G\stackrel{\Gamma}{\rightarrow}E\circ\widetilde{G}} \qquad \textbf{P1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G\|H\stackrel{\Gamma}{\rightarrow}\widetilde{G}\|H} \qquad \textbf{P2}\ \frac{H\stackrel{\Gamma}{\rightarrow}\widetilde{H}}{G\|H\stackrel{\Gamma}{\rightarrow}G\|\widetilde{H}}$$

$$\textbf{P3}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ H\stackrel{\Delta}{\rightarrow}\widetilde{H}}{G\|H\stackrel{\Gamma+\Delta}{\longrightarrow}\widetilde{G}\|\widetilde{H}} \qquad \textbf{L}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G[f]\stackrel{f(\Gamma)}{\longrightarrow}\widetilde{G}[f]} \qquad \textbf{Rs}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G},\ a,\hat{a}\not\in\mathcal{A}(\Gamma)}{G\ \mathsf{rs}\ a\stackrel{\Gamma}{\rightarrow}\widetilde{G}\ \mathsf{rs}\ a}$$

$$\textbf{I1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[G*E*F]\stackrel{\Gamma}{\rightarrow}[\widetilde{G}*E*F]} \qquad \textbf{I2}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[E*G*F]\stackrel{\Gamma}{\rightarrow}[E*\widetilde{G}*F]} \qquad \textbf{I3}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{[E*F*G]\stackrel{\Gamma}{\rightarrow}[E*F*\widetilde{G}]}$$

$$\textbf{Sy1}\ \frac{G\stackrel{\Gamma}{\rightarrow}\widetilde{G}}{G\ \mathsf{sy}\ a\stackrel{\Gamma}{\rightarrow}\widetilde{G}\ \mathsf{sy}\ a} \qquad \textbf{Sy2}\ \frac{G\ \mathsf{sy}\ a\xrightarrow{\Gamma'+\{(\alpha,\rho)\}+\{(\beta,\chi)\}}\widetilde{G}\ \mathsf{sy}\ a,\ a\in\alpha,\ \hat{a}\in\beta}{G\ \mathsf{sy}\ a\xrightarrow{\Gamma'+\{(\alpha\oplus_a\beta,\rho\cdot\chi)\}}\widetilde{G}\ \mathsf{sy}\ a}$$

Comparison of inaction, action and empty loop rules

Rules	State change	Time progress	Activities execution
Inaction rules	−	−	−
Action rules	±	+	+
Empty loop rule	−	+	−

Transition systems

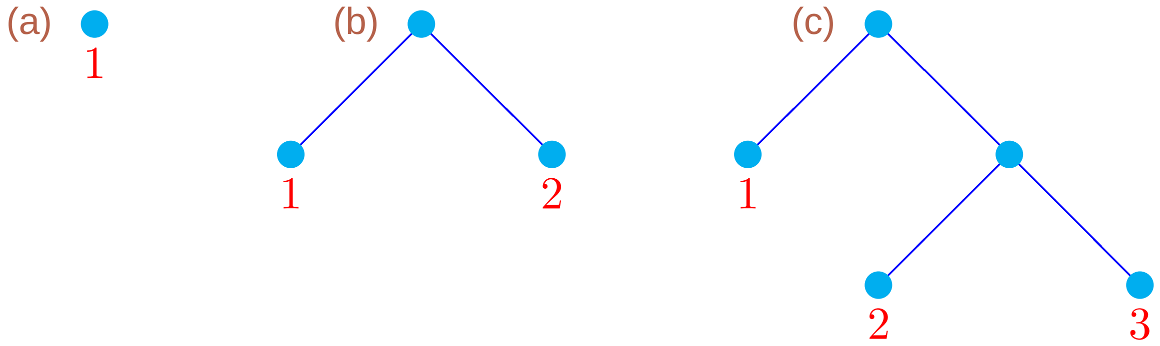
Definition 119 *Let $n \in \mathbb{N}$. The numbering of expressions is*

$$\iota ::= n \mid (\iota)(\iota).$$

Num is the set of *all numberings* of expressions.

The *content* of a numbering $\iota \in Num$ is

$$Cont(\iota) = \begin{cases} \{\iota\}, & \iota \in \mathbb{N}; \\ Cont(\iota_1) \cup Cont(\iota_2), & \iota = (\iota_1)(\iota_2). \end{cases}$$



The binary trees encoded with the numberings 1, $(1)(2)$ and $(1)((2)(3))$

$[G]_\approx = \{H \mid G \approx H\}$ is the equivalence class of a dynamic expression G w.r.t. structural equivalence.

Definition 120 *The derivation set $DR(G)$ of a dynamic expression G is the minimal set:*

- $[G]_\approx \in DR(G)$;
- *if* $[H]_\approx \in DR(G)$ *and* $\exists \Gamma\ H \xrightarrow{\Gamma} \widetilde{H}$ *then* $[\widetilde{H}]_\approx \in DR(G)$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

The set of *all multisets of activities executable from s* is
$Exec(s) = \{\Gamma \mid \exists H \in s\ \exists \widetilde{H}\ H \xrightarrow{\Gamma} \widetilde{H}\}$.

Let $\Gamma \in Exec(s) \setminus \{\emptyset\}$. The *probability that the multiset of activities Γ is ready for execution in s*:

$$PF(\Gamma, s) = \prod_{(\alpha,\rho)\in\Gamma} \rho \cdot \prod_{\{\{(\beta,\chi)\}\in Exec(s) \mid (\beta,\chi)\not\in\Gamma\}} (1-\chi).$$

In the case $\Gamma = \emptyset$ we define

$$PF(\emptyset, s) = \begin{cases} \prod_{\{(\beta,\chi)\}\in Exec(s)} (1-\chi), & Exec(s) \neq \{\emptyset\}; \\ 1, & \text{otherwise}. \end{cases}$$

Let $\Gamma \in Exec(s)$. The *probability to execute the multiset of activities Γ in s* is

$$PT(\Gamma, s) = \frac{PF(\Gamma, s)}{\sum_{\Delta\in Exec(s)} PF(\Delta, s)}.$$

The *probability to move from s to $\tilde{s}$ by executing any multiset of activities* is

$$PM(s, \tilde{s}) = \sum_{\{\Gamma \mid \exists H\in s\ \exists \widetilde{H}\in\tilde{s}\ H \xrightarrow{\Gamma} \widetilde{H}\}} PT(\Gamma, s).$$

Definition 121 *The (labeled probabilistic) transition system of a dynamic expression* G *is* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, *where*

- *the set of states is* $S_G = DR(G)$;
- *the set of labels is* $L_G = I\!N_{fin}^{\mathcal{SL}} \times (0; 1]$;
- *the set of transitions is* $\mathcal{T}_G = \{(s, (\Gamma, PT(\Gamma, s)), \tilde{s}) \mid s, \tilde{s} \in DR(G),\ \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \stackrel{\Gamma}{\to} \widetilde{H}\}$;
- *the initial state is* $s_G = [G]_\approx$.

A transition $(s, (\Gamma, \mathcal{P}), \tilde{s}) \in \mathcal{T}_G$ is written as $s \stackrel{\Gamma}{\to}_{\mathcal{P}} \tilde{s}$.

We write $s \stackrel{\Gamma}{\to} \tilde{s}$ if $\exists \mathcal{P}\ s \stackrel{\Gamma}{\to}_{\mathcal{P}} \tilde{s}$ and $s \to \tilde{s}$ if $\exists \Gamma\ s \stackrel{\Gamma}{\to} \tilde{s}$.

Definition 122 *Let* G, G' *be dynamic expressions and* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, $TS(G') = (S_{G'}, L_{G'}, \mathcal{T}_{G'}, s_{G'})$ *be their transition systems. A mapping* $\beta : S_G \to S_{G'}$ *is an isomorphism between* $TS(G)$ *and* $TS(G')$, $\beta : TS(G) \simeq TS(G')$, *if*

1. β *is a bijection s.t.* $\beta(s_G) = s_{G'}$;
2. $\forall s, \tilde{s} \in S_G\ \forall \Gamma\ s \stackrel{\Gamma}{\to}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \stackrel{\Gamma}{\to}_{\mathcal{P}} \beta(\tilde{s})$.

$TS(G)$ *and* $TS(G')$ *are isomorphic,* $TS(G) \simeq TS(G')$, *if* $\exists \beta : TS(G) \simeq TS(G')$.

For $E \in RegStatExpr$, let $TS(E) = TS(\overline{E})$.

Definition 123 G *and* G' *are equivalent w.r.t. transition systems,* $G =_{ts} G'$, *if* $TS(G) \simeq TS(G')$.

Definition 124 *The* underlying discrete time Markov chain (DTMC) *of a dynamic expression* G, $DTMC(G)$, *has the state space* $DR(G)$, *the initial state* $[G]_{\approx}$ *and transitions* $s \rightarrow_{\mathcal{P}} \tilde{s}$, *if* $s \rightarrow \tilde{s}$ *and* $\mathcal{P} = PM(s, \tilde{s})$.

For $E \in RegStatExpr$, let $DTMC(E) = DTMC(\overline{E})$.

For a dynamic expression G, a discrete random variable is associated with every state of $DTMC(G)$.

The random variables (residence time in the states) are geometrically distributed: the probability to stay in the state $s \in DR(G)$ for $k-1$ moments and leave it at the moment $k \geq 1$ is $PM(s,s)^{k-1}(1 - PM(s,s))$.

The mean value formula: the *average sojourn time in the state* s is

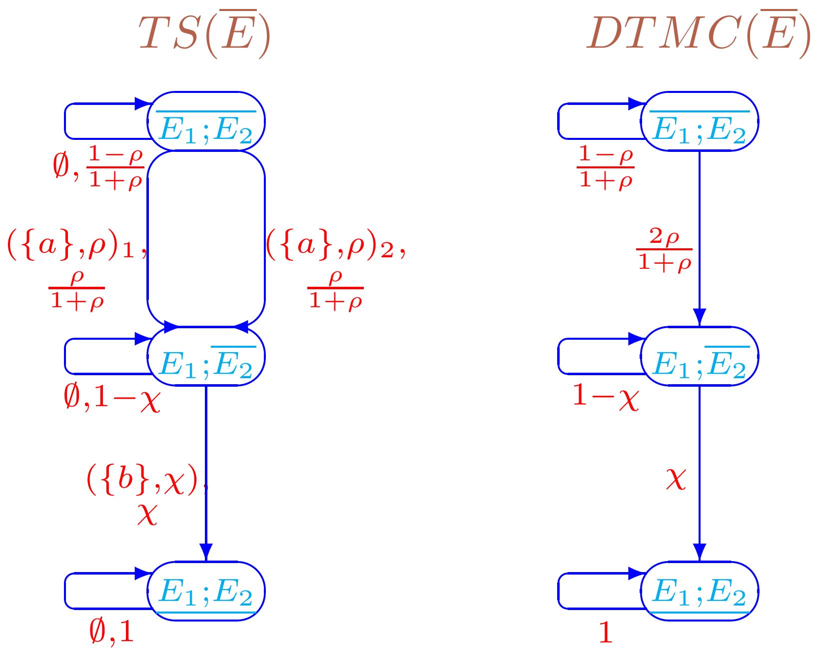
$$SJ(s) = \frac{1}{1 - PM(s,s)}.$$

The *average sojourn time vector* SJ of G has the elements $SJ(s),\ s \in DR(G)$.

Analogously: the *sojourn time variance in the state* s is

$$VAR(s) = \frac{PM(s,s)}{(1 - PM(s,s))^2}.$$

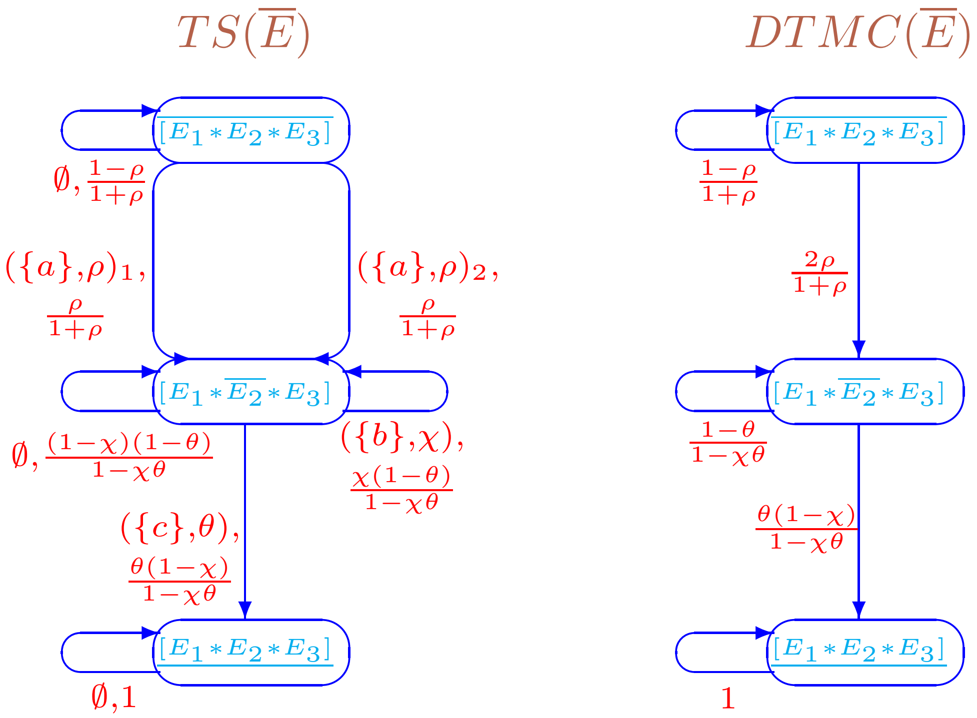
The *sojourn time variance vector* VAR of G has the elements $VAR(s),\ s \in DR(G)$.



The transition system and the underlying DTMC of $\overline{E}$ for
$E = ((\{a\}, \rho)_1[](\{a\}, \rho)_2); (\{b\}, \chi)$

Let $E_1 = (\{a\}, \rho)[](\{a\}, \rho), \ E_2 = (\{b\}, \chi)$ and $E = E_1; E_2$.

The identical activities of the composite static expression are enumerated as:
$E = ((\{a\}, \rho)_1[](\{a\}, \rho)_2); (\{b\}, \chi)$.



EXPRIT:The transition system and the underlying DTMC of $\overline{E}$ for

$$E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$$

Let $E_1 = (\{a\},\rho)[](\{a\},\rho),\ E_2 = (\{b\},\chi),\ E_3 = (\{c\},\theta)$ and $E = [E_1 * E_2 * E_3]$.

The identical activities of the composite static expression are enumerated as: $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$.

$DR(\overline{E})$ consists of $s_1 = [[\overline{E_1 * E_2 * E_3}]]_\approx,\ s_2 = [[E_1 * \overline{E_2} * E_3]]_\approx,$ $s_3 = [[\underline{E_1 * E_2 * E_3}]]_\approx$.

The average sojourn time vector of $\overline{E}$ is $SJ = \left(\frac{1+\rho}{2\rho}, \frac{1-\chi\theta}{\theta(1-\chi)}, \infty\right)$.

The sojourn time variance vector of $\overline{E}$ is

$$VAR = \left(\frac{1-\rho^2}{4\rho^2}, \frac{(1-\theta)(1-\chi\theta)}{\theta^2(1-\chi)^2}, \infty\right).$$

Denotational semantics

Labeled DTSPNs

Definition 125 *A labeled discrete time stochastic Petri net (LDTSPN) is a tuple $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$:*

- P_N *and* T_N *are finite sets of places and transitions* *($P_N \cup T_N \neq \emptyset,\ P_N \cap T_N = \emptyset$);*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbb{N}$ *is the arc weight function;*
- $\Omega_N : T_N \to (0;1)$ *is the transition probability function;*
- $L_N : T_N \to \mathcal{L}$ *is the transition labeling function;*
- $M_N \in \mathbb{N}_{fin}^{P_N}$ *is the initial marking.*

  Concurrent transition firings at discrete time moments.

  LDTSPNs have step semantics.

Let M be a marking of a LDTSPN $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$. Then $t \in Ena(M)$ fires in the next time moment with probability $\Omega_N(t)$, if no other transition is enabled in M.

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U is ready for firing in M*:

$$PF(U, M) = \prod_{t \in U} \Omega_N(t) \cdot \prod_{u \in Ena(M) \setminus U} (1 - \Omega_N(u)).$$

In the case $U = \emptyset$ we define

$$PF(\emptyset, M) = \begin{cases} \prod_{u \in Ena(M)} (1 - \Omega_N(u)) & Ena(M) \neq \emptyset; \\ 1 & \text{otherwise}. \end{cases}$$

Let $U \subseteq Ena(M)$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U fires in M*:

$$PT(U, M) = \frac{PF(U, M)}{\sum_{\{V \mid {}^\bullet V \subseteq M\}} PF(V, M)}.$$

If $U = \emptyset$ then $M = \widetilde{M}$.

Firing of U changes marking M to $\widetilde{M} = M - {}^\bullet U + U^\bullet$, $M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT(U, M)$.

We write $M \stackrel{U}{\rightarrow} \widetilde{M}$ if $\exists \mathcal{P}\ M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \rightarrow \widetilde{M}$ if $\exists U\ M \stackrel{U}{\rightarrow} \widetilde{M}$.

For $U = \{t\}$ we write $M \stackrel{t}{\rightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \stackrel{t}{\rightarrow} \widetilde{M}$.

Definition 126 *Let N be an LDTSPN.*

- *The reachability set $RS(N)$ is the minimal set of markings s.t.*
  - $M_N \in RS(N)$;
  - *if $M \in RS(N)$ and $M \rightarrow \widetilde{M}$ then $\widetilde{M} \in RS(N)$.*
- *The reachability graph $RG(N)$ is a directed labeled graph with*
  - *the set of nodes $RS(N)$;*
  - *an arc labeled by $(U, \mathcal{P})$ from node M to $\widetilde{M}$ if $M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$.*
- *The underlying Discrete Time Markov Chain (DTMC) $DTMC(N)$ is a DTMC with*
  - *the state space $RS(N)$;*
  - *a transition $M \rightarrow_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PM(M, \widetilde{M})$ is the probability to move from M to $\widetilde{M}$ by firing any set of transitions:*

$$PM(M, \widetilde{M}) = \sum_{\{U \mid M \stackrel{U}{\rightarrow} \widetilde{M}\}} PT(U, M);$$

  - *the initial state M_N.*

The *average sojourn time in the marking M* is

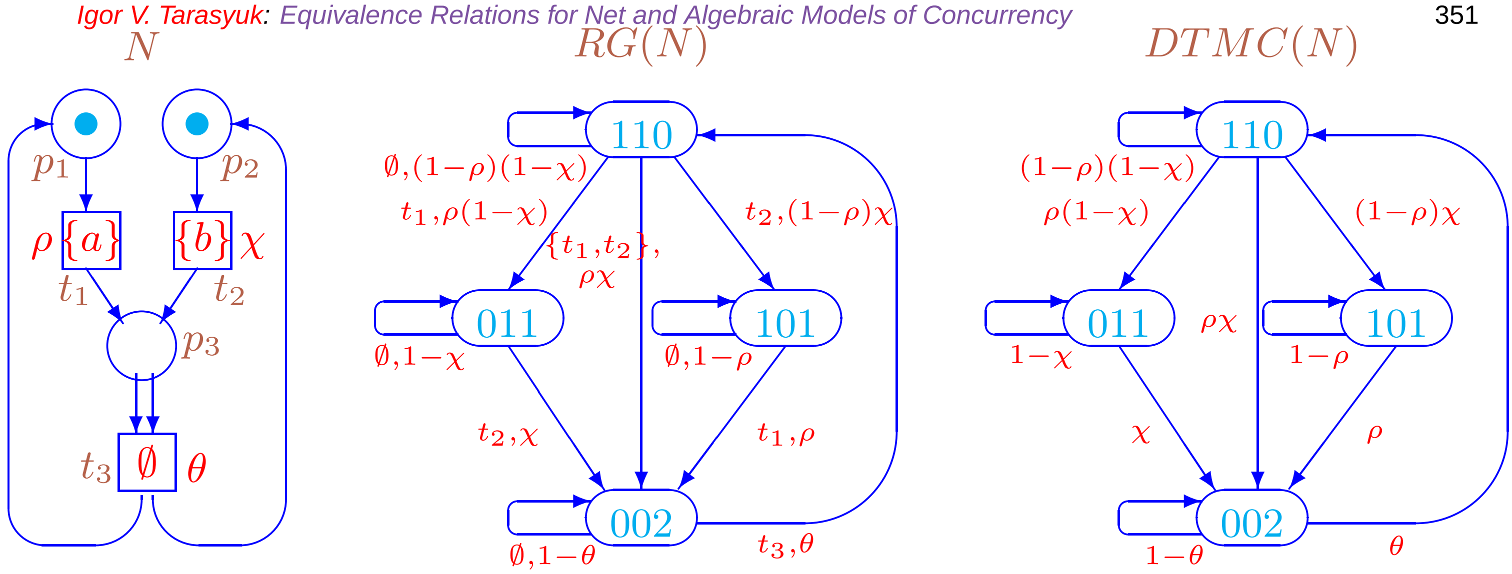
$$SJ(M) = \frac{1}{1 - PM(M, M)}.$$

The *average sojourn time vector SJ* of N has the elements $SJ(M),\ M \in RS(N)$.

The *sojourn time variance in the marking* M is

$$VAR(M) = \frac{PM(M, M)}{(1 - PM(M, M))^2}.$$

The *sojourn time variance vector* VAR of N has the elements $VAR(M),\ M \in RS(N)$.



LDTSPN, its reachability graph and the underlying DTMC

The transitions are t_1 (labeled by $\{a\}$), t_2 (labeled by $\{b\}$) and t_3 (labeled by $\emptyset$).

The transition probabilities are $\rho = \Omega_N(t_1),\ \chi = \Omega_N(t_2),\ \theta = \Omega_N(t_3)$.

$RS(N)$ consists of $M_1 = (1,1,0),\ M_2 = (0,1,1),\ M_3 = (1,0,1),$ $M_4 = (0,0,2)$.

The average sojourn time vector of N is

$$SJ = \left(\frac{1}{\rho+\chi-\rho\chi}, \frac{1}{\chi}, \frac{1}{\rho}, \frac{1}{\theta}\right).$$

The sojourn time variance vector of N:

$$VAR = \left(\frac{1-\rho-\chi+\rho\chi}{(\rho+\chi-\rho\chi)^2}, \frac{1-\chi}{\chi^2}, \frac{1-\rho}{\rho^2}, \frac{1-\theta}{\theta^2}\right).$$

The elements $\mathcal{P}_{ij}$ $(1 \le i, j \le 4)$ of (one-step) transition probability matrix (TPM) of $DTMC(N)$ are

$$\mathcal{P}_{ij} = \begin{cases} PM(s_i, s_j) & s_i \to s_j; \\ 0 & \text{otherwise.} \end{cases}$$

The (one-step) TPM of $DTMC(N)$ is

$$\mathbf{P} = \begin{pmatrix} (1-\rho)(1-\chi) & \rho(1-\chi) & \chi(1-\rho) & \rho\chi \\ 0 & 1-\chi & 0 & \chi \\ 0 & 0 & 1-\rho & \rho \\ \theta & 0 & 0 & 1-\theta \end{pmatrix}$$

The steady-state PMF ψ is a solution of

$$\begin{cases} \psi(\mathbf{P}-\mathbf{I}) = \mathbf{0} \\ \psi\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of size four and $\mathbf{0} = (0,0,0,0),\ \mathbf{1} = (1,1,1,1)$.

For $\rho = \chi = \theta$

$$\psi = \left(\frac{1}{5-3\rho}, \frac{1-\rho}{5-3\rho}, \frac{1-\rho}{5-3\rho}, \frac{2-\rho}{5-3\rho}\right).$$

The inverse of the steady-state PMF is the mean recurrence time vector

$$RC = \left(5-3\rho, \frac{5-3\rho}{1-\rho}, \frac{5-3\rho}{1-\rho}, \frac{5-3\rho}{2-\rho}\right).$$

The average time to come back to the initial marking $M_N = M_1$ in the long-term behaviour is in $(2;5)$.

Algebra of dts-boxes

Definition 127 *A discrete time stochastic Petri box (dts-box) is* $N = (P_N, T_N, W_N, \Lambda_N)$, *where*

- P_N *and* T_N *are finite sets of places and transitions, respectively, s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$;
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to I\!N$ *is a function of the weights of arcs between places and transitions and vice versa;*
- Λ_N *is the place and transition labeling function s.t.*
  - $\Lambda_N|_{P_N} : P_N \to \{\mathsf{e}, \mathsf{i}, \mathsf{x}\}$ *(it specifies entry, internal and exit places);*
  - $\Lambda_N|_{T_N} : T_N \to \{\varrho \mid \varrho \subseteq I\!N_{fin}^{\mathcal{SL}} \times \mathcal{SL}\}$ *(it associates transitions with the relabeling relations on activities).*

Moreover, $\forall t \in T_N \ {}^\bullet t \neq \emptyset \neq t^\bullet$.

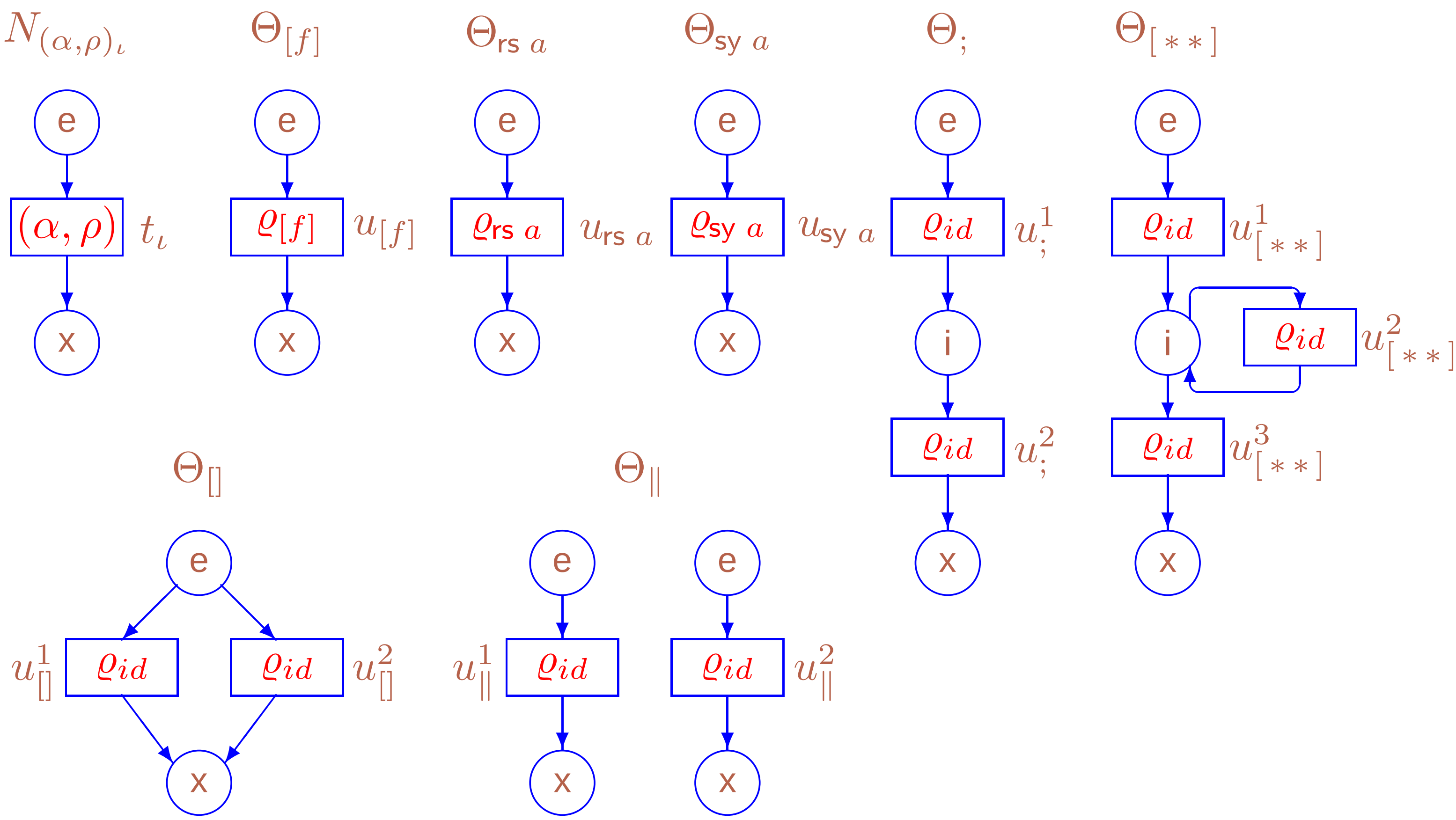
For the set of entry places of N, ${}^\circ N = \{p \in P_N \mid \Lambda_N(p) = \mathsf{e}\}$, *and the set of exit places of* N, $N^\circ = \{p \in P_N \mid \Lambda_N(p) = \mathsf{x}\}$, *it holds:*

${}^\circ N \neq \emptyset \neq N^\circ$ *and* ${}^\bullet({}^\circ N) = \emptyset = (N^\circ)^\bullet$.

A dts-box is *plain* if $\forall t \in T_N \ \Lambda_N(t) = \varrho_{(\alpha,\rho)}$, where $\varrho_{(\alpha,\rho)} = \{(\emptyset, (\alpha, \rho))\}$ is the constant relabeling, identified with (α, ρ).

A *marked plain dts-box* is a pair (N, M_N), where N is a plain dts-box and $M_N \in I\!N_{fin}^{P_N}$ is its marking.

Let $\overline{N} = (N, {}^\circ N)$ and $\underline{N} = (N, N^\circ)$.



The plain and operator dts-boxes

Definition 128 *Let* $(\alpha, \rho) \in \mathcal{SL},\ a \in Act$ *and* $E, F, K \in RegStatExpr$*. The* denotational semantics *of* $dtsPBC$ *is a mapping* Box_{dts} *from* $RegStatExpr$ *into plain dts-boxes:*

1. $Box_{dts}((\alpha, \rho)_\iota) = N_{(\alpha,\rho)_\iota}$;
2. $Box_{dts}(E \circ F) = \Theta_\circ(Box_{dts}(E), Box_{dts}(F)),\ \circ \in \{;, [], \|\}$;
3. $Box_{dts}(E[f]) = \Theta_{[f]}(Box_{dts}(E))$;
4. $Box_{dts}(E \circ a) = \Theta_{\circ a}(Box_{dts}(E)),\ \circ \in \{\text{rs}, \text{sy}\}$;
5. $Box_{dts}([E * F * K]) = \Theta_{[\,*\,*\,]}(Box_{dts}(E), Box_{dts}(F), Box_{dts}(K))$.

For $E \in RegStatExpr$, let $Box_{dts}(\overline{E}) = \overline{Box_{dts}(E)}$ and $Box_{dts}(\underline{E}) = \underline{Box_{dts}(E)}$.

We denote isomorphism of transition systems by $\simeq$,

and the same symbol denotes isomorphism of reachability graphs and DTMCs

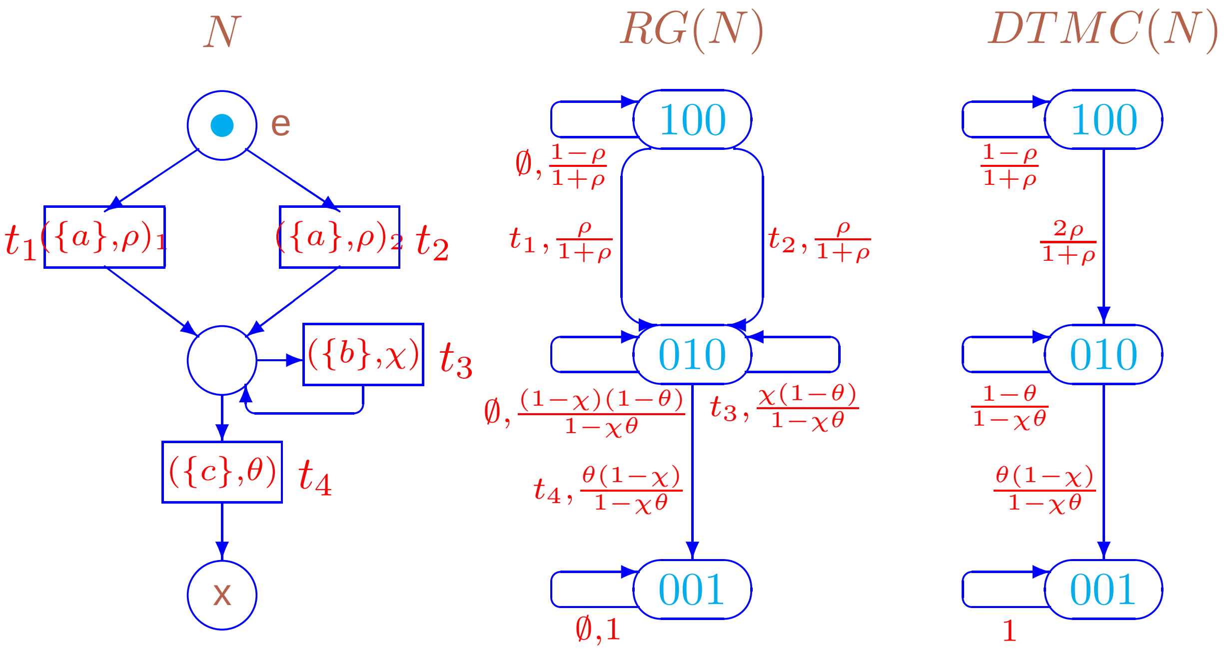
as well as isomorphism between transition systems and reachability graphs.

Theorem 34 *For any static expression* E

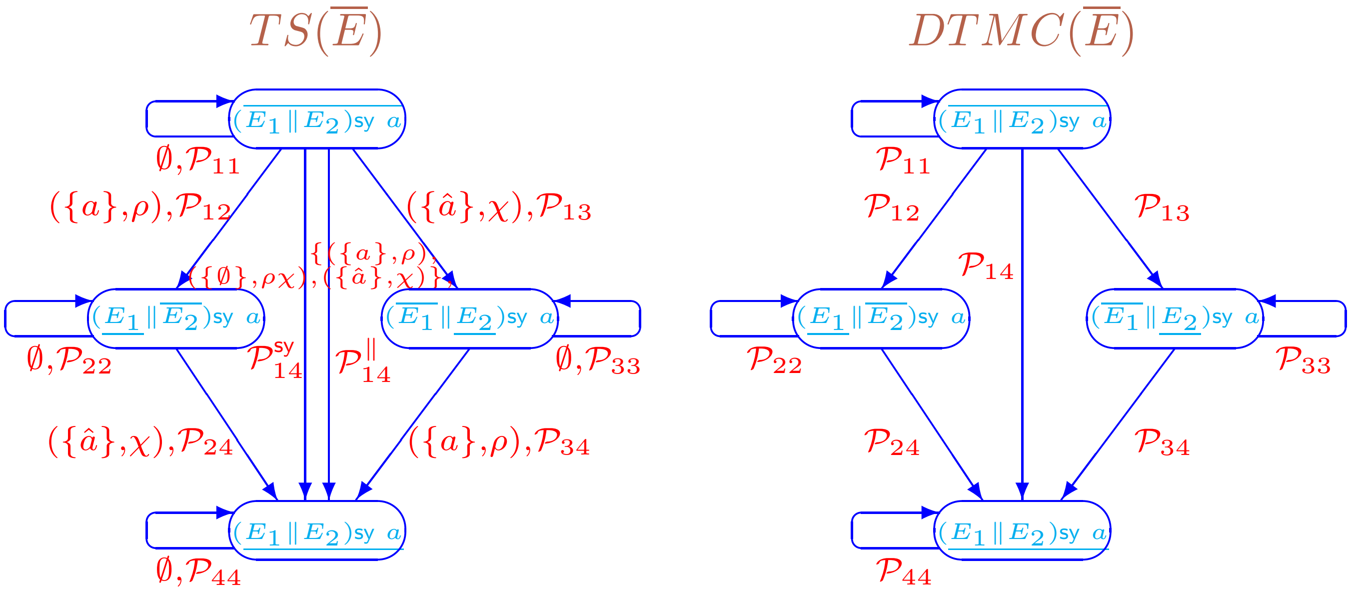
$$TS(\overline{E})\simeq RG(Box_{dts}(\overline{E})).$$

Proposition 25 *For any static expression* E

$$DTMC(\overline{E})\simeq DTMC(Box_{dts}(\overline{E})).$$

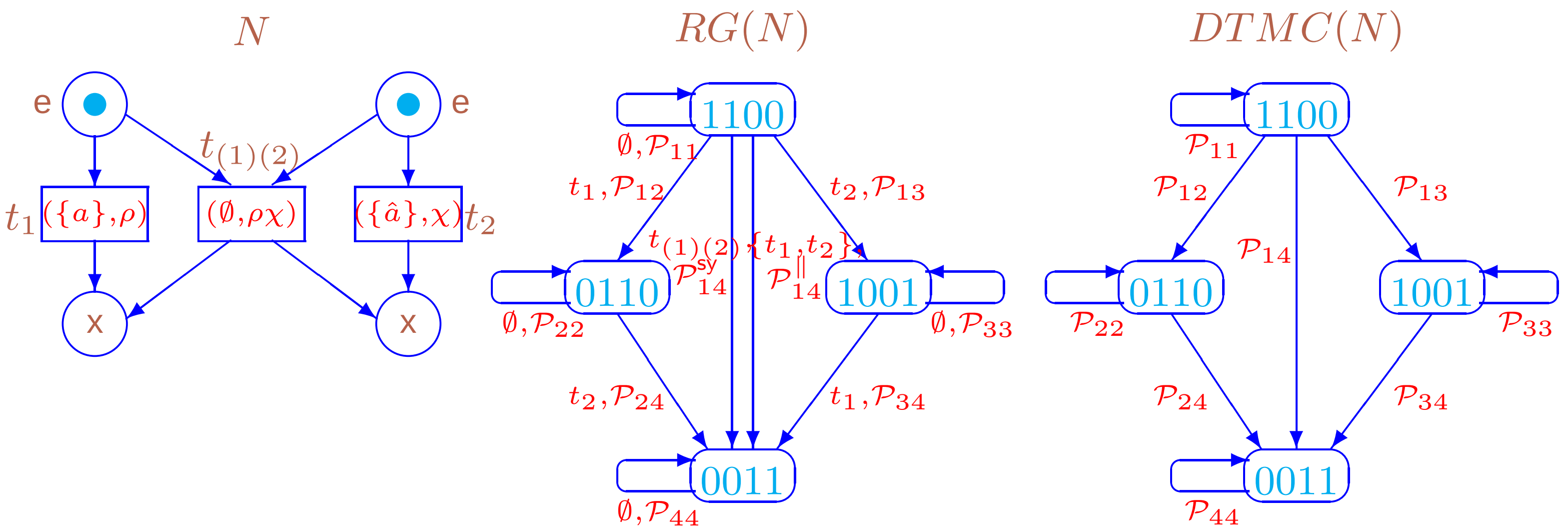


BOXIT:The marked dts-box $N = Box_{dts}(\overline{E})$ for $E = [((\{a\},\rho)_1[](\{a\},\rho)_2) * (\{b\},\chi) * (\{c\},\theta)]$, its reachability graph and the underlying DTMC



EXPR:The transition system and the underlying DTMC of $\overline{E}$ for

$$E = ((\{a\}, \rho)\|(\{\hat{a}\}, \chi)) \text{ sy } a$$



BOX:The marked dts-box $N = Box_{dts}(\overline{E})$ for

$E = ((\{a\}, \rho)\|(\{\hat{a}\}, \chi))$ sy a, its reachability graph and the underlying DTMC

The normalization factor $\mathcal{N} = \frac{1}{1-\rho^2\chi-\rho\chi^2+\rho^2\chi^2}$.

$\mathcal{P}_{11} = \mathcal{N}(1-\rho)(1-\chi)(1-\rho\chi)$ $\mathcal{P}_{12} = \mathcal{N}\rho(1-\chi)(1-\rho\chi)$

$\mathcal{P}_{13} = \mathcal{N}\chi(1-\rho)(1-\rho\chi)$ $\mathcal{P}_{14}^{\mathsf{sy}} = \mathcal{N}\rho\chi(1-\rho)(1-\chi)$

$\mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(1-\rho\chi)$ $\mathcal{P}_{22} = 1-\chi$

$\mathcal{P}_{24} = \chi$ $\mathcal{P}_{33} = 1-\rho$

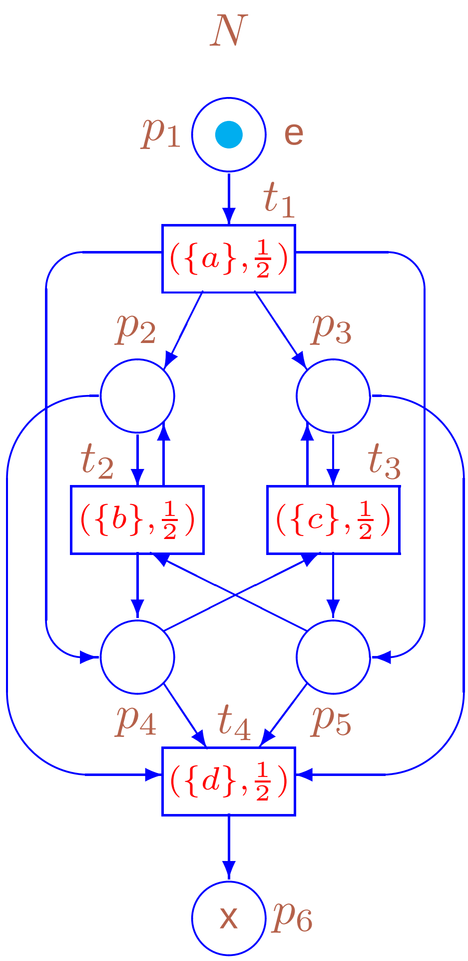
$\mathcal{P}_{34} = \rho$ $\mathcal{P}_{44} = 1$

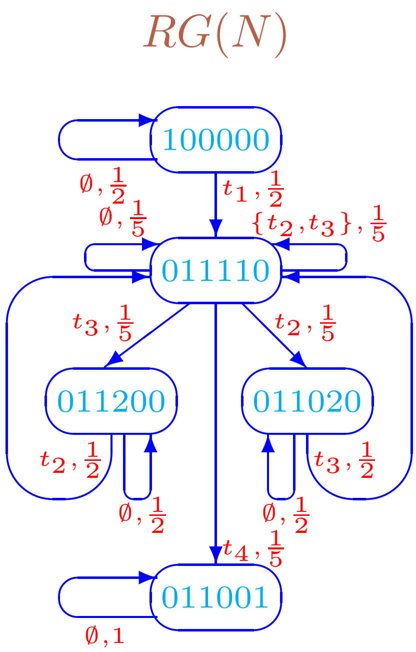
$\mathcal{P}_{14} = \mathcal{P}_{14}^{\mathsf{sy}} + \mathcal{P}_{14}^{\|} = \mathcal{N}\rho\chi(2-\rho-\chi)$

The case $\rho = \chi = \frac{1}{2}$:

$$\mathcal{P}_{11} = \mathcal{P}_{12} = \mathcal{P}_{13} = \mathcal{P}_{14}^{\|} = \frac{3}{13},\ \mathcal{P}_{14}^{\mathsf{sy}} = \frac{1}{13},$$

$$\mathcal{P}_{22} = \mathcal{P}_{24} = \mathcal{P}_{33} = \mathcal{P}_{34} = \frac{1}{2},\ \mathcal{P}_{44} = 1,\ \mathcal{P}_{14} = \frac{4}{13}.$$





The marked dts-box $N = Box_{dts}(\overline{E})$ for
$E = [((\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}) \| (\{c\},\frac{1}{2})) * (\{d\},\frac{1}{2})]$ and its reachability graph

$M_1 = (1, 0, 0, 0, 0, 0)$ is the initial marking.

$M_2 = (0, 1, 1, 1, 1, 0)$ is obtained from M_1 by firing t_1.

$M_3 = (0, 1, 1, 2, 0, 0)$ is obtained from M_2 by firing t_2 and has 2 tokens in the place p_4.

$M_4 = (0, 1, 1, 0, 2, 0)$ is obtained from M_2 by firing t_3 and has 2 tokens in the place p_5.

Concurrency in the second argument of iteration in $\overline{E}$ can lead to non-safeness of the corresponding marked dts-box N, but it is 2-bounded in the worst case.

The origin of the problem: N has as a self-loop with two subnets which can function independently.

Stochastic equivalences

Empty loops in transition systems

Let G be a dynamic expression and $s \in DR(G)$.

The *probability to stay in s due to k $(k \geq 1)$ empty loops* is $(PT(\emptyset, s))^k$.

Let $\Gamma \in Exec(s) \setminus \{\emptyset\}$. The *probability to execute the non-empty multiset of activities Γ in s after possible empty loops*:

$$PT^*(\Gamma, s) = PT(\Gamma, s) \sum_{k=0}^{\infty} (PT(\emptyset, s))^k = \frac{PT(\Gamma, s)}{1 - PT(\emptyset, s)} = EL(s) PT(\Gamma, s),$$

where $EL(s) = \frac{1}{1 - PT(\emptyset, s)}$ is the *empty loops abstraction factor*.

The *empty loops abstraction vector* EL of G has the elements $EL(s),\ s \in DR(G)$.

Definition 129 *The* (labeled probabilistic) transition system without empty loops $TS^*(G)$ *has the state space* $DR(G)$ *and the transitions* $s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$*, if* $s \stackrel{\Gamma}{\rightarrow} \tilde{s},\ \Gamma \neq \emptyset$ *and* $\mathcal{P} = PT^*(\Gamma, s)$*.*

We write $s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$ if $\exists \mathcal{P}\ s \stackrel{\Gamma}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \twoheadrightarrow \tilde{s}$ if $\exists \Gamma\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}$.

For $\Gamma = \{(\alpha, \rho)\}$ we write $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$ and $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}$.

For $E \in RegStatExpr$, let $TS^*(E) = TS^*(\overline{E})$.

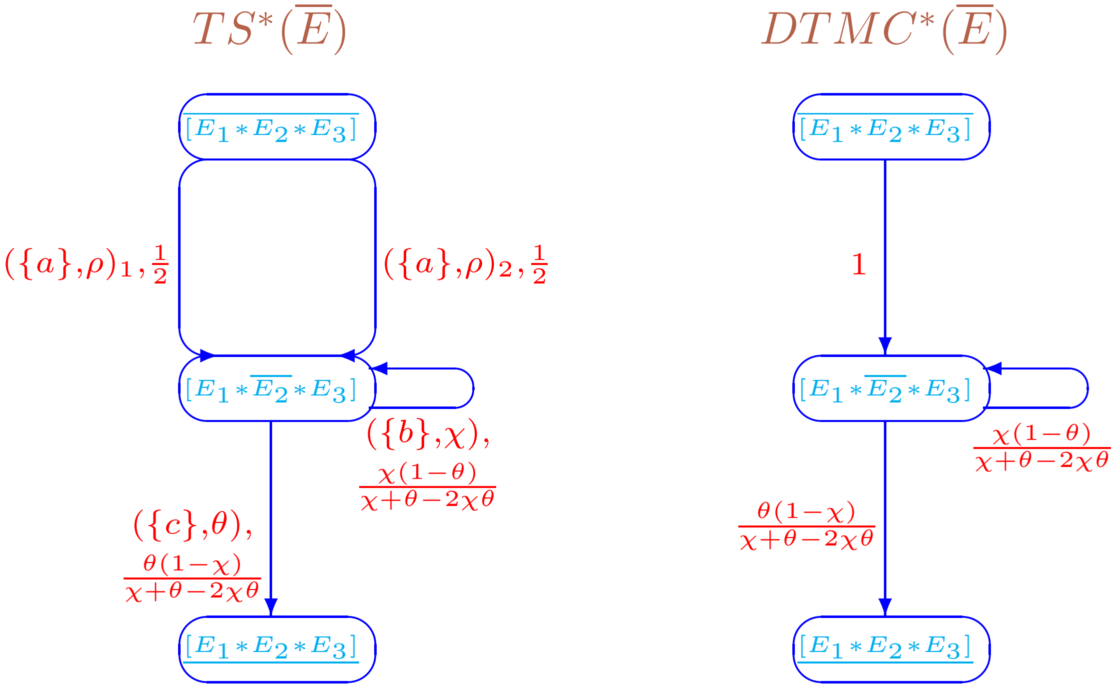
Definition 130 G *and* G' *are* equivalent w.r.t. transition systems without empty loops, $G =_{ts*} G'$*, if* $TS^*(G) \simeq TS^*(G')$*.*

Definition 131 *The* underlying DTMC without empty loops $DTMC^*(G)$ *has the state space* $DR(G)$ *and transitions* $s \twoheadrightarrow_{\mathcal{P}} \tilde{s}$, *if* $s \twoheadrightarrow \tilde{s}$, *where* $\mathcal{P} = PM^*(s, \tilde{s})$ *is the* probability to move from s to $\tilde{s}$ by executing any non-empty multiset of activities after possible empty loops*:*

$$PM^*(s, \tilde{s}) = \sum_{\{\Gamma | s \xrightarrow{\Gamma} \tilde{s}\}} PT^*(\Gamma, s) = \begin{cases} EL(s)(PM(s,s) - PT(\emptyset, s)), & s = \tilde{s}; \\ EL(s)PM(s, \tilde{s}), & \text{otherwise}, \end{cases}$$

where $PM(s,s) - PT(\emptyset, s)$ *is the* probability to stay in s due to any non-empty loop, *i.e. by executing any non-empty multiset of activities.*

For $E \in RegStatExpr$, let $DTMC^*(E) = DTMC^*(\overline{E})$.



The transition system and the underlying DTMC without empty loops of $\overline{E}$ in Figure EXPRIT

Empty loops in reachability graphs

Let N be an LDTSPN and $M \in RS(N)$.

The *probability to stay in M due to k $(k \geq 1)$ empty loops* is $(PT(\emptyset, M))^k$.

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^\bullet U \subseteq M$. The *probability that the non-empty set of transitions U fires in M after possible empty loops*:

$$PT^*(U, M) = PT(U, M) \sum_{k=0}^{\infty} (PT(\emptyset, M))^k = \frac{PT(U,M)}{1-PT(\emptyset,M)} = EL(M)PT(U, M),$$

where $EL(M) = \frac{1}{1-PT(\emptyset,M)}$ is the *empty loops abstraction factor*.

The *empty loops abstraction vector* of N, EL, has the elements $EL(M),\ M \in RS(N)$.

Definition 132 *The* reachability graph without empty loops $RG^*(N)$ *with the set of nodes* $RS(N)$ *and the set of arcs corresponding to the transitions* $M \stackrel{U}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$, *if* $M \stackrel{U}{\rightarrow} \widetilde{M}$, $U \neq \emptyset$ *and* $\mathcal{P} = PT^*(U, M)$.

We write $M \stackrel{U}{\twoheadrightarrow} \widetilde{M}$ if $\exists \mathcal{P}\ M \stackrel{U}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \twoheadrightarrow \widetilde{M}$ if $\exists U\ M \stackrel{U}{\twoheadrightarrow} \widetilde{M}$.

For $U = \{t\}$ we write $M \stackrel{t}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{M}$ and $M \stackrel{t}{\twoheadrightarrow} \widetilde{M}$.

Definition 133 *The* underlying DTMC without empty loops $DTMC^*(N)$ *has the state space* $RS(N)$ *and transitions* $M \twoheadrightarrow_{\mathcal{P}} \widetilde{M}$, *if* $M \twoheadrightarrow \widetilde{M}$, *where* $\mathcal{P} = PM^*(M, \widetilde{M})$ *is the* probability to move from M to $\widetilde{M}$ by firing any non-empty set of transitions after possible empty loops*:*

$$PM^*(M,\widetilde{M}) = \sum_{\{U \in Ena(M) \mid M \stackrel{U}{\twoheadrightarrow} \widetilde{M}\}} PT^*(U, M) = \begin{cases} EL(M)(PM(M,M) - PT(\emptyset, M)), & M = \widetilde{M}; \\ EL(M)PM(M,\widetilde{M}), & \text{otherwise}, \end{cases}$$

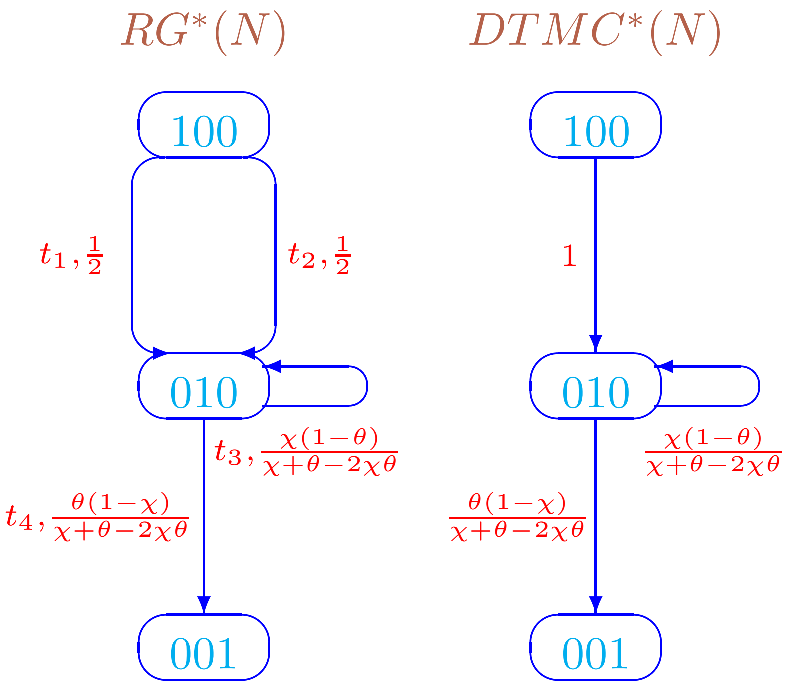
where $PM(M,M) - PT(\emptyset, M)$ *is the* probability to stay in M due to any non-empty loop, *i.e. by firing any non-empty multiset of transitions.*

Theorem 35 *For any static expression* E

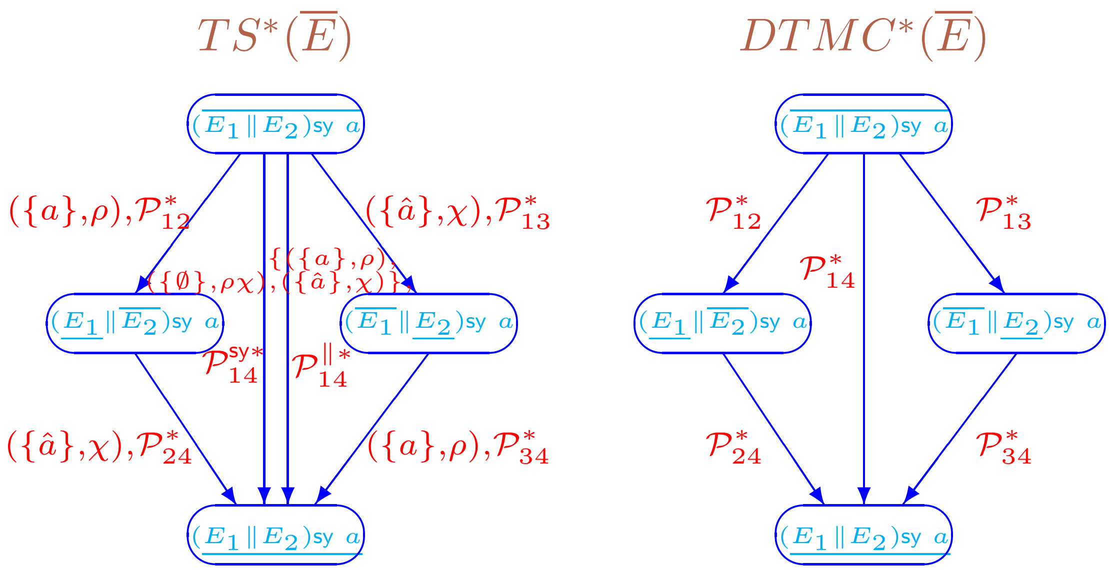
$$TS^*(\overline{E}) \simeq RG^*(Box_{dts}(\overline{E})).$$

Proposition 26 *For any static expression* E

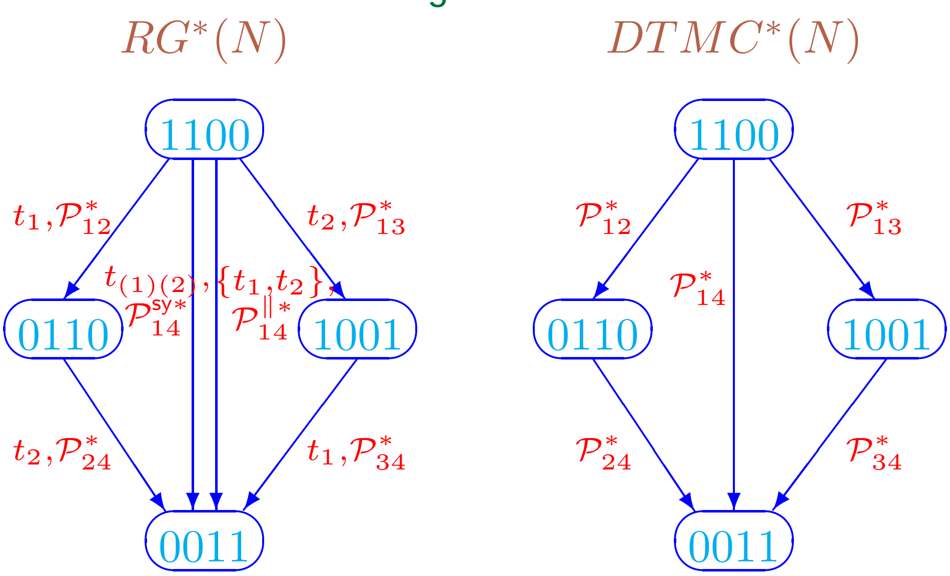
$$DTMC^*(\overline{E}) \simeq DTMC^*(Box_{dts}(\overline{E})).$$



The reachability graph and the underlying DTMC without empty loops of N in Figure BOXIT



The transition system and the underlying DTMC without empty loops of $\overline{E}$ in Figure EXPR



The reachability graph and the underlying DTMC without empty loops of N in Figure BOX

The normalization factor $\mathcal{N}^* = \frac{1}{\rho+\chi-2\rho^2\chi-2\rho\chi^2+2\rho^2\chi^2}$.

$$\mathcal{P}^*_{12} = \frac{\mathcal{P}_{12}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho(1-\chi)(1-\rho\chi)$$
$$\mathcal{P}^*_{13} = \frac{\mathcal{P}_{13}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\chi(1-\rho)(1-\rho\chi)$$
$$\mathcal{P}^{\mathsf{sy}*}_{14} = \frac{\mathcal{P}^{\mathsf{sy}}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho)(1-\chi)$$
$$\mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(1-\rho\chi)$$
$$\mathcal{P}^*_{24} = \frac{\mathcal{P}_{24}}{1-\mathcal{P}_{22}} = 1$$
$$\mathcal{P}^*_{34} = \frac{\mathcal{P}_{34}}{1-\mathcal{P}_{33}} = 1$$
$$\mathcal{P}^*_{14} = \mathcal{P}^{\mathsf{sy}*}_{14} + \mathcal{P}^{\|*}_{14} = \frac{\mathcal{P}^{\mathsf{sy}}_{14}+\mathcal{P}^{\|}_{14}}{1-\mathcal{P}_{11}} = \mathcal{N}^*\rho\chi(2-\rho-\chi)$$

The case $\rho = \chi = \frac{1}{2}$:

$$\mathcal{P}^*_{12} = \mathcal{P}^*_{13} = \mathcal{P}^{\|*}_{14} = \frac{3}{10},\ \mathcal{P}^{\mathsf{sy}*}_{14} = \frac{1}{10},\ \mathcal{P}^*_{24} = \mathcal{P}^*_{34} = 1,\ \mathcal{P}^*_{14} = \frac{2}{5}.$$

Stochastic trace equivalences

Let G be a dynamic expression, $s, \tilde{s} \in DR(G)$ and $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}$. We write $s \stackrel{(\alpha,\rho)}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = pt^*((\alpha, \rho), s)$ is the *probability to execute the activity* (α, ρ) *in* s *after possible empty loops when only one-element steps are allowed*:

$$pt^*((\alpha, \rho), s) = \frac{PT^*(\{(\alpha, \rho)\}, s)}{\sum_{\{(\beta,\chi)\}\in Exec(s)} PT^*(\{(\beta, \chi)\}, s)}.$$

For $\Gamma \in I\!N_{fin}^{\mathcal{SL}}$, we consider $\mathcal{L}(\Gamma) \in I\!N_{fin}^{\mathcal{L}}$, i.e. (possibly empty) multisets of multiactions.

Definition 134 *An* interleaving stochastic trace *of a dynamic expression* G *is a pair* $(\sigma, PT^*(\sigma))$*, where* $\sigma = \alpha_1 \cdots \alpha_n \in \mathcal{L}^*$ *and*

$$PT^*(\sigma) = \sum_{\{(\alpha_1,\rho_1),\ldots,(\alpha_n,\rho_n) \mid [G]_\approx = s_0 \stackrel{(\alpha_1,\rho_1)}{\twoheadrightarrow} s_1 \stackrel{(\alpha_2,\rho_2)}{\twoheadrightarrow} \cdots \stackrel{(\alpha_n,\rho_n)}{\twoheadrightarrow} s_n\}} \prod_{i=1}^{n} pt^*((\alpha_i, \rho_i), s_{i-1}).$$

We denote a set of all interleaving stochastic traces *of a dynamic expression* G *by* $IntStochTraces(G)$*.* G *and* G' *are* interleaving stochastic trace equivalent*,* $G \equiv_{is} G'$*, if*

$$IntStochTraces(G) = IntStochTraces(G').$$

Let $E = ((\{a\}, \frac{1}{2}) \| (\{\hat{a}\}, \frac{1}{2}))$ sy a.

$IntStochTraces(\overline{E}) = \{(\emptyset, \frac{1}{7}),\ (\{a\}, \frac{3}{7}),\ (\{\hat{a}\}, \frac{3}{7}),\ (\{a\}\{\hat{a}\}, \frac{3}{7}),$
$(\{\hat{a}\}\{a\}, \frac{3}{7})\}$.

Definition 135 *A step stochastic trace of a dynamic expression G is a pair $(\Sigma, PT^*(\Sigma))$, where $\Sigma = A_1 \cdots A_n \in (I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\})^*$ and*

$$PT^*(\Sigma) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid [G]_\approx = s_0 \xrightarrow{\Gamma_1} s_1 \xrightarrow{\Gamma_2} \cdots \xrightarrow{\Gamma_n} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\le i\le n)\}} \prod_{i=1}^{n} PT^*(\Gamma_i, s_{i-1}).$$

We denote a set of all step stochastic traces of a dynamic expression G by $StepStochTraces(G)$. G and G' are step stochastic trace equivalent, $G \equiv_{ss} G'$, if

$$StepStochTraces(G) = StepStochTraces(G').$$

Let $E = ((\{a\}, \frac{1}{2}) \| (\{\hat{a}\}, \frac{1}{2}))$ sy a.

$StepStochTraces(\overline{E}) = \{(\{\emptyset\}, \frac{1}{10}),\ (\{\{a\}\}, \frac{3}{10}),\ (\{\{\hat{a}\}\}, \frac{3}{10}),$
$(\{\{a\}\}\{\{\hat{a}\}\}, \frac{3}{10}),\ (\{\{\hat{a}\}\}\{\{a\}\}, \frac{3}{10}),\ (\{\{\hat{a}\}, \{a\}\}, \frac{3}{10})\}$.

Stochastic bisimulation equivalences

Let G be a dynamic expression and $\mathcal{H} \subseteq DR(G)$. For $s \in DR(G)$ and $A \in \mathbb{N}_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$ we write $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM_A^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via non-empty steps with the multiaction part A after possible empty loops*:

$$PM_A^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s},\ \mathcal{L}(\Gamma) = A\}} PT^*(\Gamma, s).$$

We write $s \stackrel{A}{\twoheadrightarrow} \mathcal{H}$ if $\exists \mathcal{P}\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \mathcal{H}$.

We write $s \twoheadrightarrow_{\mathcal{P}} \mathcal{H}$ if $\exists A\ s \stackrel{A}{\twoheadrightarrow} \mathcal{H}$, where $\mathcal{P} = PM^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via any non-empty steps after possible empty loops*:

$$PM^*(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}} PT^*(\Gamma, s).$$

We write $s \stackrel{\alpha}{\rightsquigarrow}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = pm_\alpha^*(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via steps with the multiaction part $\{\alpha\}$ after possible empty loops when only one-element steps are allowed*:

$$pm_\alpha^*(s, \mathcal{H}) = \sum_{\{(\alpha,\rho) \mid \exists \tilde{s} \in \mathcal{H}\ s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}\}} pt^*((\alpha, \rho), s).$$

We write $s \stackrel{\alpha}{\rightsquigarrow} \mathcal{H}$ if $\exists \mathcal{P}\ s \stackrel{\alpha}{\rightsquigarrow}_{\mathcal{P}} \mathcal{H}$.

Definition 136 *Let G and G' be dynamic expressions. An equivalence relation $\mathcal{R} \subseteq (DR(G) \cup DR(G'))^2$ is a $\star$-stochastic bisimulation between G and G', $\star \in${interleaving, step}, $\mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$, if:*

1. $([G]_\approx, [G']_\approx) \in \mathcal{R}$.
2. $(s_1, s_2) \in \mathcal{R} \Rightarrow \forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$
   - $\forall x \in \mathcal{L}$ *and* $\hookrightarrow = \rightharpoonup$, *if* $\star = i$;
   - $\forall x \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$ *and* $\hookrightarrow = \twoheadrightarrow$, *if* $\star = s$;

$$s_1 \overset{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{H} \Leftrightarrow s_2 \overset{x}{\hookrightarrow}_{\mathcal{P}} \mathcal{H}.$$

Two dynamic expressions G and G' are $\star$-stochastic bisimulation equivalent, $\star \in${interleaving, step}, $G \underline{\leftrightarrow}_{\star s} G'$, if $\exists \mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$.

$\mathcal{R}_{\star s}(G, G') = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{\star s} G'\}$, $\star \in \{i, s\}$, is the *union of all $\star$-stochastic bisimulations* between G and G', $\star \in${*interleaving, step*}.

Proposition 27 *Let G and G' be dynamic expressions and $G \underline{\leftrightarrow}_{\star s} G'$, $\star \in \{i, s\}$. Then $\mathcal{R}_{\star s}(G, G')$ is the largest $\star$-stochastic bisimulation between G and G', $\star \in${interleaving, step}.*

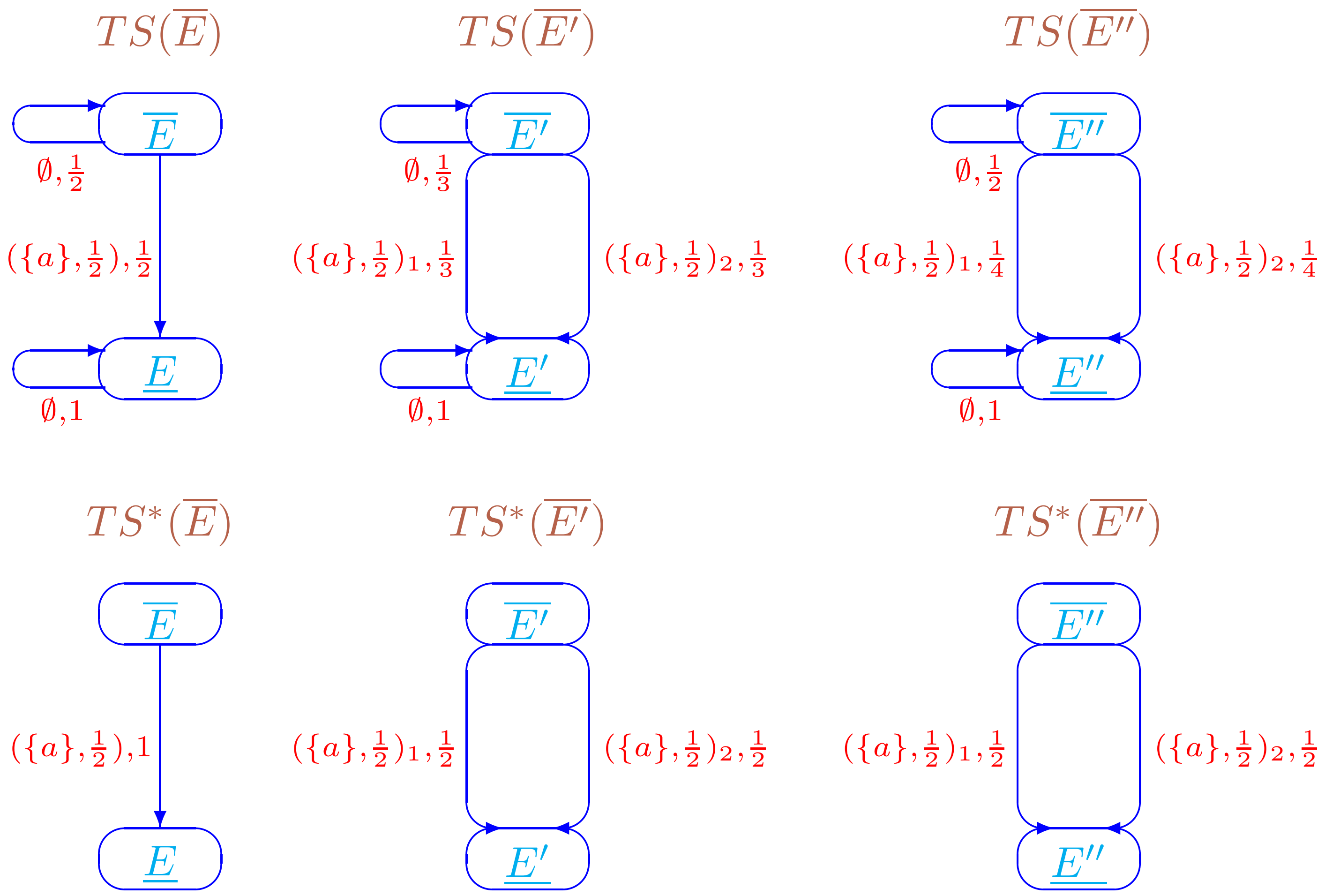
Stochastic isomorphism

Let G be a dynamic expression, $s, \tilde{s} \in DR(G)$ and $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \{\tilde{s}\}$. We write $s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s}$.

Definition 137 *Let G, G' be dynamic expressions. A mapping $\beta : DR(G) \to DR(G')$ is a* stochastic isomorphism *between G and G', $\beta : G =_{sto} G'$, if*

1. *β is a bijection s.t. $\beta([G]_{\approx}) = [G']_{\approx}$;*
2. *$\forall s, \tilde{s} \in DR(G)\ \forall A \in \mathbb{N}^{\mathcal{L}}_{fin} \setminus \{\emptyset\}\ s \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \beta(\tilde{s})$.*

G and G' are stochastically isomorphic*, $G =_{sto} G'$, if $\exists \beta : G =_{sto} G'$.*



Properties of the stochastic isomorphism based on transition systems with empty loops

$E = (\{a\}, \frac{1}{2}),\ E' = (\{a\}, \frac{1}{2})_1[](\{a\}, \frac{1}{2})_2,\ E'' = (\{a\}, \frac{1}{3})_1[](\{a\}, \frac{1}{3})_2$.

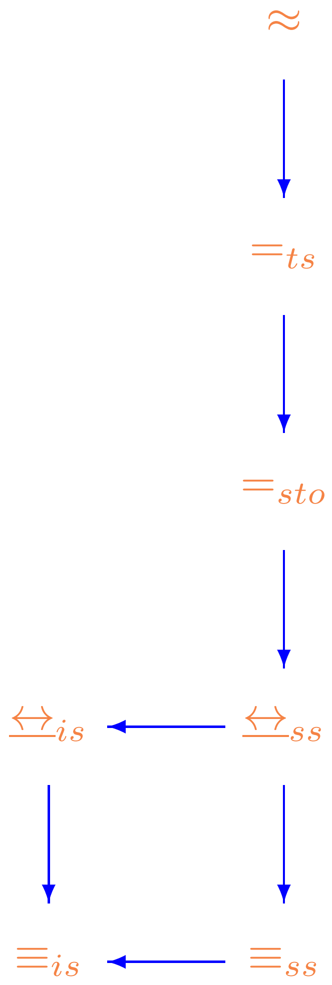
The (one-element) multisets of activities which label the transitions of $TS^*(\overline{E}),\ TS^*(\overline{E'}),\ TS^*(\overline{E''})$, and non-empty ones of $TS(\overline{E}),\ TS(\overline{E'}),\ TS(\overline{E''})$, have the same multiaction part $\{\{a\}\}$.

- $\overline{E} =_{sto} \overline{E'} =_{sto} \overline{E''}$, since the probability of the only one non-empty transition in $TS^*(\overline{E})$ is 1, the probability of both non-empty transitions in $TS^*(\overline{E'})$ and $TS^*(\overline{E''})$ is $\frac{1}{2}$, and $1 = \frac{1}{2} + \frac{1}{2}$.
- $\overline{E}$ is not equivalent to $\overline{E'}$ w.r.t. the stronger version of stochastic isomorphism, since the probability of the only one non-empty transition in $TS(\overline{E})$ is $\frac{1}{2}$, whereas the probability of both non-empty transitions in $TS(\overline{E'})$ is $\frac{1}{3}$, and $\frac{1}{2} \neq \frac{2}{3} = \frac{1}{3} + \frac{1}{3}$.
- $\overline{E'}$ is not equivalent to $\overline{E''}$ w.r.t. the stronger version of stochastic isomorphism, since the probability of both non-empty transitions in $TS(\overline{E'})$ is $\frac{1}{3}$, whereas the probability of both non-empty transitions in $TS(\overline{E''})$ is $\frac{1}{4}$, and $\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq \frac{1}{2} = \frac{1}{4} + \frac{1}{4}$.
- $\overline{E}$ is equivalent to $\overline{E''}$ w.r.t. the stronger version of stochastic isomorphism, since the probability of the only one non-empty transition in $TS(\overline{E})$ is $\frac{1}{2}$, the probability of both non-empty transitions in $TS(\overline{E''})$ is $\frac{1}{4}$, and $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$.

Interrelations of the stochastic equivalences

Proposition 28 *Let* $\star \in \{i, s\}$*. For dynamic expressions* G *and* G'*:*

1. $G \underline{\leftrightarrow}_{\star s} G' \;\Rightarrow\; G \equiv_{\star s} G'$;
2. $G =_{ts*} G' \;\Leftrightarrow\; G =_{ts} G'$.



Interrelations of the stochastic equivalences

Theorem 36 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, is, ss, sto, ts\}$*. For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \twoheadleftarrow\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\twoheadrightarrow_{\star\star}$.

Validity of the implications

- The implications $\leftrightarrow_{ss} \to \leftrightarrow_{is}$, $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$ are valid, since single activities are one-element multisets.
- The implications $\underline{\leftrightarrow}_{\star s} \to \equiv_{\star s}$, $\star \in \{i, s\}$, are valid by the proposition above.
- The implication $=_{sto} \to \underline{\leftrightarrow}_{ss}$ is proved as follows. Let $\beta : G =_{sto} G'$. Then $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, where $\mathcal{R} = \{(s, \beta(s)) \mid s \in DR(G)\}$.
- The implication $=_{ts} \to =_{sto}$ is valid, since stochastic isomorphism is that of transition systems without empty loops up to merging of transitions with labels having identical multiaction parts.
- The implication $\approx \to =_{ts}$ is valid, since the transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

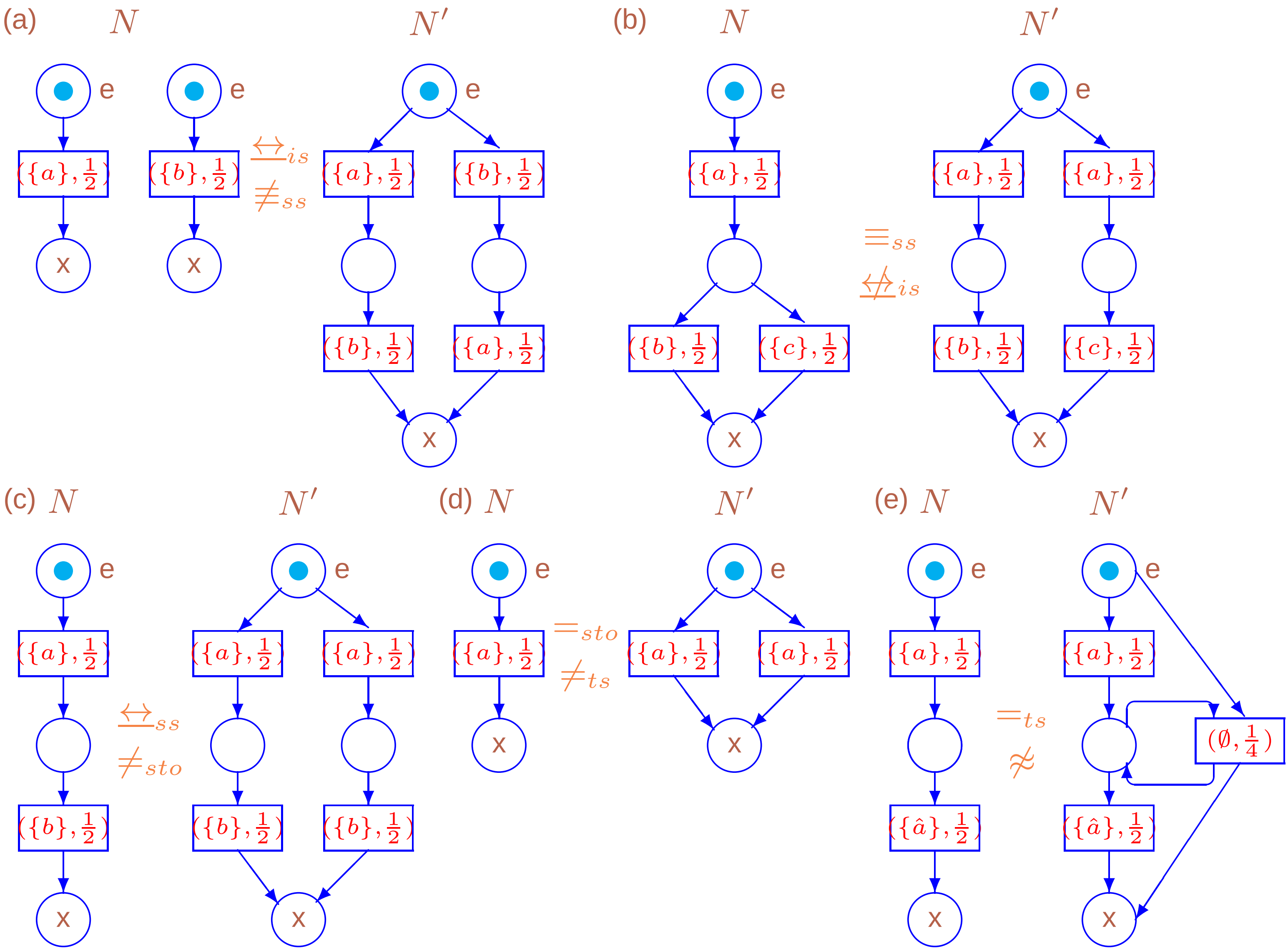
(a) Let $E = (\{a\}, \frac{1}{2})\|(\{b\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{b\}, \frac{1}{2}); (\{a\}, \frac{1}{2}))$. Then $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$, but $\overline{E} \not\equiv_{ss} \overline{E'}$, since only in $TS^*(\overline{E'})$ multiactions $\{a\}$ and $\{b\}$ cannot be executed concurrently.

(b) Let $E = (\{a\}, \frac{1}{2}); ((\{b\}, \frac{1}{2})[](\{c\}, \frac{1}{2}))$ and $E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{a\}, \frac{1}{2}); (\{c\}, \frac{1}{2}))$. Then $\overline{E} \equiv_{ss} \overline{E'}$, but $\overline{E} \not\underline{\leftrightarrow}_{is} \overline{E'}$, since only in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.

(c) Let $E = (\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})[](\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2})$. Then $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$, but $\overline{E} \neq_{sto} \overline{E'}$, since $TS^*(\overline{E'})$ has more states than $TS^*(\overline{E})$.

(d) Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2})_1[](\{a\}, \frac{1}{2})_2$. Then $\overline{E} =_{sto} \overline{E'}$, but $\overline{E} \neq_{ts} \overline{E'}$, since only $TS(\overline{E'})$ has two transitions.

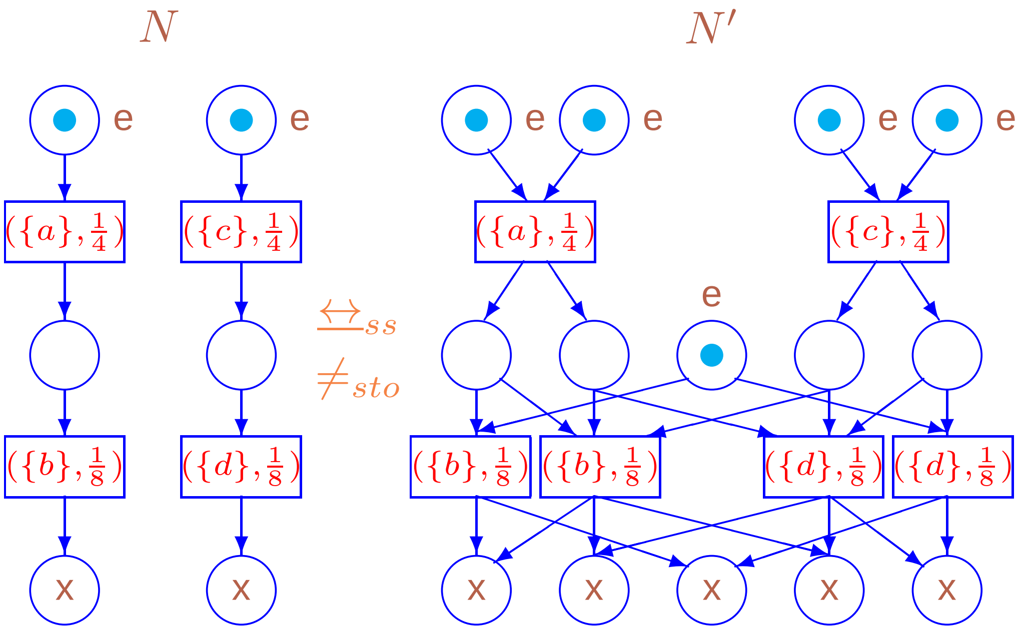
(e) Let $E = (\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E} =_{ts} \overline{E'}$, but $\overline{E} \not\approx \overline{E'}$, since $\overline{E}$ and $\overline{E'}$ cannot be reached each from other by applying inaction rules.



Dts-boxes of the dynamic expressions from equivalence examples of the theorem above

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$ for each picture (a)–(e).

Reduction modulo equivalences



Reduction of a dts-box up to $\underline{\leftrightarrow}_{ss}$

Let $E = ((\{a\},\frac{1}{2});(\{b\},\frac{1}{2}))\|((\{c\},\frac{1}{2});(\{d\},\frac{1}{2}))$ and $E' = ((((\{a,x\},\frac{1}{2});((\{b,y_1\},\frac{1}{2})[](\{b,y_2\},\frac{1}{2})))\|((\{a,\hat{x}\},\frac{1}{2});((\{b,\widehat{y_2},y_2'\},\frac{1}{2})[](\{d,v_1\},\frac{1}{2})))\|((\{c,z\},\frac{1}{2});((\{b,\widehat{y_2'}\},\frac{1}{2})[](\{d,\widehat{v_1},v_1'\},\frac{1}{2})))\|((\{c,\hat{z}\},\frac{1}{2});((\{d,\widehat{v_1'}\},\frac{1}{2})[](\{d,v_2\},\frac{1}{2})))\|((\{b,\widehat{y_1}\},\frac{1}{4})[](\{d,\widehat{v_2}\},\frac{1}{4})))$ sy x sy y_1 sy y_2 sy y_2' sy z sy v_1 sy v_1' sy v_2 rs x rs y_1 rs y_2 rs y_2' rs z rs v_1 rs v_1' rs v_2. Then $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$, but $\overline{E}\neq_{sto}\overline{E'}$, since $TS^*(\overline{E'})$ has more states than $TS^*(\overline{E})$. E is a reduction of E' w.r.t. $\underline{\leftrightarrow}_{ss}$.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

N is a reduction of N' w.r.t. the net version of $\underline{\leftrightarrow}_{ss}$.

An *autobisimulation* is a bisimulation between an expression and itself.

For a dynamic expression G and a step stochastic autobisimulation $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G$, let $\mathcal{K} \in DR(G)/_{\mathcal{R}}$ and $s_1, s_2 \in \mathcal{K}$.

We have $\forall \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}}\ \forall A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}\ s_1 \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}} \Leftrightarrow s_2 \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

The equality is valid for all $s_1, s_2 \in \mathcal{K}$, hence, we can rewrite it as $\mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM_A^*(\mathcal{K}, \widetilde{\mathcal{K}}) = PM_A^*(s_1, \widetilde{\mathcal{K}}) = PM_A^*(s_2, \widetilde{\mathcal{K}})$.

We write $\mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}$ if $\exists \mathcal{P}\ \mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$ and $\mathcal{K} \twoheadrightarrow \widetilde{\mathcal{K}}$ if $\exists A\ \mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}$.

The similar arguments: we write $\mathcal{K} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}}) = PM^*(s_1, \widetilde{\mathcal{K}}) = PM^*(s_2, \widetilde{\mathcal{K}})$.

$\mathcal{R}_{ss}(G) = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G\}$ is the *largest step stochastic autobisimulation* on G.

Definition 138 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) (labeled probabilistic) transition system without empty loops *of a dynamic expression G is* $TS_{\underline{\leftrightarrow}_{ss}}^*(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$*, where*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\mathcal{R}_{ss}(G)}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq (I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}) \times (0; 1]$;
- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{K}, (A, PM_A^*(\mathcal{K}, \widetilde{\mathcal{K}})), \widetilde{\mathcal{K}}) \mid \mathcal{K}, \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}_{ss}(G)},\ \mathcal{K} \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = [[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$.

The transition $(\mathcal{K}, (A, \mathcal{P}), \widetilde{\mathcal{K}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{K} \stackrel{A}{\twoheadrightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

For $E \in RegStatExpr$, let $TS_{\underline{\leftrightarrow}_{ss}}^*(E) = TS_{\underline{\leftrightarrow}_{ss}}^*(\overline{E})$.

Definition 139 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) underlying DTMC without empty loops *of a dynamic expression G, $DTMC_{\underline{\leftrightarrow}_{ss}}^*(G)$, has the state space $DR(G)/_{\mathcal{R}_{ss}(G)}$, the initial state $[[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$ and the transitions $\mathcal{K} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}})$.*

For $E \in RegStatExpr$, let $DTMC_{\underline{\leftrightarrow}_{ss}}^*(E) = DTMC_{\underline{\leftrightarrow}_{ss}}^*(\overline{E})$.

Logical characterization

Logic iPML

Definition 140 *$\top$ is the truth, $\alpha \in \mathcal{L}$, $\mathcal{P} \in (0; 1]$. A formula of $iPML$:*

$$\Phi ::= \top \mid \neg\Phi \mid \Phi \wedge \Phi \mid \nabla_\alpha \mid \langle\alpha\rangle_{\mathcal{P}}\Phi$$

iPML is the set of *all formulas of the logic $iPML$*.

Definition 141 *Let G be a dynamic expression and $s \in DR(G)$. The satisfaction relation $\models_G \subseteq DR(G) \times \mathbf{iPML}$:*

1. *$s \models_G \top$ — always;*
2. *$s \models_G \neg\Phi$, if $s \not\models_G \Phi$;*
3. *$s \models_G \Phi \wedge \Psi$, if $s \models_G \Phi$ and $s \models_G \Psi$;*
4. *$s \models_G \nabla_\alpha$, if not $s \xrightarrow{\alpha} DR(G)$;*
5. *$s \models_G \langle\alpha\rangle_{\mathcal{P}}\Phi$, if $\exists \mathcal{H} \subseteq DR(G)\ s \xrightarrow{\alpha}_{\mathcal{Q}} \mathcal{H},\ \mathcal{Q} \geq \mathcal{P}$ and $\forall \tilde{s} \in \mathcal{H}\ \tilde{s} \models_G \Phi$.*

$\langle\alpha\rangle\Phi = \exists\mathcal{P}\ \langle\alpha\rangle_{\mathcal{P}}\Phi$. $\langle\alpha\rangle_{\mathcal{Q}}\Phi$ implies $\langle\alpha\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

We write $G \models_G \Phi$, if $[G]_\approx \models_G \Phi$.

Definition 142 *G and G' are logically equivalent in $iPML$, $G =_{iPML} G'$, if $\forall \Phi \in \mathbf{iPML}\ G \models_G \Phi \Leftrightarrow G' \models_{G'} \Phi$.*

Let G be a dynamic expression and $s \in DR(G),\ \alpha \in \mathcal{L}$.

The set of states reached from s by execution of α, the *image set*, is
$Image(s, \alpha) = \{\tilde{s} \mid \exists \{(\alpha, \rho)\} \in Exec(s)\ s \stackrel{(\alpha,\rho)}{\twoheadrightarrow} \tilde{s}\}$.

A dynamic expression G is an *image-finite* one, if
$\forall s \in DR(G)\ \forall \alpha \in \mathcal{L}\ |Image(s, \alpha)| < \infty$.

Theorem 37 *For image-finite dynamic expressions G and G'*

$$G \underline{\leftrightarrow}_{is} G' \ \Leftrightarrow\ G =_{iPML} G'.$$

Let $E = (\{a\}, \frac{1}{2}); ((\{b\}, \frac{1}{2})[](\{c\}, \frac{1}{2}))$ and
$E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{a\}, \frac{1}{2}); (\{c\}, \frac{1}{2}))$. Then $\overline{E} \neq_{iPML} \overline{E'}$, because for $\Phi = \langle \{a\} \rangle_1 \langle \{b\} \rangle_{\frac{1}{2}} \top$ we have $\overline{E} \models_{\overline{E}} \Phi$, but $\overline{E'} \not\models_{\overline{E'}} \Phi$, since only in $TS^*(\overline{E'})$ a multiaction $\{a\}$ can be executed so that no multiaction $\{b\}$ can occur afterwards.

Logic sPML

Definition 143 $\top$ *is the truth,* $A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$, $\mathcal{P} \in (0;1]$.

A formula *of* $sPML$*:*

$$\Phi ::= \top \mid \neg\Phi \mid \Phi\wedge\Phi \mid \nabla_A \mid \langle A\rangle_{\mathcal{P}}\Phi$$

sPML is the set of *all formulas of the logic* $sPML$.

Definition 144 *Let* G *be a dynamic expression and* $s \in DR(G)$*. The* satisfaction relation $\models_G \subseteq DR(G) \times \mathbf{sPML}$*:*

1. $s \models_G \top$ *— always;*
2. $s \models_G \neg\Phi$*, if* $s \not\models_G \Phi$*;*
3. $s \models_G \Phi\wedge\Psi$*, if* $s \models_G \Phi$ *and* $s \models_G \Psi$*;*
4. $s \models_G \nabla_A$*, if not* $s \stackrel{A}{\to}$ $DR(G)$*;*
5. $s \models_G \langle A\rangle_{\mathcal{P}}\Phi$*, if* $\exists\mathcal{H} \subseteq DR(G)$ $s \stackrel{A}{\to}_{\mathcal{Q}} \mathcal{H}$, $\mathcal{Q} \geq \mathcal{P}$ *and* $\forall\tilde{s} \in \mathcal{H}$ $\tilde{s} \models_G \Phi$.

$\langle A\rangle\Phi = \exists\mathcal{P}\ \langle A\rangle_{\mathcal{P}}\Phi$. $\langle A\rangle_{\mathcal{Q}}\Phi$ implies $\langle A\rangle_{\mathcal{P}}\Phi$, if $\mathcal{Q} \geq \mathcal{P}$.

We write $G \models_G \Phi$, if $[G]_{\approx} \models_G \Phi$.

Definition 145 G *and* G' *are* logically equivalent *in* $sPML$, $G =_{sPML} G'$*, if* $\forall\Phi \in \mathbf{sPML}\ G \models_G \Phi \Leftrightarrow G' \models_{G'} \Phi$.

Let G be a dynamic expression and $s \in DR(G),\ A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}$.

The set of states reached from s by execution of A, the *image set*, is
$Image(s, A) = \{\tilde{s} \mid \exists \Gamma \in Exec(s)\ \mathcal{L}(\Gamma) = A,\ s \stackrel{\Gamma}{\twoheadrightarrow} \tilde{s}\}$.

A dynamic expression G is an *image-finite* one, if
$\forall s \in DR(G)\ \forall A \in I\!N_{fin}^{\mathcal{L}} \setminus \{\emptyset\}\ |Image(s, A)| < \infty$.

Theorem 38 *For image-finite dynamic expressions G and G'*

$$G \underline{\leftrightarrow}_{ss} G' \ \Leftrightarrow\ G =_{sPML} G'.$$

Let $E = (\{a\}, \frac{1}{2}) \| (\{b\}, \frac{1}{2})$ and
$E' = ((\{a\}, \frac{1}{2}); (\{b\}, \frac{1}{2}))[]((\{b\}, \frac{1}{2}); (\{a\}, \frac{1}{2}))$. Then $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$ but $\overline{E} \neq_{sPML} \overline{E'}$, because for $\Phi = \langle\{a, b\}\rangle_{\frac{1}{3}} \top$ we have $\overline{E} \models_{\overline{E}} \Phi$, but $\overline{E'} \not\models_{\overline{E'}} \Phi$, since only in $TS^*(\overline{E'})$ multiactions $\{a\}$ and $\{b\}$ cannot be executed concurrently.

Stationary behaviour

Theoretical background

The elements $\mathcal{P}^*_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of *(one-step) transition probability matrix (TPM)* $\mathbf{P}^*$ for $DTMC^*(G)$:

$$\mathcal{P}^*_{ij} = \begin{cases} PM^*(s_i, s_j), & s_i \twoheadrightarrow s_j; \\ 0, & \text{otherwise}. \end{cases}$$

The *transient (k-step, $k \in I\!N$) probability mass function (PMF)* $\psi^*[k] = (\psi^*_1[k], \ldots, \psi^*_n[k])$ for $DTMC^*(G)$ is calculated as

$$\psi^*[k] = \psi^*[0](\mathbf{P}^*)^k.$$

where $\psi^*[0] = (\psi^*_1[0], \ldots, \psi^*_n[0])$ is the *initial PMF*:

$$\psi^*_i[0] = \begin{cases} 1, & s_i = [G]_\approx; \\ 0, & \text{otherwise}. \end{cases}$$

$\psi^*[k+1] = \psi^*[k]\mathbf{P}^*, \ k \in I\!N$.

The *steady-state PMF* $\psi^* = (\psi^*_1, \ldots, \psi^*_n)$ for $DTMC^*(G)$ is a solution of

$$\begin{cases} \psi^*(\mathbf{P}^* - \mathbf{I}) = \mathbf{0} \\ \psi^*\mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of order n, $\mathbf{0}$ is a vector of n values 0, $\mathbf{1}$ is that of n values 1.

When $DTMC^*(G)$ has the single steady state, $\psi^* = \lim_{k\to\infty} \psi^*[k]$.

For $s \in DR(G)$ with $s = s_i$ $(1 \leq i \leq n)$ we define
$\psi^*[k](s) = \psi_i^*[k]$ $(k \in I\!N)$ and $\psi^*(s) = \psi_i^*$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G),\ S, \widetilde{S} \subseteq DR(G)$.
The following performance indices (measures) are based on the steady-state PMF.

- The *average recurrence (return) time in the state s* (i.e. the number of discrete time units or steps required for this) is $\frac{1}{\psi^*(s)}$.
- The *fraction of residence time in the state s* is $\psi^*(s)$.
- The *fraction of residence time in the set of states $S \subseteq DR(G)$* or the *probability of the event determined by a condition that is true for all states from S* is $\sum_{s \in S} \psi^*(s)$.
- The *relative fraction of residence time in the set of states S w.r.t. that in $\widetilde{S}$* is $\frac{\sum_{s \in S} \psi^*(s)}{\sum_{\tilde{s} \in \widetilde{S}} \psi^*(\tilde{s})}$.
- The *steady-state probability to perform a step with a multiset of activities Δ* is $\sum_{s \in DR(G)} \psi^*(s) \sum_{\{\Gamma \mid \Delta \subseteq \Gamma\}} PT^*(\Gamma, s)$.
- The *probability of the event determined by a reward function r on the states* is $\sum_{s \in DR(G)} \psi^*(s) r(s)$, where $\forall s \in DR(G)\ 0 \leq r(s) \leq 1$.

Theorem 39 *Let G be a dynamic expression and EL be its empty loops abstraction vector. The steady-state PMFs ψ for $DTMC(G)$ and ψ^* for $DTMC^*(G)$ are related as: $\forall s \in DR(G)$*

$$\psi(s) = \frac{\psi^*(s) EL(s)}{\sum_{\tilde{s} \in DR(G)} \psi^*(\tilde{s}) EL(\tilde{s})}.$$

Steady state and equivalences

For $s \in DR(G)$ with $s = s_i$ $(1 \le i \le n)$ we define
$\psi^*[k](s) = \psi_i^*[k]$ $(k \in \mathbb{N})$ and $\psi^*(s) = \psi_i^*$.

Proposition 29 *Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ and ψ^* be the steady-state PMF for $DTMC^*(G)$, ψ'^* be the steady-state PMF for $DTMC^*(G')$. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

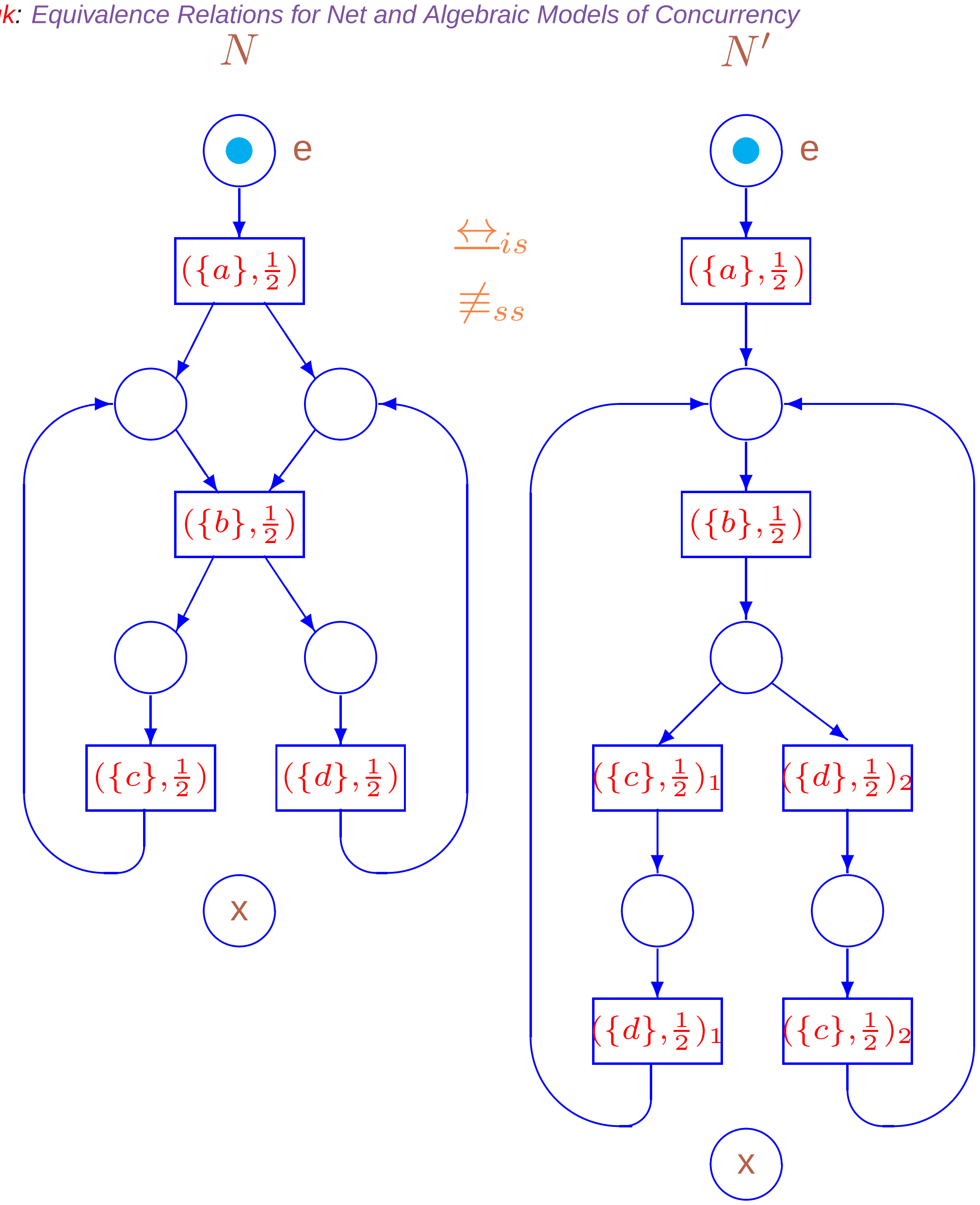
$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \psi'^*(s').$$

The result of the proposition above is valid if we replace steady-state probabilities with transient ones.

Let G be a dynamic expression. The transient PMF $\psi^*_{\underline{\leftrightarrow}_{ss}}[k]$ ($k \in \mathbb{N}$) and the steady-state PMF $\psi^*_{\underline{\leftrightarrow}_{ss}}$ for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(G)$ are defined like the corresponding notions $\psi^*[k]$ and ψ^* for $DTMC^*(G)$.

By the proposition above: $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)}\ \psi^*_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) = \sum_{s \in \mathcal{K}} \psi^*(s)$.

$\mathsf{Stop} = (\{c\}, \frac{1}{2})$ rs c is the process that performs empty loops with probability 1 and never terminates.



$\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady-state probabilities to enter into an equivalence class

Let $E = [(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); ((\{c\},\frac{1}{2}) \| (\{d\},\frac{1}{2}))) * \mathsf{Stop}]$ and $E' = [(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (((\{c\},\frac{1}{2})_1; (\{d\},\frac{1}{2})_1) [] ((\{d\},\frac{1}{2})_2; (\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]$. We have $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$. $DR(\overline{E})$ consists of

$s_1 = [[\overline{(\{a\},\frac{1}{2})} * ((\{b\},\frac{1}{2}); ((\{c\},\frac{1}{2}) \| (\{d\},\frac{1}{2}))) * \mathsf{Stop}]]_{\approx}$,

$s_2 = [[(\{a\},\frac{1}{2}) * (\overline{(\{b\},\frac{1}{2})}; ((\{c\},\frac{1}{2}) \| (\{d\},\frac{1}{2}))) * \mathsf{Stop}]]_{\approx}$,

$s_3 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (\overline{(\{c\},\frac{1}{2})} \| \overline{(\{d\},\frac{1}{2})})) * \mathsf{Stop}]]_{\approx}$,

$s_4 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (\underline{(\{c\},\frac{1}{2})} \| \overline{(\{d\},\frac{1}{2})})) * \mathsf{Stop}]]_{\approx}$,

$s_5 = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2}); (\overline{(\{c\},\frac{1}{2})} \| \underline{(\{d\},\frac{1}{2})})) * \mathsf{Stop}]]_{\approx}$.

$DR(\overline{E'})$ consists of

$s_1' = [[\overline{(\{a\},\frac{1}{2})} * ((\{b\},\frac{1}{2});(((\{c\},\frac{1}{2})_1;(\{d\},\frac{1}{2})_1)[]((\{d\},\frac{1}{2})_2;$
$(\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]]_\approx,$

$s_2' = [[(\{a\},\frac{1}{2}) * (\overline{(\{b\},\frac{1}{2})};(((\{c\},\frac{1}{2})_1;(\{d\},\frac{1}{2})_1)[]((\{d\},\frac{1}{2})_2;$
$(\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]]_\approx,$

$s_3' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});\overline{(((\{c\},\frac{1}{2})_1;(\{d\},\frac{1}{2})_1)[]((\{d\},\frac{1}{2})_2;}$
$\overline{(\{c\},\frac{1}{2})_2))}) * \mathsf{Stop}]]_\approx,$

$s_4' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});(((\{c\},\frac{1}{2})_1;\overline{(\{d\},\frac{1}{2})_1})[]((\{d\},\frac{1}{2})_2;$
$(\{c\},\frac{1}{2})_2))) * \mathsf{Stop}]]_\approx,$

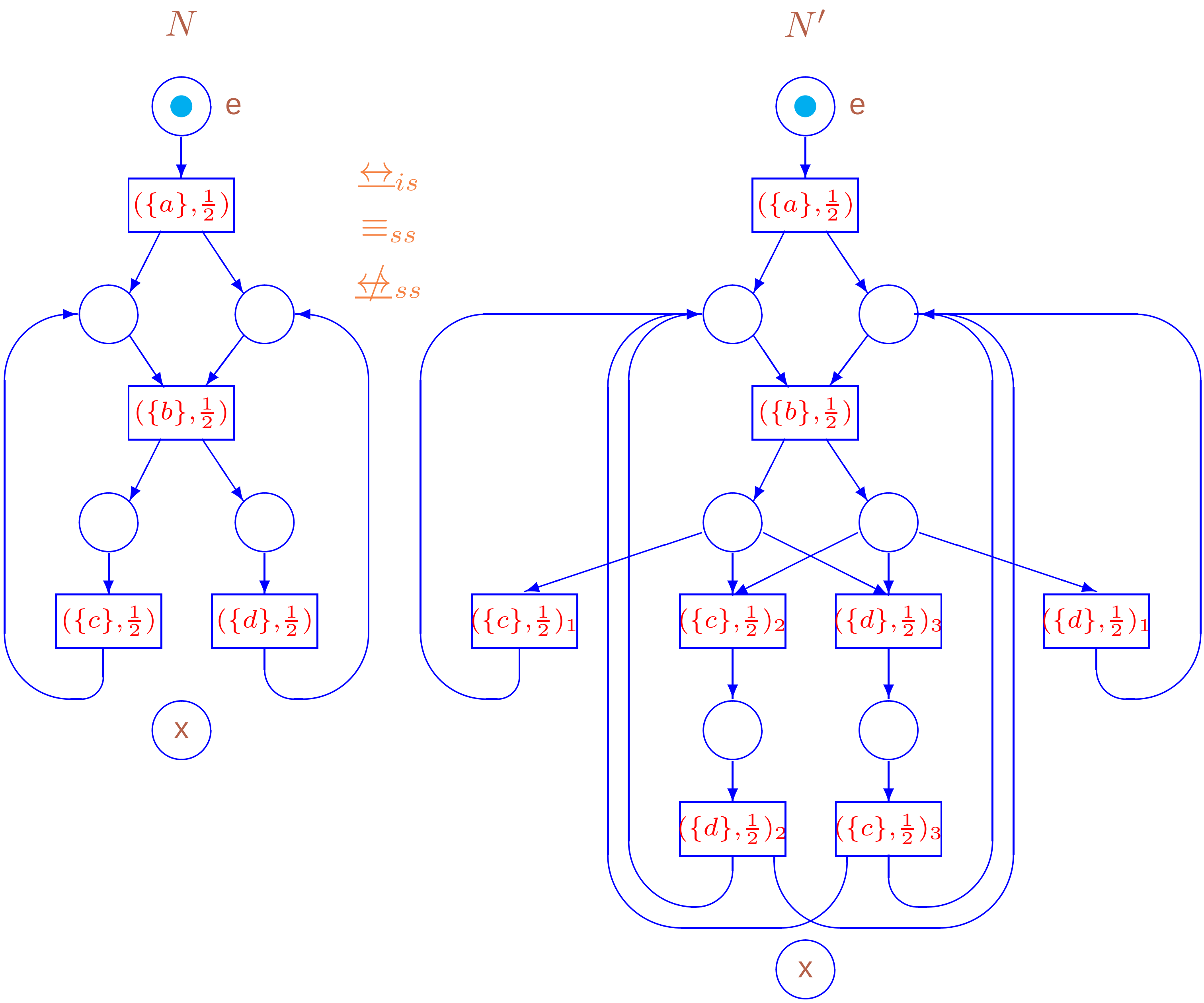
$s_5' = [[(\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2});(((\{c\},\frac{1}{2})_1;(\{d\},\frac{1}{2})_1)[]((\{d\},\frac{1}{2})_2;$
$\overline{(\{c\},\frac{1}{2})_2)}))) * \mathsf{Stop}]]_\approx.$

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ \psi'^* = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right).$$

Consider $\mathcal{H} = \{s_3, s_3'\}$. We have $\sum_{s \in \mathcal{H} \cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$, whereas $\sum_{s' \in \mathcal{H} \cap DR(\overline{E'})} \psi'^*(s') = \psi'^*(s_3') = \frac{1}{3}$. Thus, $\underline{\leftrightarrow}_{is}$ does not guarantee a coincidence of steady-state probabilities to enter into an equivalence class.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.



The intersection of $\underline{\leftrightarrow}_{is}$ and $\equiv_{ss}$ does not guarantee a coincidence of steady-state probabilities to enter into an equivalence class

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2}) \| (\{d\}, \frac{1}{2}))) * \mathsf{Stop}]$ and $E' = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1)) [] (((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2) [] ((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]$.

We have $\overline{E} \underline{\leftrightarrow}_{is} \overline{E'}$ and $\overline{E} \equiv_{ss} \overline{E'}$.

$DR(\overline{E})$ is as in the previous example.

$DR(\overline{E'})$ consists of

$s'_1 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1))[](((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2)[]((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]]_\approx$,
$s'_2 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1))[](((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2)[]((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]]_\approx$,
$s'_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1))[](((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2)[]((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]]_\approx$,
$s'_4 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1))[](((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2)[]((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]]_\approx$,
$s'_5 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1))[](((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2)[]((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]]_\approx$,
$s'_6 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1))[](((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2)[]((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]]_\approx$,
$s'_7 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); (((\{c\}, \frac{1}{2})_1 \| (\{d\}, \frac{1}{2})_1))[](((\{c\}, \frac{1}{2})_2; (\{d\}, \frac{1}{2})_2)[]((\{d\}, \frac{1}{2})_3; (\{c\}, \frac{1}{2})_3))))) * \mathsf{Stop}]]_\approx$.

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right),\ \psi'^* = \left(0, \frac{13}{38}, \frac{13}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}, \frac{3}{38}\right).$$

Consider $\mathcal{H} = \{s_3, s'_3\}$. We have $\sum_{s \in \mathcal{H} \cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{3}{8}$, whereas $\sum_{s' \in \mathcal{H} \cap DR(\overline{E'})} \psi'^*(s') = \psi'^*(s'_3) = \frac{13}{38}$. Thus, $\underline{\leftrightarrow}_{is}$ plus $\equiv_{ss}$ do not guarantee a coincidence of steady-state probabilities to enter into an equivalence class.

In figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

Definition 146 *A derived step trace of a dynamic expression G is*
$\Sigma = A_1 \cdots A_n \in (\mathbb{N}_{fin}^{\mathcal{L}} \setminus \{\emptyset\})^*$*, where*
$\exists s \in DR(G)\ s \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i) = A_i\ (1 \leq i \leq n)$.

The probability to execute the derived step trace Σ in s:

$$PT^*(\Sigma, s) =$$
$$\sum_{\{\Gamma_1,\ldots,\Gamma_n \mid s=s_0 \stackrel{\Gamma_1}{\twoheadrightarrow} s_1 \stackrel{\Gamma_2}{\twoheadrightarrow} \cdots \stackrel{\Gamma_n}{\twoheadrightarrow} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\leq i\leq n)\}} \prod_{i=1}^n PT^*(\Gamma_i, s_{i-1}).$$

Theorem 40 *Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ and ψ^* be the steady-state PMF for $DTMC^*(G)$, ψ'^* be the steady-state PMF for $DTMC^*(G')$ and Σ be a derived step trace of G and G'. Then*
$\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$

$$\sum_{s \in \mathcal{H} \cap DR(G)} \psi^*(s) PT^*(\Sigma, s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \psi'^*(s') PT^*(\Sigma, s').$$

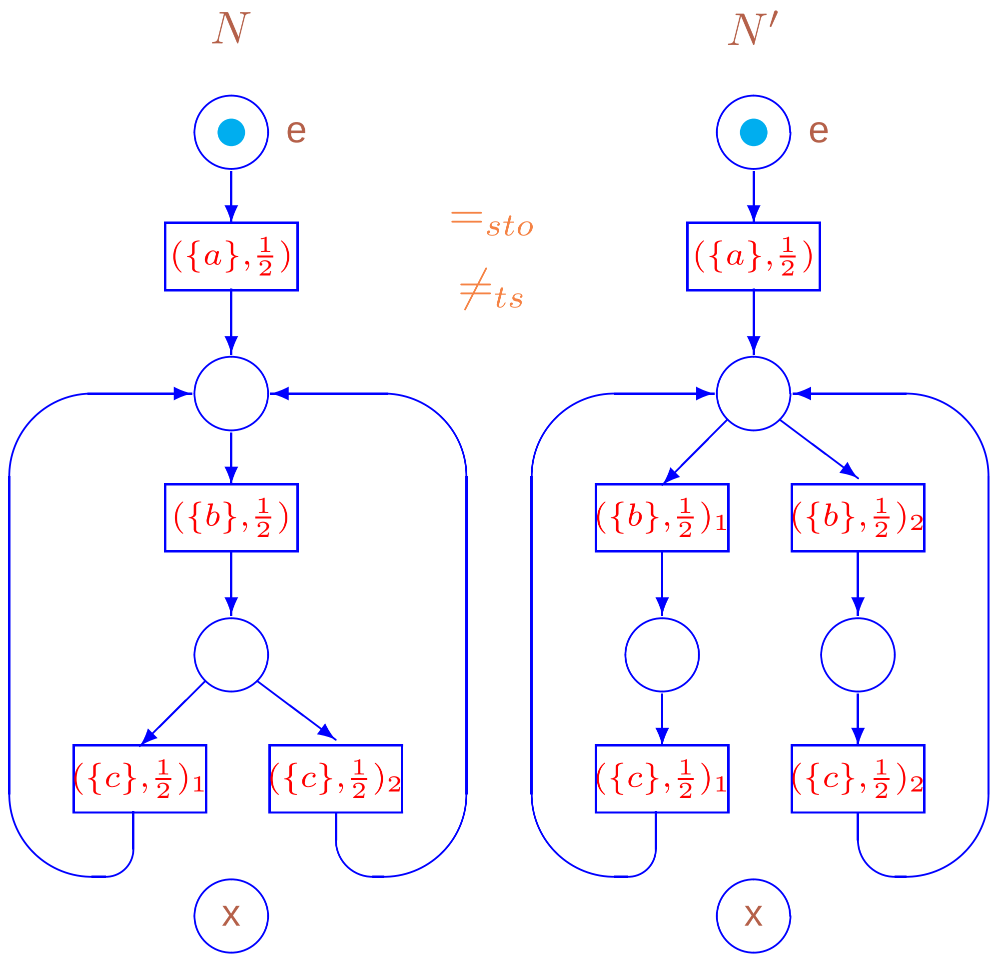
The result of the theorem above is valid

if we replace steady-state probabilities with transient ones.

By the theorem above: $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)}$

$$\psi^*_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) PT^*(\Sigma, \mathcal{K}) = \sum_{s \in \mathcal{K}} \psi^*(s) PT^*(\Sigma, s),$$

where $\forall s \in \mathcal{K}\ PT^*(\Sigma, \mathcal{K}) = PT^*(\Sigma, s)$.



$\underline{\leftrightarrow}_{ss}$ preserves steady-state behaviour in the equivalence classes

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2})_1 [] (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$ and
$E' = [(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1) [] ((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$.
We have $\overline{E} =_{sto} \overline{E'}$, hence, $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$.

$DR(\overline{E})$ consists of

$s_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{2})_1 [] (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; ((\{c\}, \frac{1}{2})_1 [] (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_\approx$,
$s_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); \overline{((\{c\}, \frac{1}{2})_1 [] (\{c\}, \frac{1}{2})_2)}) * \mathsf{Stop}]]_\approx$.

$DR(\overline{E'})$ consists of

$s_1' = [[\overline{(\{a\}, \frac{1}{2})} * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))*$ Stop$]]_{\approx}$,

$s_2' = [[(\{a\}, \frac{1}{2}) * \overline{(((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))}*$ Stop$]]_{\approx}$,

$s_3' = [[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; \overline{(\{c\}, \frac{1}{2})_1})[]((\{b\}, \frac{1}{2})_2; (\{c\}, \frac{1}{2})_2))*$ Stop$]]_{\approx}$,

$s_4' = [[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{2})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{2})_2; \overline{(\{c\}, \frac{1}{2})_2}))*$ Stop$]]_{\approx}$.

The steady-state PMFs ψ^* for $DTMC^*(\overline{E})$ and ψ'^* for $DTMC^*(\overline{E'})$ are

$$\psi^* = \left(0, \frac{1}{2}, \frac{1}{2}\right),\ \psi'^* = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Consider $\mathcal{H} = \{s_3, s_3', s_4'\}$. The steady-state probabilities for $\mathcal{H}$ coincide: $\sum_{s \in \mathcal{H} \cap DR(\overline{E})} \psi^*(s) = \psi^*(s_3) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \psi'^*(s_3') + \psi'^*(s_4') = \sum_{s' \in \mathcal{H} \cap DR(\overline{E'})} \psi'^*(s')$.

Let $\Sigma = \{\{c\}\}$. The steady-state probabilities to enter into the equivalence class $\mathcal{H}$ and start the derived step trace Σ from it coincide:
$\psi^*(s_3)(PT^*(\{(\{c\}, \frac{1}{2})_1\}, s_3) + PT^*(\{(\{c\}, \frac{1}{2})_2\}, s_3)) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2} = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 = \psi'^*(s_3')PT^*(\{(\{c\}, \frac{1}{2})_1\}, s_3') + \psi'^*(s_4')PT^*(\{(\{c\}, \frac{1}{2})_2\}, s_4')$.

In the figure above $N = Box_{dts}(\overline{E})$ and $N' = Box_{dts}(\overline{E'})$.

Simplification of performance analysis

The method of performance analysis simplification.

1. The system under investigation is specified by a static expression of $dtsPBC$.
2. The transition system without empty loops of the expression is constructed.
3. After examining this transition system for self-similarity and symmetry, a step stochastic autobisimulation equivalence for the expression is determined.
4. The quotient underlying DTMC without empty loops of the expression is constructed from the quotient transition system without empty loops.
5. The steady-state probabilities and performance indices based on this DTMC are calculated.

Equivalence-based simplification of performance evaluation

The limitation of the method: the expressions with underlying DTMCs containing one closed communication class of states, which is ergodic, to ensure uniqueness of the stationary distribution.

If a DTMC contains several closed communication classes of states that are all ergodic: several stationary distributions may exist, depending on the initial PMF.

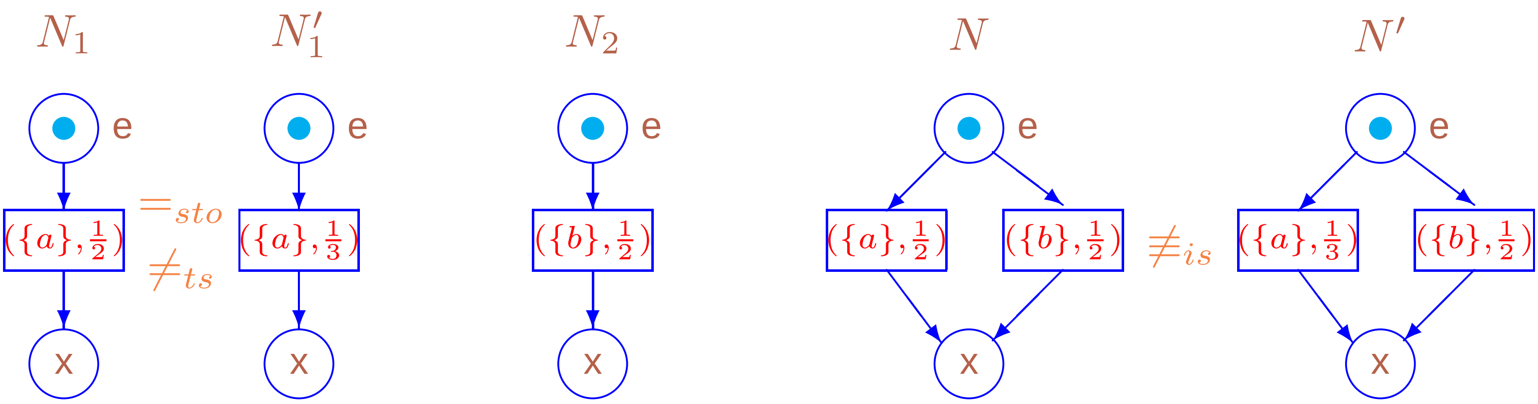
The general steady-state probabilities are then calculated as
the sum of the stationary probabilities of all the ergodic classes of states,
weighted by the probabilities to enter these classes,
starting from the initial state and passing through transient states.

The underlying DTMC of each process expression has one initial PMF (that at the time moment 0): the stationary distribution is unique.

It is worth applying the method to the systems with similar subprocesses.

Preservation by algebraic operations

Definition 147 *Let $\leftrightarrow$ be an equivalence of dynamic expressions. Static expressions E and E' are* equivalent w.r.t. $\leftrightarrow$, *$E \leftrightarrow E'$, if $\overline{E} \leftrightarrow \overline{E'}$.*



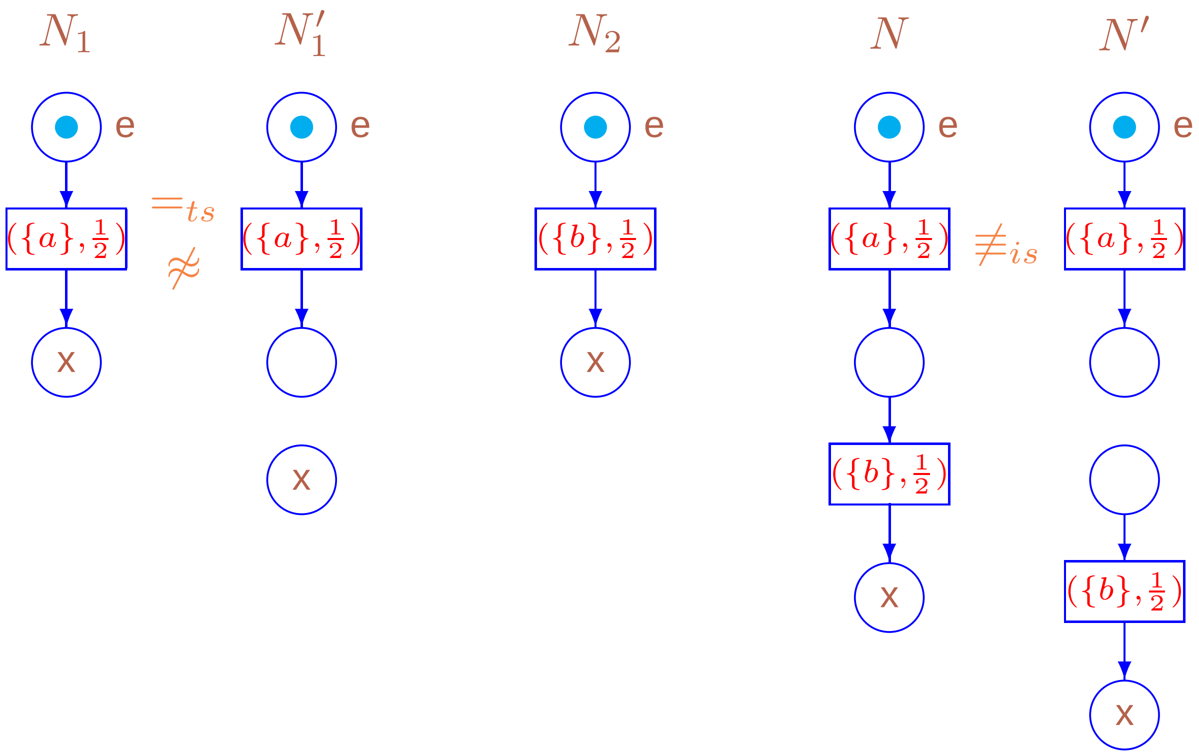
SC1: The equivalences between $\equiv_{is}$ and $=_{sto}$ are not congruences

Let $E = (\{a\}, \frac{1}{2})$, $E' = (\{a\}, \frac{1}{3})$ and $F = (\{b\}, \frac{1}{2})$. We have $\overline{E} =_{sto} \overline{E'}$, since both $TS^*(\overline{E})$ and $TS^*(\overline{E'})$ have the transitions with the multiaction part of labels $\{a\}$ and probability 1. On the other hand, $\overline{E[]F} \not\equiv_{is} \overline{E'[]F}$, since only in $TS^*(\overline{E'[]F})$ the probabilities of the transitions with the multiaction parts of labels $\{a\}$ and $\{b\}$ are different ($\frac{1}{3}$ and $\frac{2}{3}$, respectively). Thus, no equivalence between $\equiv_{is}$ and $=_{sto}$ is a congruence.

In the figure above
$N_1 = Box_{dts}(\overline{E})$, $N_1' = Box_{dts}(\overline{E'})$, $N_2 = Box_{dts}(\overline{F})$ and
$N = Box_{dts}(\overline{E[]F})$, $N' = Box_{dts}(\overline{E'[]F})$.

Proposition 30 *Let* $\star \in \{is, ss\}$, $\star\star \in \{sto, ts\}$. *The equivalences* $\equiv_\star$, $\underline{\leftrightarrow}_\star$, $=_{\star\star}$ *are not preserved by algebraic operations.*

Proposition 31 *The equivalence* $\approx$ *is preserved by algebraic operations.*



SC2: The equivalences between $\equiv_{is}$ and $=_{ts}$ are not congruences

Let $E = (\{a\}, \frac{1}{2})$, $E' = (\{a\}, \frac{1}{2})$; Stop and $F = (\{b\}, \frac{1}{2})$. We have $\overline{E} =_{ts} \overline{E'}$, since both $TS(\overline{E})$ and $TS(\overline{E'})$ have the transitions with the multiaction part of labels $\{a\}$ and probability $\frac{1}{2}$. On the other hand, $\overline{E; F} \not\equiv_{is} \overline{E'; F}$, since only in $TS^*(\overline{E'; F})$ no other transition can fire after the transition with the multiaction part of label $\{a\}$. Thus, no equivalence between $\equiv_{is}$ and $=_{ts}$ is a congruence.

In the figure above
$N_1 = Box_{dts}(\overline{E})$, $N_1' = Box_{dts}(\overline{E'})$, $N_2 = Box_{dts}(\overline{F})$ and
$N = Box_{dts}(\overline{E; F})$, $N' = Box_{dts}(\overline{E'; F})$.

For an analogue of $=_{ts}$ to be a congruence, we have to equip transition systems with two extra transitions skip and redo as in [MVC02].

The equivalences between $\equiv_{is}$ and $=_{sto}$ defined on the basis of the enriched transition systems will still be non-congruences by Example SC1.

Rules for skip and redo: skipping and redoing all executions.

Let $E \in RegStatExpr$.

Rules for skip and redo

$$\textbf{Sk}\ \overline{E} \stackrel{\mathsf{skip}}{\rightarrow} \underline{E} \qquad \textbf{Rd}\ \underline{E} \stackrel{\mathsf{redo}}{\rightarrow} \overline{E}$$

Definition 148 *Let E be a static expression and $TS(\overline{E}) = (S, L, \mathcal{T}, s)$. The (labeled probabilistic) sr-transition system of $\overline{E}$ is a quadruple $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$, where*

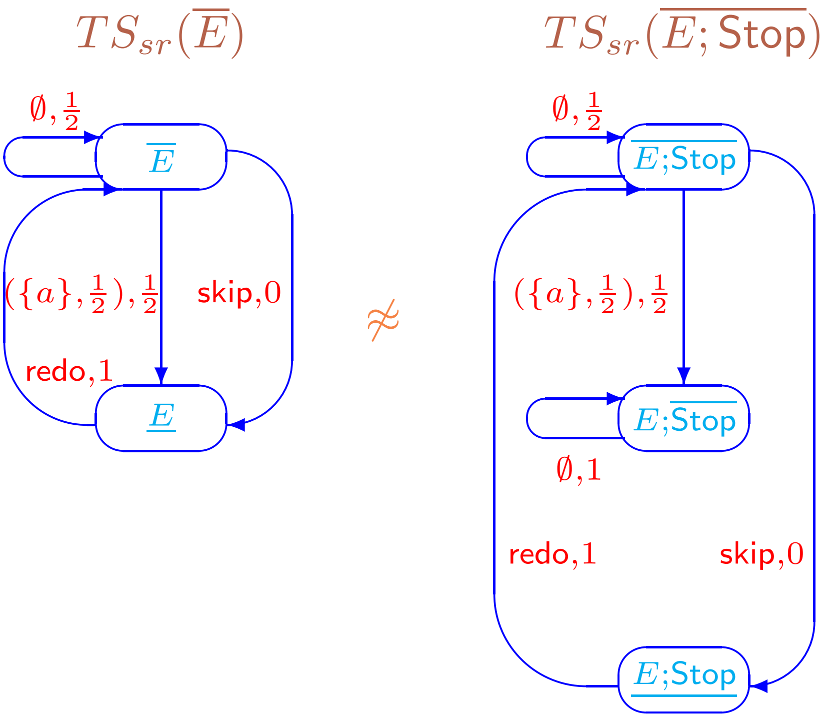
- $S_{sr} = S \cup \{[\underline{E}]_\approx\}$;
- $L_{sr} \subseteq (I\!N_{fin}^{\mathcal{SL}} \times (0;1]) \cup \{(\mathsf{skip}, 0), (\mathsf{redo}, 1)\}$;
- $\mathcal{T}_{sr} = \mathcal{T} \setminus \{([\underline{E}]_\approx, (\emptyset, 1), [\underline{E}]_\approx)\} \cup \{([\overline{E}]_\approx, (\mathsf{skip}, 0), [\underline{E}]_\approx), ([\underline{E}]_\approx, (\mathsf{redo}, 1), [\overline{E}]_\approx)\}$;
- $s_{sr} = s$.

Definition 149 *Let E, E' be static expressions and $TS_{sr}(\overline{E}) = (S_{sr}, L_{sr}, \mathcal{T}_{sr}, s_{sr})$, $TS_{sr}(\overline{E'}) = (S'_{sr}, L'_{sr}, \mathcal{T}'_{sr}, s'_{sr})$ be their sr-transition systems. A mapping $\beta : S_{sr} \to S'_{sr}$ is an* isomorphism *between $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E'})$, $\beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, if*

1. *β is a bijection s.t. $\beta(s_{sr}) = s'_{sr}$ and $\beta([\underline{E}]_\approx) = [\underline{E'}]_\approx$;*
2. *$\forall s, \tilde{s} \in S_{sr}\ \forall \Gamma\ s \xrightarrow{\Gamma}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \xrightarrow{\Gamma}_{\mathcal{P}} \beta(\tilde{s})$.*

Two sr-transition systems $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E'})$ are isomorphic*, $TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$, if $\exists \beta : TS_{sr}(\overline{E}) \simeq TS_{sr}(\overline{E'})$.*

For $E \in RegStatExpr$, let $TS_{sr}(E) = TS_{sr}(\overline{E})$.



TSSR: The sr-transition systems of $\overline{E}$ and $\overline{E;\text{Stop}}$ for $E = (\{a\}, \frac{1}{2})$

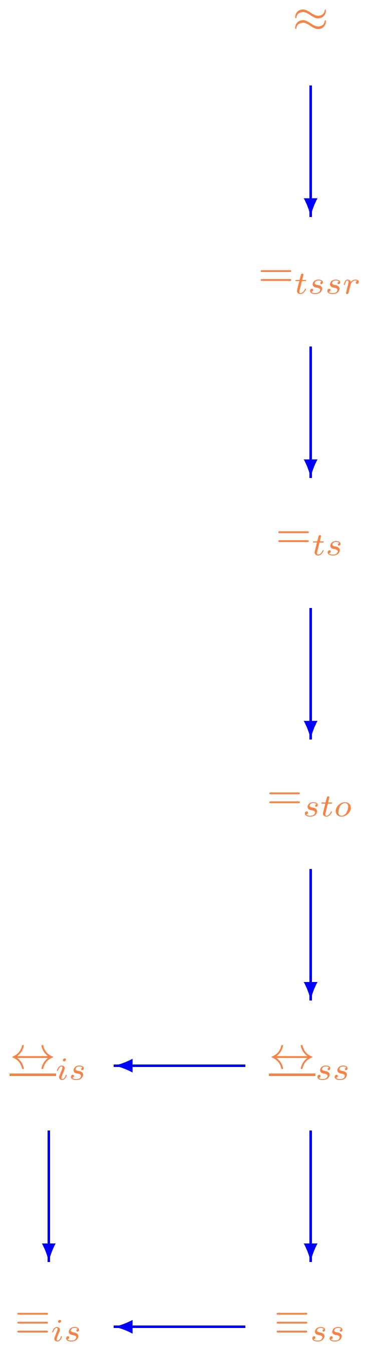
Let $E = (\{a\}, \frac{1}{2})$. In the figure above the transition systems $TS_{sr}(\overline{E})$ and $TS_{sr}(\overline{E;\text{Stop}})$ are presented.

In the latter sr-transition system the final state can be reached by the transition $(\text{skip}, 0)$ only from the initial state.

Definition 150 $\overline{E}$ *and* $\overline{E'}$ *are* isomorphic w.r.t. sr-transition systems, $\overline{E}=_{tssr}\overline{E'}$, *if* $TS_{sr}(\overline{E})\simeq TS_{sr}(\overline{E'})$.

sr-transition systems without empty loops can be defined and the equivalence $=_{tssr*}$ based on them.

The coincidence of $=_{tssr}$ and $=_{tssr*}$ can be proved as for $=_{ts}$ and $=_{ts*}$.



Interrelations of the stochastic equivalences and the new congruence

Theorem 41 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}, =, \approx\}$ *and* $\star, \star\star \in \{\_, is, ss, sto, ts, tssr\}$. *For dynamic expressions* G *and* G'

$$G \leftrightarrow_{\star} G' \;\Rightarrow\; G \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} G'$$

iff in the graph in figure above there exists a directed path from $\leftrightarrow_{\star}$ *to* $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.

Validity of the implications

- The implication $=_{tssr} \to =_{ts}$ is valid, since sr-transition systems have more states and transitions than usual ones.
- The implication $\approx \to =_{tssr}$ is valid, since the sr-transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

- Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2});\mathsf{Stop}$. We have $\overline{E} =_{ts} \overline{E'}$ (see example with Figure SC2). On the other hand, $\overline{E} \neq_{tssr} \overline{E'}$, since only in $TS_{sr}(\overline{E'})$ after the transition with multiaction part of label $\{a\}$ we do not reach the final state (see Figure TSSR).
- Let $E = (\{a\}, \frac{1}{2})$ and $E' = ((\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E} =_{tssr} \overline{E'}$, since $\overline{E} =_{ts} \overline{E'}$ by the last example from the equivalence interrelations theorem, and the final states of both $TS_{sr}(\overline{E'})$ and $TS_{sr}(\overline{E'})$ are reachable from the others with "normal" transitions (not with skip only). On the other hand, $\overline{E} \not\approx \overline{E'}$.

Theorem 42 *Let* $a \in Act$ *and* $E, E', F \in RegStatExpr$. *If* $\overline{E} =_{tssr} \overline{E'}$ *then*

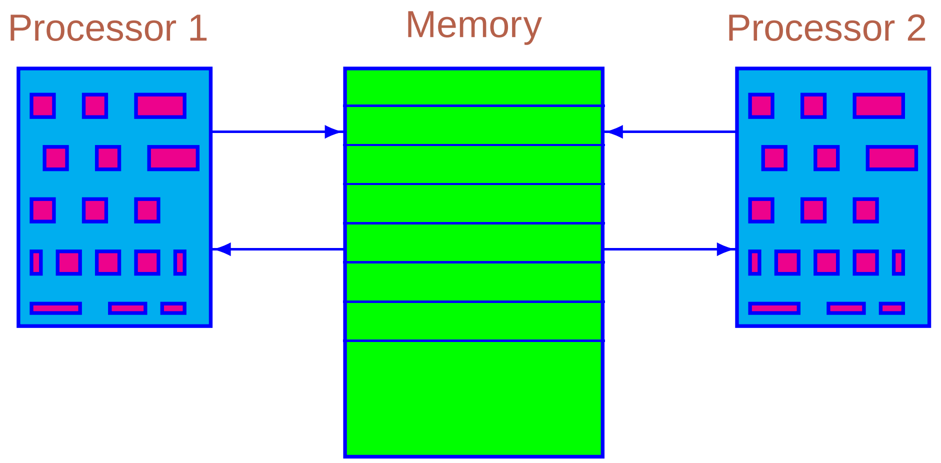
1. $\overline{E \circ F} =_{tssr} \overline{E' \circ F}$, $\overline{F \circ E} =_{tssr} \overline{F \circ E'}$, $\circ \in \{;, [], \|\}$;
2. $\overline{E[f]} =_{tssr} \overline{E'[f]}$;
3. $\overline{E \circ a} =_{tssr} \overline{E' \circ a}$, $\circ \in \{\mathrm{rs}, \mathrm{sy}\}$;
4. $\overline{[E * F * K]} =_{tssr} \overline{[E' * F * K]}$, $\overline{[F * E * K]} =_{tssr} \overline{[F * E' * K]}$, $\overline{[F * K * E]} =_{tssr} \overline{[F * K * E']}$.

Case studies

Shared memory system

The standard system

A model of two processors accessing a common shared memory [MBCDF95]



The diagram of the shared memory system

After activation of the system (turning the computer on), two processors are active, and the common memory is available. Each processor can request an access to the memory.

When a processor starts an acquisition of the memory, another processor waits until the former one ends its operations, and the system returns to the state with both active processors and the available memory.

a corresponds to the system activation.

r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i.

b_i and e_i correspond to the beginning and the end of the common memory access of processor i.

The other actions are used for communication purpose only.

The static expression of the first processor is
$E_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r_1\}, \frac{1}{2}); (\{b_1, y_1\}, \frac{1}{2}); (\{e_1, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is
$E_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r_2\}, \frac{1}{2}); (\{b_2, y_2\}, \frac{1}{2}); (\{e_2, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is
$E_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the shared memory system with two processors is
$E = (E_1 \| E_2 \| E_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

Effect of synchronization

The synchronization of $(\{b_i, y_i\}, \frac{1}{2})$ and $(\{\widehat{y_i}\}, \frac{1}{2})$ produces $(\{b_i\}, \frac{1}{4})$ $(1 \leq i \leq 2)$.

The synchronization of $(\{e_i, z_i\}, \frac{1}{2})$ and $(\{\widehat{z_i}\}, \frac{1}{2})$ produces $(\{e_i\}, \frac{1}{4})$ $(1 \leq i \leq 2)$.

The synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a, \widehat{x_2}\}, \frac{1}{4})$,

Synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_2\}, \frac{1}{2})$ produces $(\{a, \widehat{x_1}\}, \frac{1}{4})$.

Synchronization of $(\{a, \widehat{x_2}\}, \frac{1}{4})$ and $(\{x_2\}, \frac{1}{2})$, as well as $(\{a, \widehat{x_1}\}, \frac{1}{4})$ and $(\{x_1\}, \frac{1}{2})$
produces $(\{a\}, \frac{1}{8})$.

$DR(\overline{E})$ consists of

$$\begin{aligned}
s_1 = [([\overline{(\{x_1\}, \tfrac{1}{2})} * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [\overline{(\{x_2\}, \tfrac{1}{2})} * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [\overline{(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2})} * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}]) \\
\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}$$

$$\begin{aligned}
s_2 = [([(\{x_1\}, \tfrac{1}{2}) * (\overline{(\{r_1\}, \tfrac{1}{2})}; (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [(\{x_2\}, \tfrac{1}{2}) * (\overline{(\{r_2\}, \tfrac{1}{2})}; (\{b_2, y_2\}, \tfrac{1}{2}); (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}]) \\
\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}$$

$$\begin{aligned}
s_3 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); \overline{(\{b_1, y_1\}, \tfrac{1}{2})}; (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [(\{x_2\}, \tfrac{1}{2}) * (\overline{(\{r_2\}, \tfrac{1}{2})}; (\{b_2, y_2\}, \tfrac{1}{2}); (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * ((\overline{(\{\widehat{y_1}\}, \tfrac{1}{2})}; (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}]) \\
\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}$$

$$\begin{aligned}
s_4 = [([(\{x_1\}, \tfrac{1}{2}) * (\overline{(\{r_1\}, \tfrac{1}{2})}; (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{b_2, y_2\}, \tfrac{1}{2})}; (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[](\overline{(\{\widehat{y_2}\}, \tfrac{1}{2})}; (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}]) \\
\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}$$

$$\begin{aligned}
s_5 = [([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); \overline{(\{e_1, z_1\}, \tfrac{1}{2})}) * \mathsf{Stop}] \\
\| [(\{x_2\}, \tfrac{1}{2}) * (\overline{(\{r_2\}, \tfrac{1}{2})}; (\{b_2, y_2\}, \tfrac{1}{2}); (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}] \\
\| [(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_1}\}, \tfrac{1}{2})})[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}]) \\
\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},
\end{aligned}$$

$$s_6 = [(([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); \overline{(\{b_1, y_1\}, \tfrac{1}{2})}; (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); \overline{(\{b_2, y_2\}, \tfrac{1}{2})}; (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\overline{(\{\widehat{y_1}\}, \tfrac{1}{2})}; (\{\widehat{z_1}\}, \tfrac{1}{2}))[](\overline{(\{\widehat{y_2}\}, \tfrac{1}{2})}; (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_7 = [(([(\{x_1\}, \tfrac{1}{2}) * (\overline{(\{r_1\}, \tfrac{1}{2})}; (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); \overline{(\{e_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_8 = [(([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); \overline{(\{e_1, z_1\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); (\{e_2, z_2\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_1}\}, \tfrac{1}{2})})[]((\{\widehat{y_2}\}, \tfrac{1}{2}); (\{\widehat{z_2}\}, \tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_9 = [(([(\{x_1\}, \tfrac{1}{2}) * ((\{r_1\}, \tfrac{1}{2}); (\{b_1, y_1\}, \tfrac{1}{2}); (\{e_1, z_1\}, \tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\}, \tfrac{1}{2}) * ((\{r_2\}, \tfrac{1}{2}); (\{b_2, y_2\}, \tfrac{1}{2}); \overline{(\{e_2, z_2\}, \tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{a, \widehat{x_1}, \widehat{x_2}\}, \tfrac{1}{2}) * (((\{\widehat{y_1}\}, \tfrac{1}{2}); (\{\widehat{z_1}\}, \tfrac{1}{2}))[]((\{\widehat{y_2}\}, \tfrac{1}{2}); \overline{(\{\widehat{z_2}\}, \tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx}.$$

Interpretation of the states

s_1: the initial state,

s_2: the system is activated and the memory is not requested,

s_3: the memory is requested by the first processor,

s_4: the memory is requested by the second processor,

s_5: the memory is allocated to the first processor,

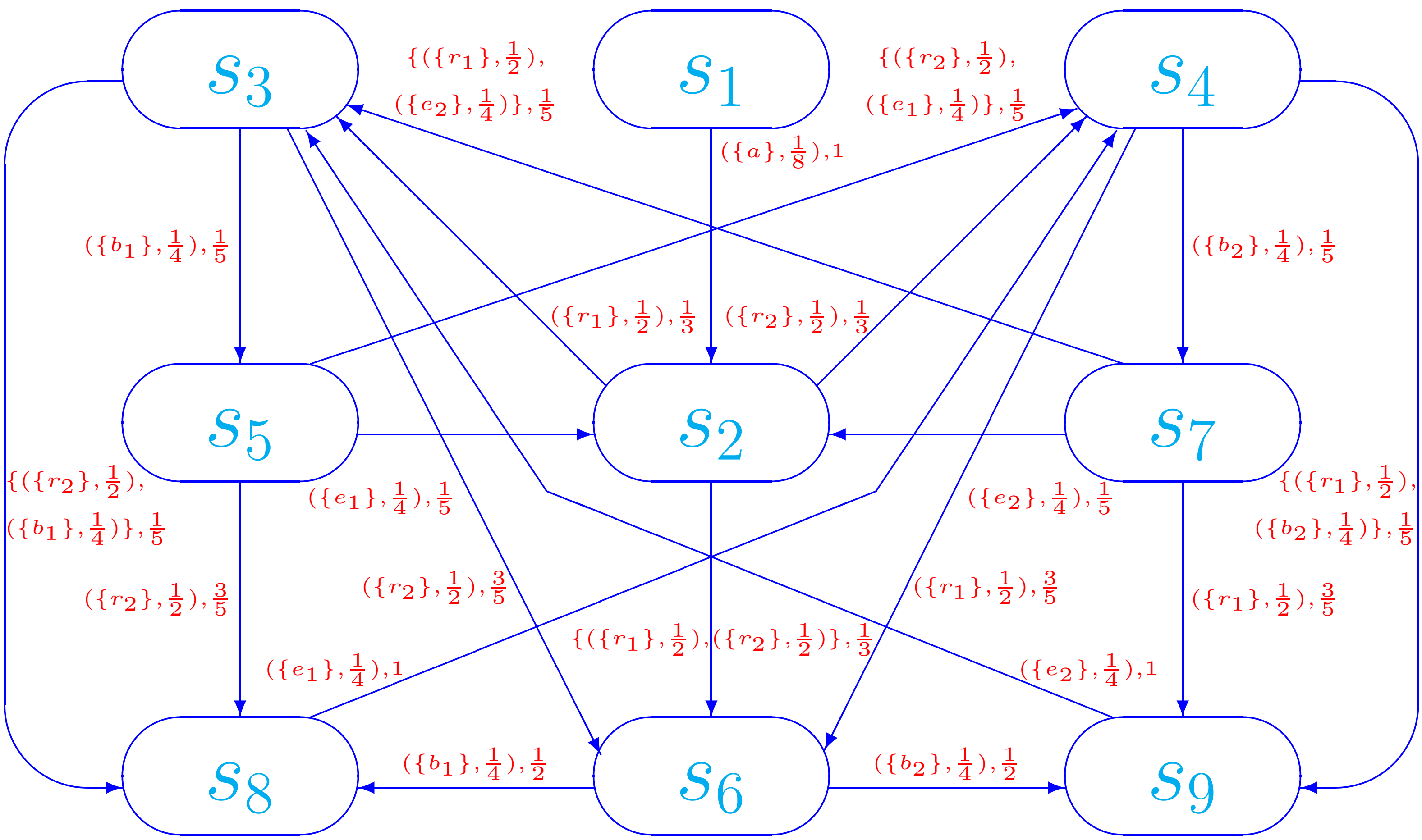
s_6: the memory is requested by two processors,

s_7: the memory is allocated to the second processor,

s_8: the memory is allocated to the first processor and the memory is requested by the second processor,

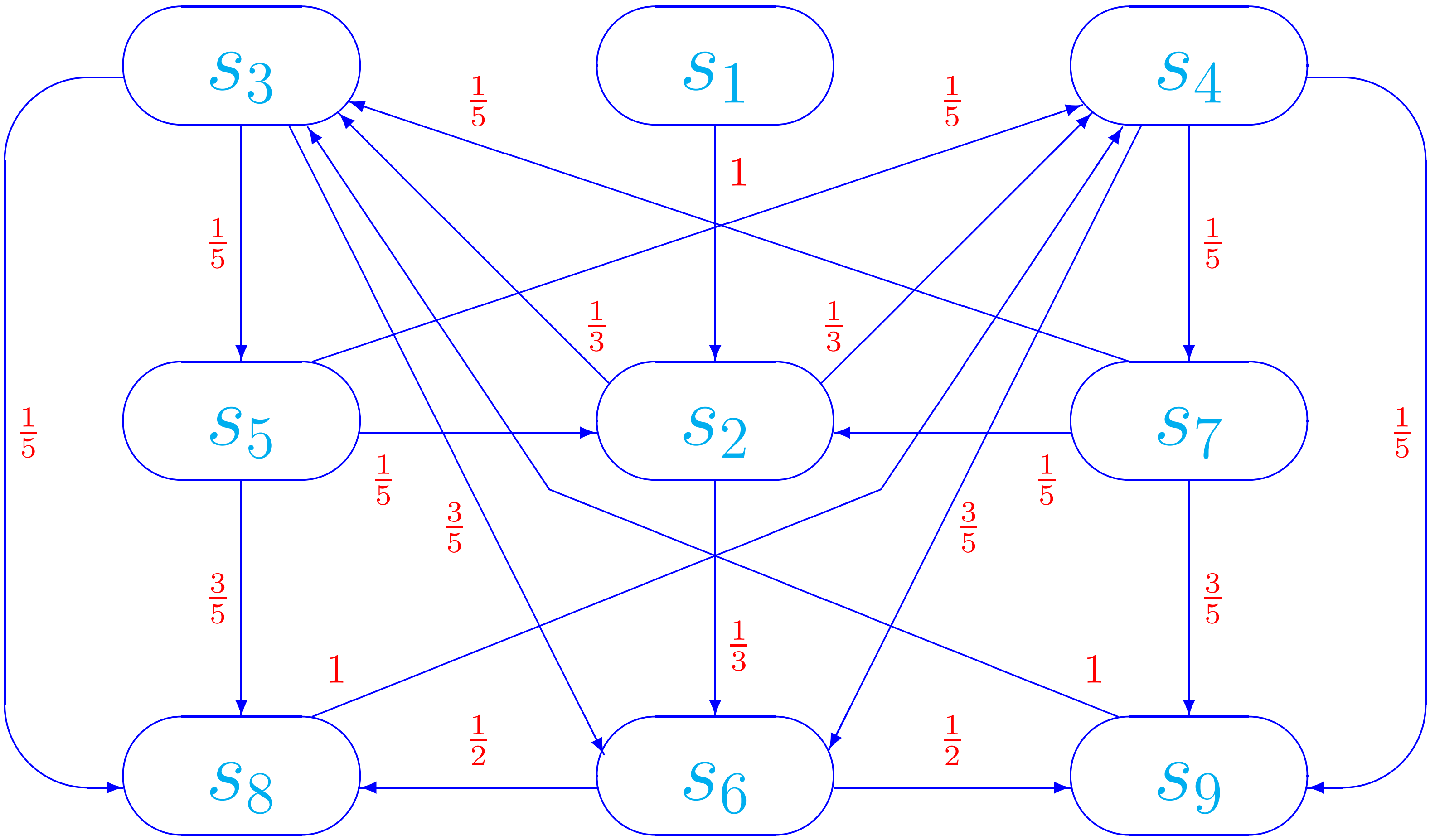
s_9: the memory is allocated to the second processor and the memory is requested by the first processor.

$TS^*(\overline{E})$



The transition system without empty loops of the shared memory system

$DTMC^*(\overline{E})$



The underlying DTMC without empty loops of the shared memory system

The TPM for $DTMC^*(\overline{E})$ is

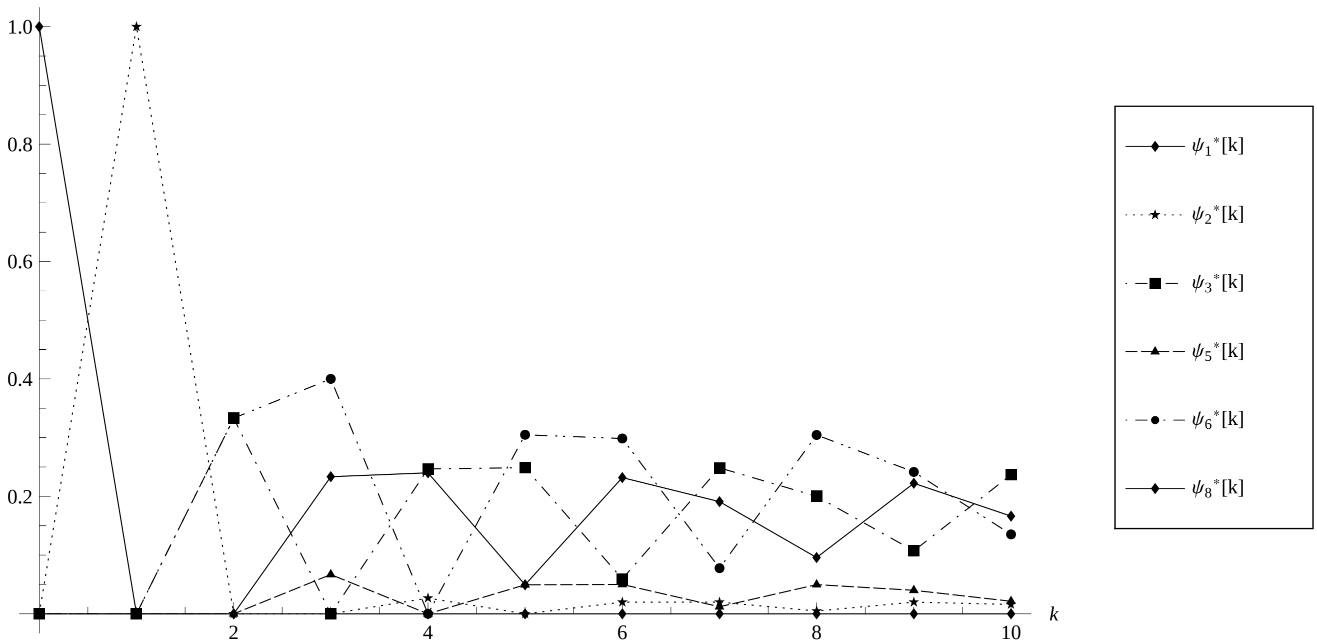
$$\mathbf{P}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The steady-state PMF for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{3}{209}, \frac{75}{418}, \frac{75}{418}, \frac{15}{418}, \frac{46}{209}, \frac{15}{418}, \frac{35}{209}, \frac{35}{209}\right).$$

Transient and steady-state probabilities of the shared memory system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2^*[k]$	0	1	0	0	0.0267	0	0.0197	0.0199	0.0047	0.0199	0.0160	0.0144
$\psi_3^*[k]$	0	0	0.3333	0	0.2467	0.2489	0.0592	0.2484	0.2000	0.1071	0.2368	0.1794
$\psi_5^*[k]$	0	0	0	0.0667	0	0.0493	0.0498	0.0118	0.0497	0.0400	0.0214	0.0359
$\psi_6^*[k]$	0	0	0.3333	0.4000	0	0.3049	0.2987	0.0776	0.3047	0.2416	0.1351	0.2201
$\psi_8^*[k]$	0	0	0	0.2333	0.2400	0.0493	0.2318	0.1910	0.0956	0.2221	0.1662	0.1675



Transient probabilities alteration diagram of the shared memory system

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.

Performance indices

- The average recurrence time in the state s_2, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{209}{3} = 69\frac{2}{3}$.
- The common memory is available in the states s_2, s_3, s_4, s_6 only.

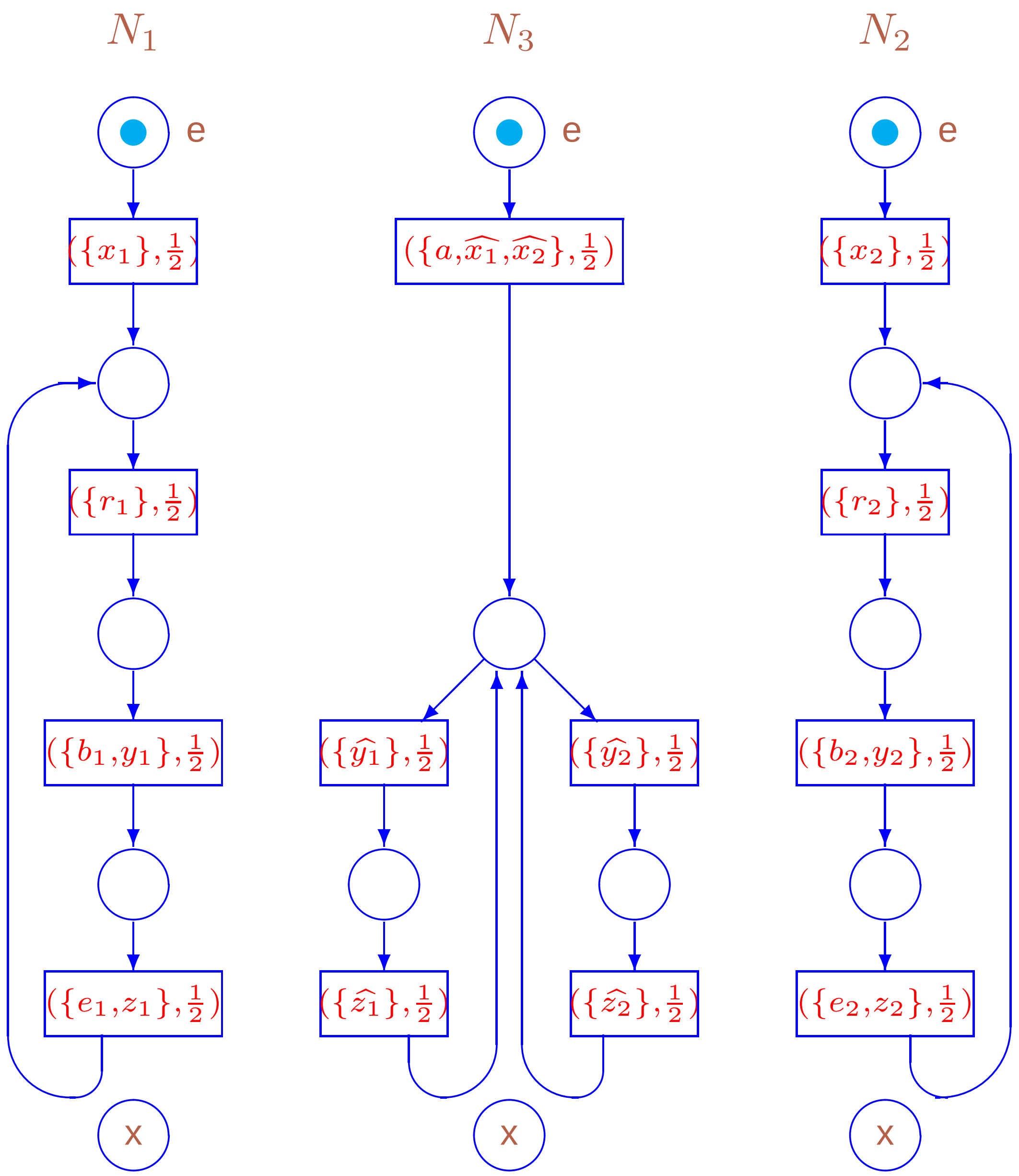
  The steady-state probability that the memory is available is $\psi_2^* + \psi_3^* + \psi_4^* + \psi_6^* = \frac{124}{209}$.

  The steady-state probability that the memory is used, the *shared memory utilization*, is $1 - \frac{124}{209} = \frac{85}{209}$.
- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is only possible from the states s_2, s_4, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

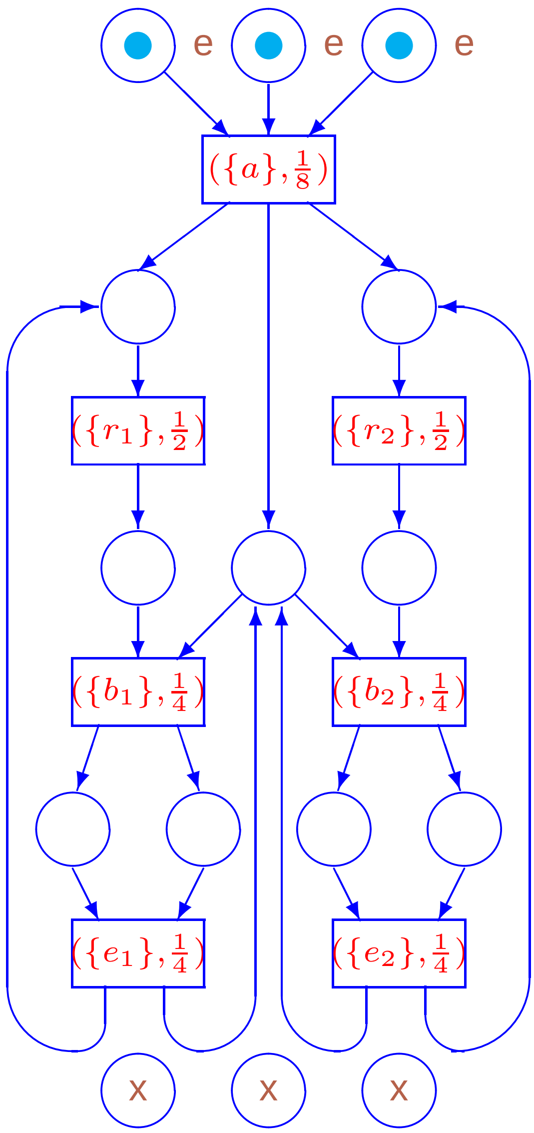
  The *steady-state probability of the shared memory request from the first processor* is

  $$\psi_2^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_4^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_7^* \sum_{\{\Gamma|(\{r_1\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) = \frac{3}{209}\left(\frac{1}{3} + \frac{1}{3}\right) + \frac{75}{418}\left(\frac{3}{5} + \frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5} + \frac{1}{5}\right) = \frac{38}{209}.$$



The marked dts-boxes of two processors and shared memory

N



The marked dts-box of the shared memory system

The abstract system

The static expression of the first processor is
$F_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_1\}, \frac{1}{2}); (\{e, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is
$F_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{b, y_2\}, \frac{1}{2}); (\{e, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is $F_3 =$
$[(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, \frac{1}{2}); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, \frac{1}{2}); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract shared memory system with two processors is $F = (F_1 \| F_2 \| F_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$.

$DTMC^*(\overline{F}) \simeq DTMC^*(\overline{E})$, thus, the TPM and the steady-state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

Performance indices

The first and second performance indices are the same for the standard and abstract systems.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \frac{1}{2})$ is only possible from the states s_2, s_3, s_4, s_5, s_7.

  The request probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{r\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from a processor* is $\psi_2^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_3) + \psi_4^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_4) + \psi_5^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_5) + \psi_7^* \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT^*(\Gamma, s_7) = \frac{3}{209}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) + \frac{75}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{75}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5}+\frac{1}{5}\right) + \frac{15}{418}\left(\frac{3}{5}+\frac{1}{5}\right) = \frac{75}{209}$.

The quotient of the abstract system

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6\}$, where

$\mathcal{K}_1 = \{s_1\}$ (the initial state),

$\mathcal{K}_2 = \{s_2\}$ (the system is activated and the memory is not requested),

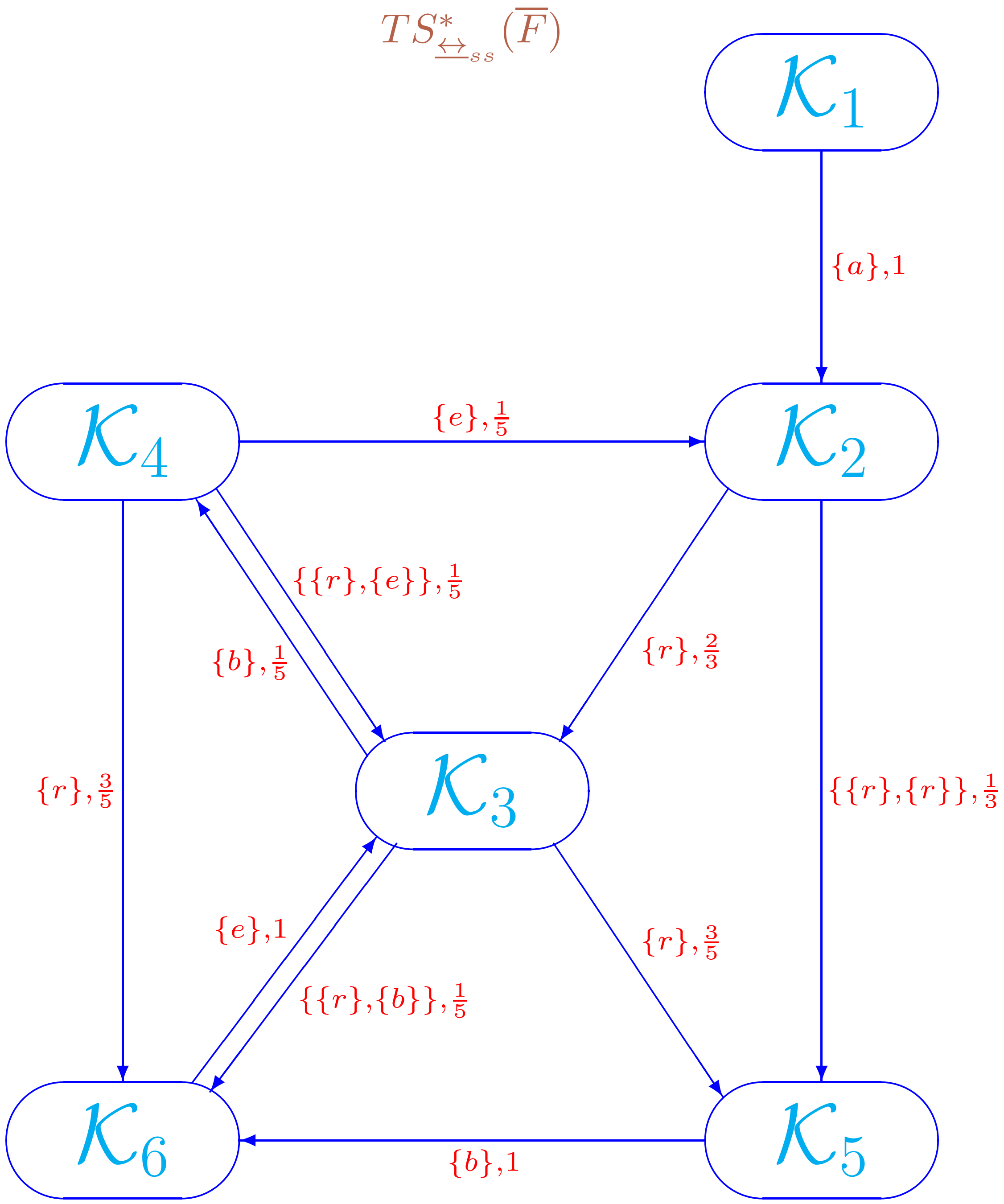
$\mathcal{K}_3 = \{s_3, s_4\}$ (the memory is requested by one processor),

$\mathcal{K}_4 = \{s_5, s_7\}$ (the memory is allocated to a processor),

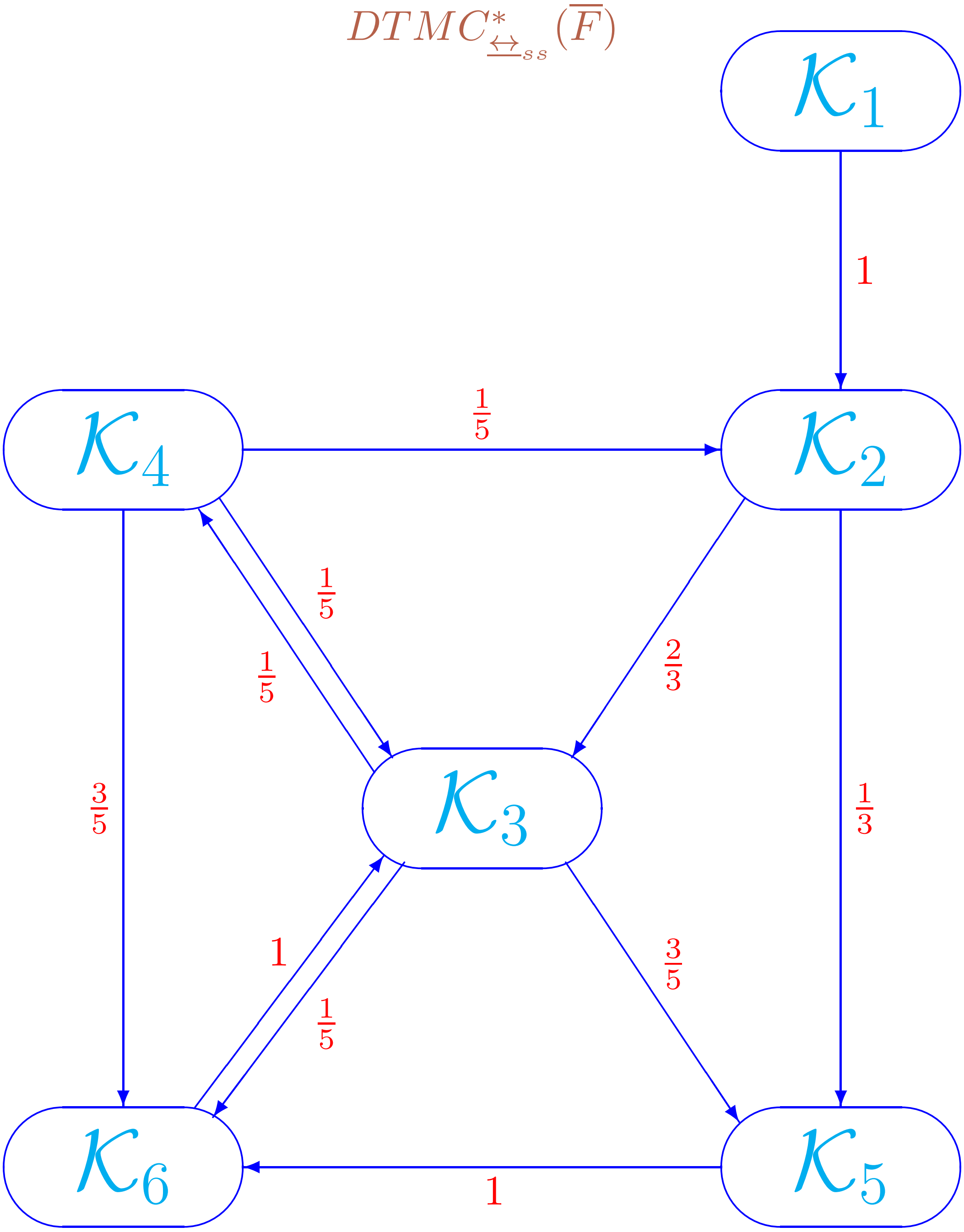
$\mathcal{K}_5 = \{s_6\}$ (the memory is requested by two processors),

$\mathcal{K}_6 = \{s_8, s_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6\}$ and
$DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3, \mathcal{K}_5\}$.



The quotient transition system without empty loops of the abstract shared memory system



The quotient underlying DTMC without empty loops of the abstract shared memory system

The TPM for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

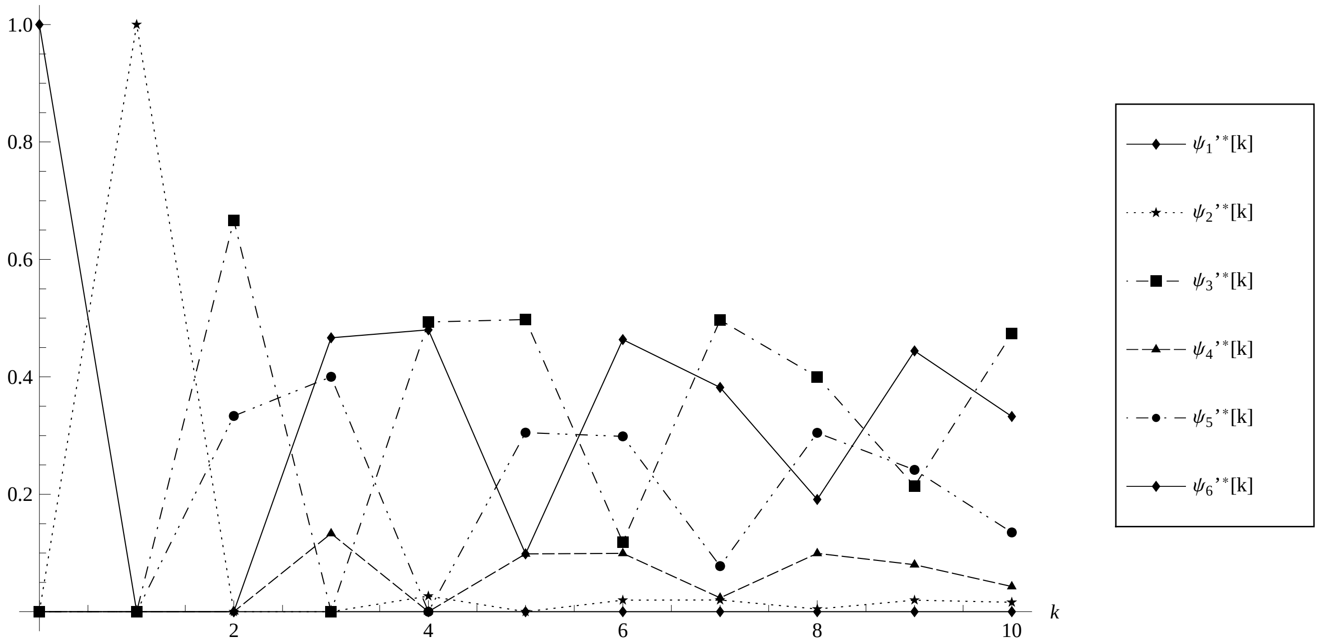
$$\mathbf{P}'^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF ψ'^* for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\psi'^* = \left(0, \frac{3}{209}, \frac{75}{209}, \frac{15}{209}, \frac{46}{209}, \frac{70}{209}\right).$$

Transient and steady-state probabilities of the quotient abstract shared memory system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1'^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2'^*[k]$	0	1	0	0	0.0267	0	0.0197	0.0199	0.0047	0.0199	0.0160	0.0144
$\psi_3'^*[k]$	0	0	0.6667	0	0.4933	0.4978	0.1184	0.4967	0.4001	0.2142	0.4735	0.3589
$\psi_4'^*[k]$	0	0	0	0.1333	0	0.0987	0.0996	0.0237	0.0993	0.0800	0.0428	0.0718
$\psi_5'^*[k]$	0	0	0.3333	0.4000	0	0.3049	0.2987	0.0776	0.3047	0.2416	0.1351	0.2201
$\psi_6'^*[k]$	0	0	0	0.4667	0.4800	0.0987	0.4636	0.3821	0.1912	0.4443	0.3325	0.3349



Transient probabilities alteration diagram of the quotient abstract shared memory system

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{{\psi'_2}^*} = \frac{209}{3} = 69\frac{2}{3}$.
- The common memory is available in the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5$ only.

  The steady-state probability that the memory is available is
  ${\psi'_2}^* + {\psi'_3}^* + {\psi'_5}^* = \frac{3}{209} + \frac{75}{209} + \frac{46}{209} = \frac{124}{209}$.
  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{124}{209} = \frac{85}{209}$.
- The common memory request of a processor $\{r\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$.

  The request probability in each of the states is a sum of execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is ${\psi'_2}^* \sum_{\{A,\mathcal{K}|\{r\}\in A,\ \mathcal{K}_2 \stackrel{A}{\to} \mathcal{K}\}} PM_A^*(\mathcal{K}_2, \mathcal{K}) +$
  ${\psi'_3}^* \sum_{\{A,\mathcal{K}|\{r\}\in A,\ \mathcal{K}_3 \stackrel{A}{\to} \mathcal{K}\}} PM_A^*(\mathcal{K}_3, \mathcal{K}) +$
  ${\psi'_4}^* \sum_{\{A,\mathcal{K}|\{r\}\in A,\ \mathcal{K}_4 \stackrel{A}{\to} \mathcal{K}\}} PM_A^*(\mathcal{K}_4, \mathcal{K}) =$
  $\frac{3}{209}\left(\frac{2}{3} + \frac{1}{3}\right) + \frac{75}{209}\left(\frac{3}{5} + \frac{1}{5}\right) + \frac{15}{209}\left(\frac{3}{5} + \frac{1}{5}\right) = \frac{75}{209}$.

The performance indices are the same for the complete and quotient abstract shared memory systems.

The coincidence of the first and second performance indices illustrates the proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces
$\{\{r\}\}, \{\{r\},\{r\}\}, \{\{r\},\{b\}\}, \{\{r\},\{e\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

The generalized system

The static expression of the first processor is
$K_1 = [(\{x_1\},\rho) * ((\{r_1\},\rho);(\{b_1,y_1\},\rho);(\{e_1,z_1\},\rho)) * \mathsf{Stop}]$.

The static expression of the second processor is
$K_2 = [(\{x_2\},\rho) * ((\{r_2\},\rho);(\{b_2,y_2\},\rho);(\{e_2,z_2\},\rho)) * \mathsf{Stop}]$.

The static expression of the shared memory is
$K_3 = [(\{a,\widehat{x_1},\widehat{x_2}\},\rho) * (((\{\widehat{y_1}\},\rho);(\{\widehat{z_1}\},\rho))[]((\{\widehat{y_2}\},\rho);(\{\widehat{z_2}\},\rho)))* \mathsf{Stop}]$.

The static expression of the generalized shared memory system with two processors is
$K = (K_1 \| K_2 \| K_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

Interpretation of the states

$\tilde{s}_1$: the initial state,

$\tilde{s}_2$: the system is activated and the memory is not requested,

$\tilde{s}_3$: the memory is requested by the first processor,

$\tilde{s}_4$: the memory is requested by the second processor,

$\tilde{s}_5$: the memory is allocated to the first processor,

$\tilde{s}_6$: the memory is requested by two processors,

$\tilde{s}_7$: the memory is allocated to the second processor,

$\tilde{s}_8$: the memory is allocated to the first processor and the memory is requested by the second processor,

$\tilde{s}_9$: the memory is allocated to the second processor and the memory is requested by the first processor.

The TPM for $DTMC^*(\overline{K})$ is $\widetilde{\mathbf{P}}^* =$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\rho}{2-\rho} & \frac{1-\rho}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{1-\rho^2}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The steady-state PMF for $DTMC^*(\overline{K})$ is

$$\tilde{\psi}^* = \frac{1}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}(0, 2\rho^2(2-\rho)(1-\rho)^2,$$
$$(2-p)(1-p+p^2)^2, (2-p)(1-p+p^2)^2, \rho(2-\rho-4\rho^2+4\rho^3-\rho^4),$$
$$2(2+\rho-5\rho^2+\rho^3+\rho^4), \rho(2-\rho-4\rho^2+4\rho^3-\rho^4),$$
$$2+3\rho-6\rho^2+\rho^3+\rho^4, 2+3\rho-6\rho^2+\rho^3+\rho^4).$$

Performance indices

- The average recurrence time in the state $\tilde{s}_2$, where no processor requests the memory, the *average system run-through*, is
$\frac{1}{\tilde{\psi}_2^*} = \frac{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}{\rho^2(2-\rho)(1-\rho)^2}$.

- The common memory is available only in the states $\tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_6$.

  The steady-state probability that the memory is available is
  $\tilde{\psi}_2^* + \tilde{\psi}_3^* + \tilde{\psi}_4^* + \tilde{\psi}_6^* =$
  $\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} + \frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)} +$
  $\frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)} + \frac{2+\rho-5\rho^2+\rho^3+\rho^4}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} =$
  $\frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is
  $1 - \frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} = \frac{2+5\rho-7\rho^2-3\rho^3+5\rho^4-\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

- The common memory request of the first processor $(\{r_1\}, \rho)$ is only possible from the states $\tilde{s}_2, \tilde{s}_4, \tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r_1\}, \rho)$.

  The *steady-state probability of the shared memory request from the first processor* is
  $\tilde{\psi}_2^* \sum_{\{\Gamma|(\{r_1\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_2) +$
  $\tilde{\psi}_4^* \sum_{\{\Gamma|(\{r_1\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_4) +$
  $\tilde{\psi}_7^* \sum_{\{\Gamma|(\{r_1\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_7) =$
  $\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho}{2-\rho} + \frac{\rho}{2-\rho}\right) +$
  $\frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2} + \frac{\rho^2}{1+\rho-\rho^2}\right) +$
  $\frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2} + \frac{\rho^2}{1+\rho-\rho^2}\right) =$
  $\frac{2+3\rho-4\rho^2-2\rho^3+2\rho^4}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}$.

The abstract generalized system and its reduction

The static expression of the first processor is

$L_1 = [(\{x_1\}, \rho) * ((\{r\}, \rho); (\{b, y_1\}, \rho); (\{e, z_1\}, \rho)) * \mathsf{Stop}]$.

The static expression of the second processor is

$L_2 = [(\{x_2\}, \rho) * ((\{r\}, \rho); (\{b, y_2\}, \rho); (\{e, z_2\}, \rho)) * \mathsf{Stop}]$.

The static expression of the shared memory is

$L_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \rho) * (((\{\widehat{y_1}\}, \rho); (\{\widehat{z_1}\}, \rho))[]((\{\widehat{y_2}\}, \rho); (\{\widehat{z_2}\}, \rho)))* \mathsf{Stop}]$.

The static expression of the abstract shared memory generalized system with two processors is

$L = (L_1 \| L_2 \| L_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{L})$ resembles $DR(\overline{K})$, and $TS^*(\overline{L})$ is similar to $TS^*(\overline{K})$.

$DTMC^*(\overline{L}) \simeq DTMC^*(\overline{K})$, thus, the TPM and the steady-state PMF for $DTMC^*(\overline{L})$ and $DTMC^*(\overline{K})$ coincide.

Performance indices

The first and second performance indices are the same for the generalized system and its abstraction.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \rho)$ is only possible from the states $\tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5, \tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r\}, \rho)$.

  The *steady-state probability of the shared memory request from a processor* is $\tilde{\psi}_2^* \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_2) + \tilde{\psi}_3^* \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_3) + \tilde{\psi}_4^* \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_4) + \tilde{\psi}_5^* \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_5) + \tilde{\psi}_7^* \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT^*(\Gamma, \tilde{s}_7) =$
  $\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho}{2-\rho}+\frac{1-\rho}{2-\rho}+\frac{\rho}{2-\rho}\right)+$
  $\frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)+$
  $\frac{(2-\rho)(1+\rho-\rho^2)^2}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)+$
  $\frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)+$
  $\frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{2(6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5)}\left(\frac{1-\rho^2}{1+\rho-\rho^2}+\frac{\rho^2}{1+\rho-\rho^2}\right)=$
  $\frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

The quotient of the abstract system

$DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_5, \widetilde{\mathcal{K}}_6\}$, where

$\widetilde{\mathcal{K}}_1 = \{\tilde{s}_1\}$ (the initial state),

$\widetilde{\mathcal{K}}_2 = \{\tilde{s}_2\}$ (the system is activated and the memory is not requested),

$\widetilde{\mathcal{K}}_3 = \{\tilde{s}_3, \tilde{s}_4\}$ (the memory is requested by one processor),

$\widetilde{\mathcal{K}}_4 = \{\tilde{s}_5, \tilde{s}_7\}$ (the memory is allocated to a processor),

$\widetilde{\mathcal{K}}_5 = \{\tilde{s}_6\}$ (the memory is requested by two processors),

$\widetilde{\mathcal{K}}_6 = \{\tilde{s}_8, \tilde{s}_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).

$DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6\}$ and $DR_V(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5\}$.

The TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}^{*}(\overline{L})$ is

$$\widetilde{\mathbf{P}}'^{*} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2(1-\rho)}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 \\ 0 & 0 & 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{1-\rho^2}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}^{*}(\overline{L})$ is

$$\tilde{\psi}'^{*} = \frac{1}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}(0, \rho^2(2-\rho)(1-\rho)^2,$$
$$(2-\rho)(1+\rho-\rho^2)^2, \rho(2-\rho-4\rho^2+4\rho^3-\rho^4),$$
$$2+\rho-5\rho^2+\rho^3+\rho^4, 2+3\rho-6\rho^2+\rho^3+\rho^4).$$

Performance indices

- The average recurrence time in the state $\widetilde{\mathcal{K}}_2$, where no processor requests the memory, the *average system run-through*, is
$\frac{1}{\tilde{\psi}_2'^*} = \frac{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}{\rho^2(2-\rho)(1-\rho)^2}$.

- The common memory is available only in the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5$.

  The steady-state probability that the memory is available is
$\tilde{\psi}_2'^* + \tilde{\psi}_3'^* + \tilde{\psi}_5'^* =$
$\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} + \frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} +$
$\frac{2+\rho-5\rho^2+\rho^3+\rho^4}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} = \frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is
$1 - \frac{4+4\rho-7\rho^2-7\rho^3+9\rho^4-2\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5} = \frac{2+5\rho-7\rho^2-3\rho^3+5\rho^4-\rho^5}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

- The common memory request of a processor $\{r\}$ is only possible from the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is $\tilde{\psi}_2'^{*} \sum_{\{A,\widetilde{\mathcal{K}}\mid\{r\}\in A,\ \widetilde{\mathcal{K}}_2 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}) +$
  $\tilde{\psi}_3'^{*} \sum_{\{A,\widetilde{\mathcal{K}}\mid\{r\}\in A,\ \widetilde{\mathcal{K}}_3 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}) +$
  $\tilde{\psi}_4'^{*} \sum_{\{A,\widetilde{\mathcal{K}}\mid\{r\}\in A,\ \widetilde{\mathcal{K}}_4 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}) =$
  $\frac{\rho^2(2-\rho)(1-\rho)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{2(1-\rho)}{2-\rho} + \frac{\rho}{2-\rho}\right) +$
  $\frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho^2}{1+\rho-\rho^2} + \frac{\rho^2}{1+\rho-\rho^2}\right) +$
  $\frac{\rho(2-\rho-4\rho^2+4\rho^3-\rho^4)}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}\left(\frac{1-\rho^2}{1+\rho-\rho^2} + \frac{\rho^2}{1+\rho-\rho^2}\right) =$
  $\frac{(2-\rho)(1+\rho-\rho^2)^2}{6+9\rho-14\rho^2-10\rho^3+14\rho^4-3\rho^5}$.

The performance indices are the same for the complete and quotient abstract generalized shared memory systems.

The coincidence of the first and second performance indices illustrates the proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

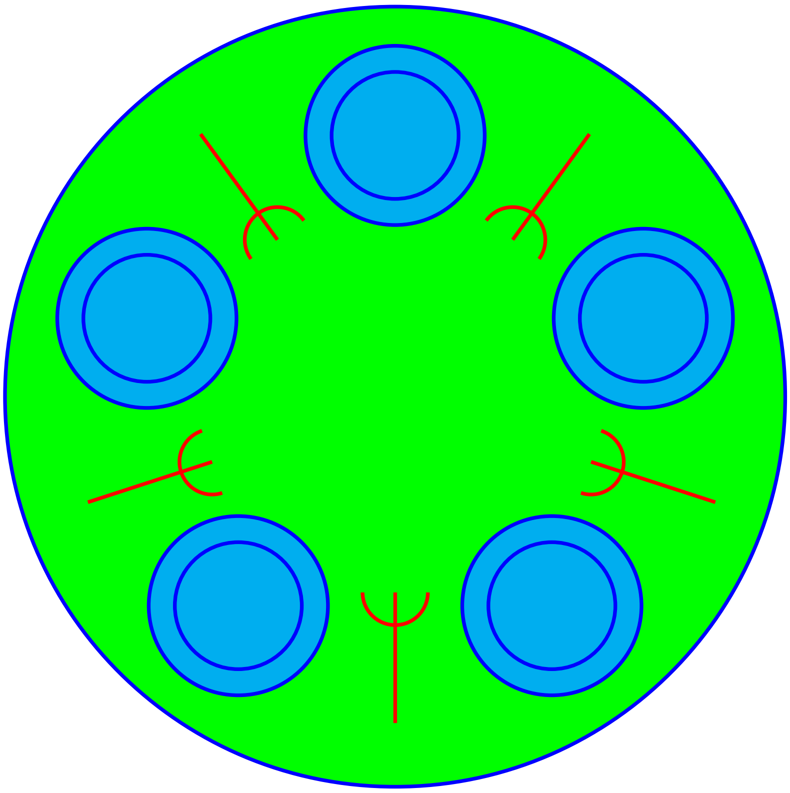
one should apply its result to the derived step traces
$\{\{r\}\}, \{\{r\},\{r\}\}, \{\{r\},\{b\}\}, \{\{r\},\{e\}\}$ of $\overline{L}$ and itself,

and sum the left and right parts of the three resulting equalities.

Dining philosophers system

The standard system

A model of five dining philosophers [P81]



The diagram of the dining philosophers system

After activation of the system (the philosophers come in the dining room), five forks appear on the table.

If the left and right forks available for a philosopher, he takes them simultaneously and begins eating.

At the end of eating, the philosopher places both his forks simultaneously back on the table.

a corresponds to the system activation.

b_i and e_i correspond to the beginning and the end of eating of philosopher i $(1 \leq i \leq 5)$.

The other actions are used for communication purpose only.

The expression of each philosopher includes two alternative subexpressions: the second one specifies a resource (fork) sharing with the right neighbor.

Arbitrary number of philosophers

The most interesting: the maximal sets of philosophers which can dine together.

The system with 1 philosopher: the only maximal set is $\emptyset$.

The system with 2 philosophers: the maximal sets are $\{1\}$, $\{2\}$.

The system with 3 philosophers: the maximal sets are $\{1\}$, $\{2\}$, $\{3\}$.

The system with 4 philosophers: the maximal sets are $\{1,3\}$, $\{2,4\}$.

The system with 5 philosophers: the maximal sets are
$\{1,3\}$, $\{1,4\}$, $\{2,4\}$, $\{2,5\}$, $\{3,5\}$.

The system with 6 philosophers: the maximal sets are
$\{1,4\}$, $\{2,5\}$, $\{3,6\}$, $\{1,3,5\}$, $\{2,4,6\}$.

The system with 7 philosophers: the maximal sets are
$\{1,3,5\}$, $\{1,3,6\}$, $\{1,4,6\}$, $\{2,4,6\}$, $\{2,4,7\}$, $\{2,5,7\}$, $\{3,5,7\}$.

A nontrivial behaviour: at least 5 philosophers occupy the table.

The neighbors cannot dine together: the maximal number of the dining persons for the system with n philosophers will be $\lfloor \frac{n}{2} \rfloor$.

If the philosopher i belongs to some maximal set then the philosopher $i(\text{mod } n) + 1$ belongs to the
next one.

- n is an even number: 2 maximal sets of $\frac{n}{2}$ persons,

  i.e. the philosophers numbered with all odd natural numbers $\leq n$

  and those numbered with all even natural numbers $\leq n$.

- n is an odd number: n maximal sets of $\frac{n-1}{2}$ persons,

  since from a maximal set one can "shift" clockwise $n-1$ times by one element modulo n until the next maximal set will coincide with the initial one.

The static expression of the philosopher i $(1 \leq i \leq 4)$ is
$E_i = [(\{x_i\}, \frac{1}{2}) * ((((\{b_i, \widehat{y_i}\}, \frac{1}{2}); (\{e_i, \widehat{z_i}\}, \frac{1}{2}))[](((\{y_{i+1}\}, \frac{1}{2});$
$(\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is
$E_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * ((((\{b_5, \widehat{y_5}\}, \frac{1}{2}); (\{e_5, \widehat{z_5}\}, \frac{1}{2}))[]$
$((\{y_1\}, \frac{1}{2}); (\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the dining philosophers system is
$E = (E_1 \| E_2 \| E_3 \| E_4 \| E_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

Effect of synchronization

Synchronization of $(\{b_i, y_i\}, \frac{1}{2})$ and $(\{\widehat{y_i}\}, \frac{1}{2})$ produces
$(\{b_i\}, \frac{1}{4})$ $(1 \leq i \leq 5)$.

Synchronization of $(\{e_i, z_i\}, \frac{1}{2})$ and $(\{\widehat{z_i}\}, \frac{1}{2})$ produces
$(\{e_i\}, \frac{1}{4})$ $(1 \leq i \leq 5)$.

Synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{2})$ and $(\{x_1\}, \frac{1}{2})$ produces
$(\{a, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{4})$.

Synchronization of $(\{a, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{4})$ and $(\{x_2\}, \frac{1}{2})$ produces
$(\{a, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{8})$.

Synchronization of $(\{a, \widehat{x_3}, \widehat{x_4}\}, \frac{1}{8})$ and $(\{x_3\}, \frac{1}{2})$ produces $(\{a, \widehat{x_4}\}, \frac{1}{16})$.

Synchronization of $(\{a, \widehat{x_4}\}, \frac{1}{16})$ and $(\{x_4\}, \frac{1}{2})$ produces $(\{a\}, \frac{1}{32})$.

$DR(\overline{E})$ consists of

$s_1 =$

$$[([\overline{(\{x_1\},\tfrac{1}{2})} * (((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2})))* \text{Stop}]\|[\overline{(\{x_2\},\tfrac{1}{2})} * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2})))* \text{Stop}]\|[\overline{(\{x_3\},\tfrac{1}{2})} * (((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2})))* \text{Stop}]\|[\overline{(\{x_4\},\tfrac{1}{2})} * (((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2})))* \text{Stop}]\|[\overline{(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2})} * (((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2}); (\{z_1\},\tfrac{1}{2}))) * \text{Stop}]) \text{ sy } x_1 \text{ sy } x_2 \text{ sy } x_3 \text{ sy } x_4 \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_3 \text{ sy } y_4 \text{ sy } y_5 \text{ sy } z_1 \text{ sy } z_2 \text{ sy } z_3 \text{ sy } z_4 \text{ sy } z_5 \text{ rs } x_1 \text{ rs } x_2 \text{ rs } x_3 \text{ rs } x_4 \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_3 \text{ rs } y_4 \text{ rs } y_5 \text{ rs } z_1 \text{ rs } z_2 \text{ rs } z_3 \text{ rs } z_4 \text{ rs } z_5]_\approx,$$

$s_2 =$

$$[([(\{x_1\},\tfrac{1}{2}) * (\overline{((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2}))})* \text{Stop}]\|[(\{x_2\},\tfrac{1}{2}) * (\overline{((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2}))})* \text{Stop}]\|[(\{x_3\},\tfrac{1}{2}) * (\overline{((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2}))})* \text{Stop}]\|[(\{x_4\},\tfrac{1}{2}) * (\overline{((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2}))})* \text{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * (\overline{((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2}); (\{z_1\},\tfrac{1}{2}))}) * \text{Stop}]) \text{ sy } x_1 \text{ sy } x_2 \text{ sy } x_3 \text{ sy } x_4 \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_3 \text{ sy } y_4 \text{ sy } y_5 \text{ sy } z_1 \text{ sy } z_2 \text{ sy } z_3 \text{ sy } z_4 \text{ sy } z_5 \text{ rs } x_1 \text{ rs } x_2 \text{ rs } x_3 \text{ rs } x_4 \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_3 \text{ rs } y_4 \text{ rs } y_5 \text{ rs } z_1 \text{ rs } z_2 \text{ rs } z_3 \text{ rs } z_4 \text{ rs } z_5]_\approx,$$

$s_3 =$

$$[([(\{x_1\},\tfrac{1}{2}) * (((\{b_1,\widehat{y_1}\},\tfrac{1}{2});\overline{(\{e_1,\widehat{z_1}\},\tfrac{1}{2})})[]((\{y_2\},\tfrac{1}{2});(\{z_2\},\tfrac{1}{2})))* \text{Stop}]\|[(\{x_2\},\tfrac{1}{2}) * (((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[](\overline{(\{y_3\},\tfrac{1}{2})};(\{z_3\},\tfrac{1}{2})))* \text{Stop}]\|[(\{x_3\},\tfrac{1}{2}) * (\overline{((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2}))})* \text{Stop}]\|[(\{x_4\},\tfrac{1}{2}) * ((\overline{(\{b_4,\widehat{y_4}\},\tfrac{1}{2})};(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2})))* \text{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2}) * (((\{b_5,\widehat{y_5}\},\tfrac{1}{2});(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2}); \overline{(\{z_1\},\tfrac{1}{2})})) * \text{Stop}]) \text{ sy } x_1 \text{ sy } x_2 \text{ sy } x_3 \text{ sy } x_4 \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_3 \text{ sy } y_4 \text{ sy } y_5 \text{ sy } z_1 \text{ sy } z_2 \text{ sy } z_3 \text{ sy } z_4 \text{ sy } z_5 \text{ rs } x_1 \text{ rs } x_2 \text{ rs } x_3 \text{ rs } x_4 \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_3 \text{ rs } y_4 \text{ rs } y_5 \text{ rs } z_1 \text{ rs } z_2 \text{ rs } z_3 \text{ rs } z_4 \text{ rs } z_5]_\approx,$$

$s_4 =$

$[(([(\{x_1\}, \frac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \frac{1}{2}); \overline{(\{e_1, \widehat{z_1}\}, \frac{1}{2})})[]((\{y_2\}, \frac{1}{2}); (\{z_2\}, \frac{1}{2})))*$
$\mathsf{Stop}] \| [(\{x_2\}, \frac{1}{2}) * \overline{(((\{b_2, \widehat{y_2}\}, \frac{1}{2}); (\{e_2, \widehat{z_2}\}, \frac{1}{2}))[]((\{y_3\}, \frac{1}{2}); (\{z_3\}, \frac{1}{2})))}*$
$\mathsf{Stop}] \| [(\{x_3\}, \frac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \frac{1}{2}); (\{e_3, \widehat{z_3}\}, \frac{1}{2}))[]((\{y_4\}, \frac{1}{2}); \overline{(\{z_4\}, \frac{1}{2})}))*$
$\mathsf{Stop}] \| [(\{x_4\}, \frac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \frac{1}{2}); \overline{(\{e_4, \widehat{z_4}\}, \frac{1}{2})})[]((\{y_5\}, \frac{1}{2}); (\{z_5\}, \frac{1}{2})))*$
$\mathsf{Stop}] \| [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \frac{1}{2}); (\{e_5, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2});$
$\overline{(\{z_1\}, \frac{1}{2})})) * \mathsf{Stop}])$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5
sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4
rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs $z_5]_\approx$,

$s_5 =$

$[(([(\{x_1\}, \frac{1}{2}) * (((\{b_1, \widehat{y_1}\}, \frac{1}{2}); \overline{(\{e_1, \widehat{z_1}\}, \frac{1}{2})})[]((\{y_2\}, \frac{1}{2}); (\{z_2\}, \frac{1}{2})))*$
$\mathsf{Stop}] \| [(\{x_2\}, \frac{1}{2}) * (((\{b_2, \widehat{y_2}\}, \frac{1}{2}); (\{e_2, \widehat{z_2}\}, \frac{1}{2}))[]((\{y_3\}, \frac{1}{2}); \overline{(\{z_3\}, \frac{1}{2})}))*$
$\mathsf{Stop}] \| [(\{x_3\}, \frac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \frac{1}{2}); \overline{(\{e_3, \widehat{z_3}\}, \frac{1}{2})})[]((\{y_4\}, \frac{1}{2}); (\{z_4\}, \frac{1}{2})))*$
$\mathsf{Stop}] \| [(\{x_4\}, \frac{1}{2}) * \overline{(((\{b_4, \widehat{y_4}\}, \frac{1}{2}); (\{e_4, \widehat{z_4}\}, \frac{1}{2}))[]((\{y_5\}, \frac{1}{2}); (\{z_5\}, \frac{1}{2})))}*$
$\mathsf{Stop}] \| [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \frac{1}{2}); (\{e_5, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2});$
$\overline{(\{z_1\}, \frac{1}{2})})) * \mathsf{Stop}])$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5
sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4
rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs $z_5]_\approx$,

$s_6 =$

$[(([(\{x_1\}, \frac{1}{2}) * \overline{(((\{b_1, \widehat{y_1}\}, \frac{1}{2}); (\{e_1, \widehat{z_1}\}, \frac{1}{2}))[]((\{y_2\}, \frac{1}{2}); (\{z_2\}, \frac{1}{2})))}*$
$\mathsf{Stop}] \| [(\{x_2\}, \frac{1}{2}) * (((\overline{\{b_2, \widehat{y_2}\}, \frac{1}{2}}); (\{e_2, \widehat{z_2}\}, \frac{1}{2}))[]((\{y_3\}, \frac{1}{2}); (\{z_3\}, \frac{1}{2})))*$
$\mathsf{Stop}] \| [(\{x_3\}, \frac{1}{2}) * (((\{b_3, \widehat{y_3}\}, \frac{1}{2}); (\{e_3, \widehat{z_3}\}, \frac{1}{2}))[]((\{y_4\}, \frac{1}{2}); \overline{(\{z_4\}, \frac{1}{2})}))*$
$\mathsf{Stop}] \| [(\{x_4\}, \frac{1}{2}) * (((\{b_4, \widehat{y_4}\}, \frac{1}{2}); \overline{(\{e_4, \widehat{z_4}\}, \frac{1}{2})})[]((\{y_5\}, \frac{1}{2}); (\{z_5\}, \frac{1}{2})))*$
$\mathsf{Stop}] \| [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * (((\{b_5, \widehat{y_5}\}, \frac{1}{2}); (\{e_5, \widehat{z_5}\}, \frac{1}{2}))[]\overline{(\{y_1\}, \frac{1}{2})};$
$((\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}])$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5
sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4
rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs $z_5]_\approx$,

$s_7 =$

$$[([(\{x_1\},\frac{1}{2}) * ((\overline{(\{b_1,\widehat{y_1}\},\frac{1}{2})};(\{e_1,\widehat{z_1}\},\frac{1}{2}))[\,]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2})))*$$
$$\text{Stop}]\|[(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[\,]((\{y_3\},\frac{1}{2});\overline{(\{z_3\},\frac{1}{2})}))*$$
$$\text{Stop}]\|[(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});\overline{(\{e_3,\widehat{z_3}\},\frac{1}{2})})[\,]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2})))*$$
$$\text{Stop}]\|[(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[\,](\overline{(\{y_5\},\frac{1}{2})};(\{z_5\},\frac{1}{2})))*$$
$$\text{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * \overline{(((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[\,]((\{y_1\},\frac{1}{2});}$$
$$\overline{(\{z_1\},\frac{1}{2}))}) * \text{Stop}]) \text{ sy } x_1 \text{ sy } x_2 \text{ sy } x_3 \text{ sy } x_4 \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_3 \text{ sy } y_4 \text{ sy } y_5$$
$$\text{sy } z_1 \text{ sy } z_2 \text{ sy } z_3 \text{ sy } z_4 \text{ sy } z_5 \text{ rs } x_1 \text{ rs } x_2 \text{ rs } x_3 \text{ rs } x_4 \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_3 \text{ rs } y_4$$
$$\text{rs } y_5 \text{ rs } z_1 \text{ rs } z_2 \text{ rs } z_3 \text{ rs } z_4 \text{ rs } z_5]_\approx,$$

$s_8 =$

$$[([(\{x_1\},\frac{1}{2}) * (((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[\,]((\{y_2\},\frac{1}{2});\overline{(\{z_2\},\frac{1}{2})}))*$$
$$\text{Stop}]\|[(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});\overline{(\{e_2,\widehat{z_2}\},\frac{1}{2})})[\,]((\{y_3\},\frac{1}{2});(\{z_3\},\frac{1}{2})))*$$
$$\text{Stop}]\|[(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});(\{e_3,\widehat{z_3}\},\frac{1}{2}))[\,]((\{y_4\},\frac{1}{2});\overline{(\{z_4\},\frac{1}{2})}))*$$
$$\text{Stop}]\|[(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});\overline{(\{e_4,\widehat{z_4}\},\frac{1}{2})})[\,]((\{y_5\},\frac{1}{2});(\{z_5\},\frac{1}{2})))*$$
$$\text{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * \overline{(((\{b_5,\widehat{y_5}\},\frac{1}{2});(\{e_5,\widehat{z_5}\},\frac{1}{2}))[\,]((\{y_1\},\frac{1}{2});}$$
$$\overline{(\{z_1\},\frac{1}{2}))}) * \text{Stop}]) \text{ sy } x_1 \text{ sy } x_2 \text{ sy } x_3 \text{ sy } x_4 \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_3 \text{ sy } y_4 \text{ sy } y_5$$
$$\text{sy } z_1 \text{ sy } z_2 \text{ sy } z_3 \text{ sy } z_4 \text{ sy } z_5 \text{ rs } x_1 \text{ rs } x_2 \text{ rs } x_3 \text{ rs } x_4 \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_3 \text{ rs } y_4$$
$$\text{rs } y_5 \text{ rs } z_1 \text{ rs } z_2 \text{ rs } z_3 \text{ rs } z_4 \text{ rs } z_5]_\approx,$$

$s_9 =$

$$[([(\{x_1\},\frac{1}{2}) * (\overline{((\{b_1,\widehat{y_1}\},\frac{1}{2});(\{e_1,\widehat{z_1}\},\frac{1}{2}))[\,]((\{y_2\},\frac{1}{2});(\{z_2\},\frac{1}{2}))})*$$
$$\text{Stop}]\|[(\{x_2\},\frac{1}{2}) * (((\{b_2,\widehat{y_2}\},\frac{1}{2});(\{e_2,\widehat{z_2}\},\frac{1}{2}))[\,]((\{y_3\},\frac{1}{2});\overline{(\{z_3\},\frac{1}{2})}))*$$
$$\text{Stop}]\|[(\{x_3\},\frac{1}{2}) * (((\{b_3,\widehat{y_3}\},\frac{1}{2});\overline{(\{e_3,\widehat{z_3}\},\frac{1}{2})})[\,]((\{y_4\},\frac{1}{2});(\{z_4\},\frac{1}{2})))*$$
$$\text{Stop}]\|[(\{x_4\},\frac{1}{2}) * (((\{b_4,\widehat{y_4}\},\frac{1}{2});(\{e_4,\widehat{z_4}\},\frac{1}{2}))[\,]((\{y_5\},\frac{1}{2});\overline{(\{z_5\},\frac{1}{2})}))*$$
$$\text{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\frac{1}{2}) * (((\{b_5,\widehat{y_5}\},\frac{1}{2});\overline{(\{e_5,\widehat{z_5}\},\frac{1}{2})})[\,]((\{y_1\},\frac{1}{2});$$
$$(\{z_1\},\frac{1}{2}))) * \text{Stop}]) \text{ sy } x_1 \text{ sy } x_2 \text{ sy } x_3 \text{ sy } x_4 \text{ sy } y_1 \text{ sy } y_2 \text{ sy } y_3 \text{ sy } y_4 \text{ sy } y_5$$
$$\text{sy } z_1 \text{ sy } z_2 \text{ sy } z_3 \text{ sy } z_4 \text{ sy } z_5 \text{ rs } x_1 \text{ rs } x_2 \text{ rs } x_3 \text{ rs } x_4 \text{ rs } y_1 \text{ rs } y_2 \text{ rs } y_3 \text{ rs } y_4$$
$$\text{rs } y_5 \text{ rs } z_1 \text{ rs } z_2 \text{ rs } z_3 \text{ rs } z_4 \text{ rs } z_5]_\approx,$$

$s_{10} =$

$$[([(\{x_1\},\tfrac{1}{2})*((((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});\overline{(\{z_2\},\tfrac{1}{2})}))*$$
$$\mathsf{Stop}]\|[(\{x_2\},\tfrac{1}{2})*((((\{b_2,\widehat{y_2}\},\tfrac{1}{2});\overline{(\{e_2,\widehat{z_2}\},\tfrac{1}{2})})[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2})))*$$
$$\mathsf{Stop}]\|[(\{x_3\},\tfrac{1}{2})*((((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[](\overline{(\{y_4\},\tfrac{1}{2})};(\{z_4\},\tfrac{1}{2})))*$$
$$\mathsf{Stop}]\|[(\{x_4\},\tfrac{1}{2})*\overline{((((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});(\{z_5\},\tfrac{1}{2})))}*$$
$$\mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2})*(((\overline{(\{b_5,\widehat{y_5}\},\tfrac{1}{2})};(\{e_5,\widehat{z_5}\},\tfrac{1}{2}))[]((\{y_1\},\tfrac{1}{2});$$
$$(\{z_1\},\tfrac{1}{2})))*\mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5$$
$$\mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4$$
$$\mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_\approx,$$

$s_{11} =$

$$[([(\{x_1\},\tfrac{1}{2})*((((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[](\overline{(\{y_2\},\tfrac{1}{2})};(\{z_2\},\tfrac{1}{2})))*$$
$$\mathsf{Stop}]\|[(\{x_2\},\tfrac{1}{2})*\overline{((((\{b_2,\widehat{y_2}\},\tfrac{1}{2});(\{e_2,\widehat{z_2}\},\tfrac{1}{2}))[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2})))}*$$
$$\mathsf{Stop}]\|[(\{x_3\},\tfrac{1}{2})*(((\overline{(\{b_3,\widehat{y_3}\},\tfrac{1}{2})};(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2})))*$$
$$\mathsf{Stop}]\|[(\{x_4\},\tfrac{1}{2})*((((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});\overline{(\{z_5\},\tfrac{1}{2})}))*$$
$$\mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2})*((((\{b_5,\widehat{y_5}\},\tfrac{1}{2});\overline{(\{e_5,\widehat{z_5}\},\tfrac{1}{2})})[]((\{y_1\},\tfrac{1}{2});$$
$$(\{z_1\},\tfrac{1}{2})))*\mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5$$
$$\mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4$$
$$\mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_\approx,$$

$s_{12} =$

$$[([(\{x_1\},\tfrac{1}{2})*((((\{b_1,\widehat{y_1}\},\tfrac{1}{2});(\{e_1,\widehat{z_1}\},\tfrac{1}{2}))[]((\{y_2\},\tfrac{1}{2});\overline{(\{z_2\},\tfrac{1}{2})}))*$$
$$\mathsf{Stop}]\|[(\{x_2\},\tfrac{1}{2})*((((\{b_2,\widehat{y_2}\},\tfrac{1}{2});\overline{(\{e_2,\widehat{z_2}\},\tfrac{1}{2})})[]((\{y_3\},\tfrac{1}{2});(\{z_3\},\tfrac{1}{2})))*$$
$$\mathsf{Stop}]\|[(\{x_3\},\tfrac{1}{2})*\overline{((((\{b_3,\widehat{y_3}\},\tfrac{1}{2});(\{e_3,\widehat{z_3}\},\tfrac{1}{2}))[]((\{y_4\},\tfrac{1}{2});(\{z_4\},\tfrac{1}{2})))}*$$
$$\mathsf{Stop}]\|[(\{x_4\},\tfrac{1}{2})*((((\{b_4,\widehat{y_4}\},\tfrac{1}{2});(\{e_4,\widehat{z_4}\},\tfrac{1}{2}))[]((\{y_5\},\tfrac{1}{2});\overline{(\{z_5\},\tfrac{1}{2})}))*$$
$$\mathsf{Stop}]\|[(\{a,\widehat{x_1},\widehat{x_2},\widehat{x_2},\widehat{x_4}\},\tfrac{1}{2})*((((\{b_5,\widehat{y_5}\},\tfrac{1}{2});\overline{(\{e_5,\widehat{z_5}\},\tfrac{1}{2})})[]((\{y_1\},\tfrac{1}{2});$$
$$(\{z_1\},\tfrac{1}{2})))*\mathsf{Stop}])\ \mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ x_3\ \mathsf{sy}\ x_4\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ y_3\ \mathsf{sy}\ y_4\ \mathsf{sy}\ y_5$$
$$\mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{sy}\ z_3\ \mathsf{sy}\ z_4\ \mathsf{sy}\ z_5\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ x_3\ \mathsf{rs}\ x_4\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ y_3\ \mathsf{rs}\ y_4$$
$$\mathsf{rs}\ y_5\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2\ \mathsf{rs}\ z_3\ \mathsf{rs}\ z_4\ \mathsf{rs}\ z_5]_\approx.$$

Interpretation of the states

s_1: the initial state,

s_2: the system is activated and no philosophers dine,

s_3: philosopher 1 dines,

s_4: philosophers 1 and 4 dine,

s_5: philosophers 1 and 3 dine,

s_6: philosopher 4 dines,

s_7: philosopher 3 dines,

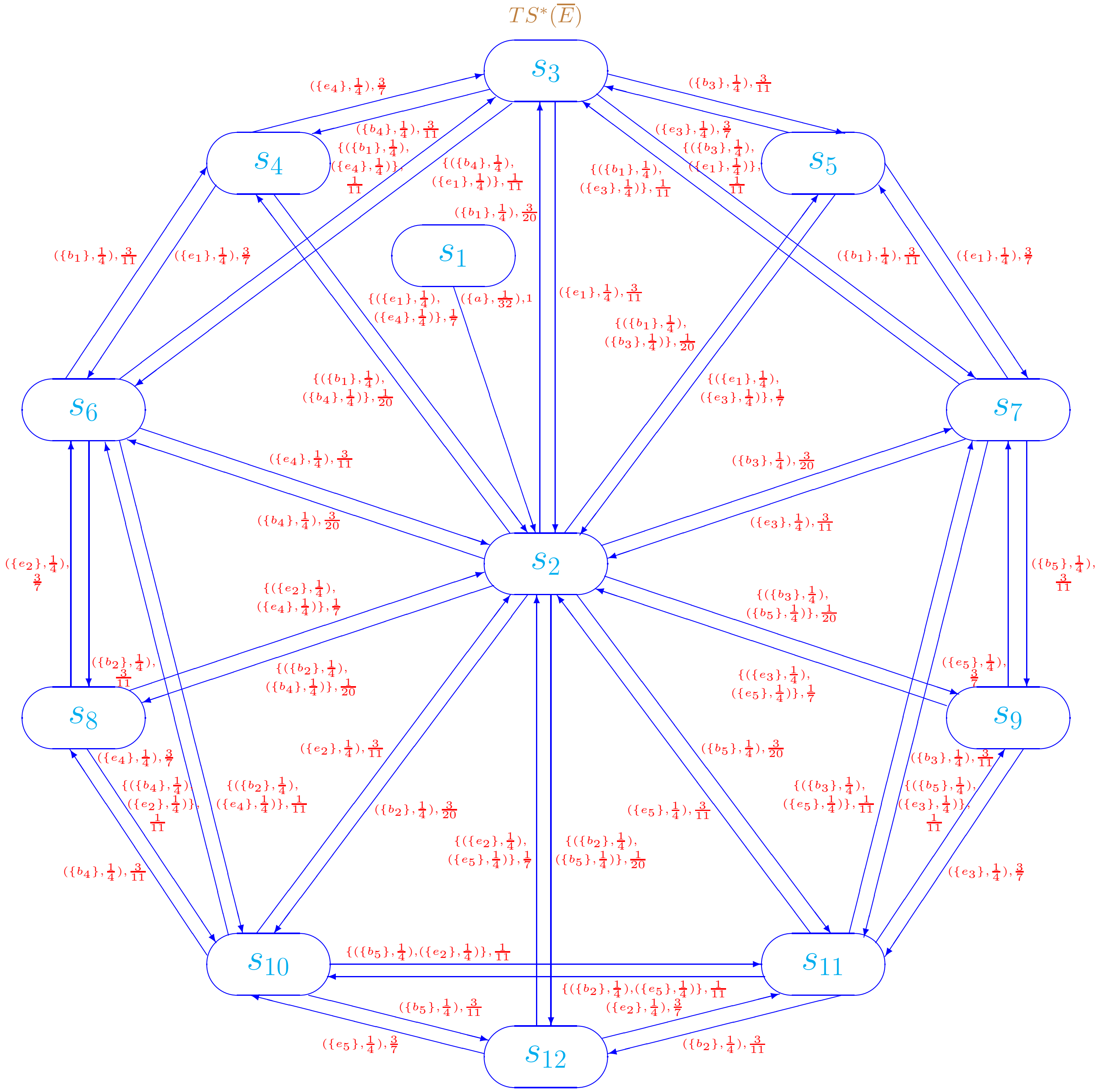
s_8: philosophers 2 and 4 dine,

s_9: philosophers 3 and 5 dine,

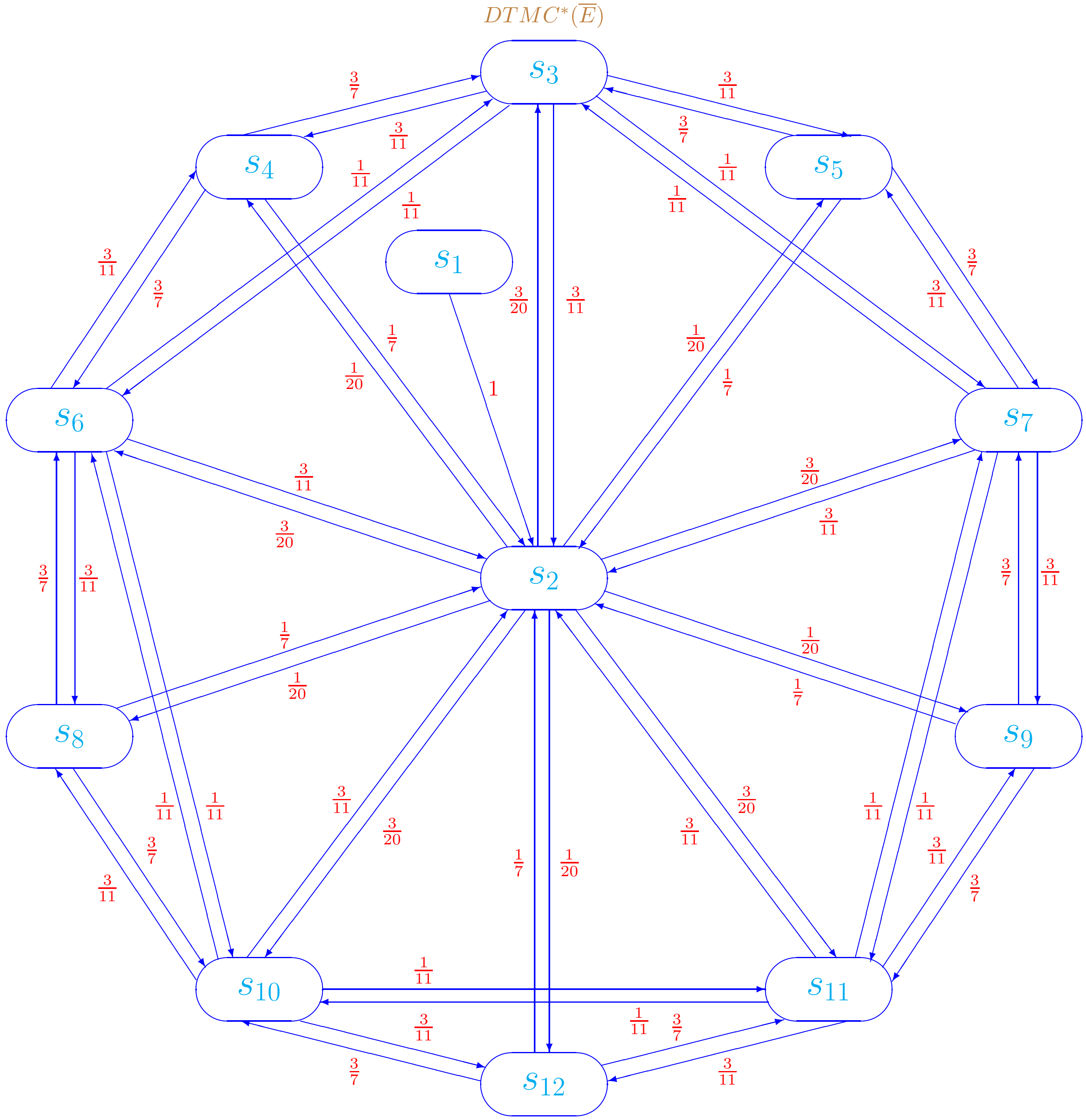
s_{10}: philosopher 2 dines,

s_{11}: philosopher 5 dine,

s_{12}: philosophers 2 and 5 dine.



The transition system without empty loops of the dining philosophers system



The underlying DTMC without empty loops of the dining philosophers system

The TPM for $DTMC^*(\overline{E})$ is

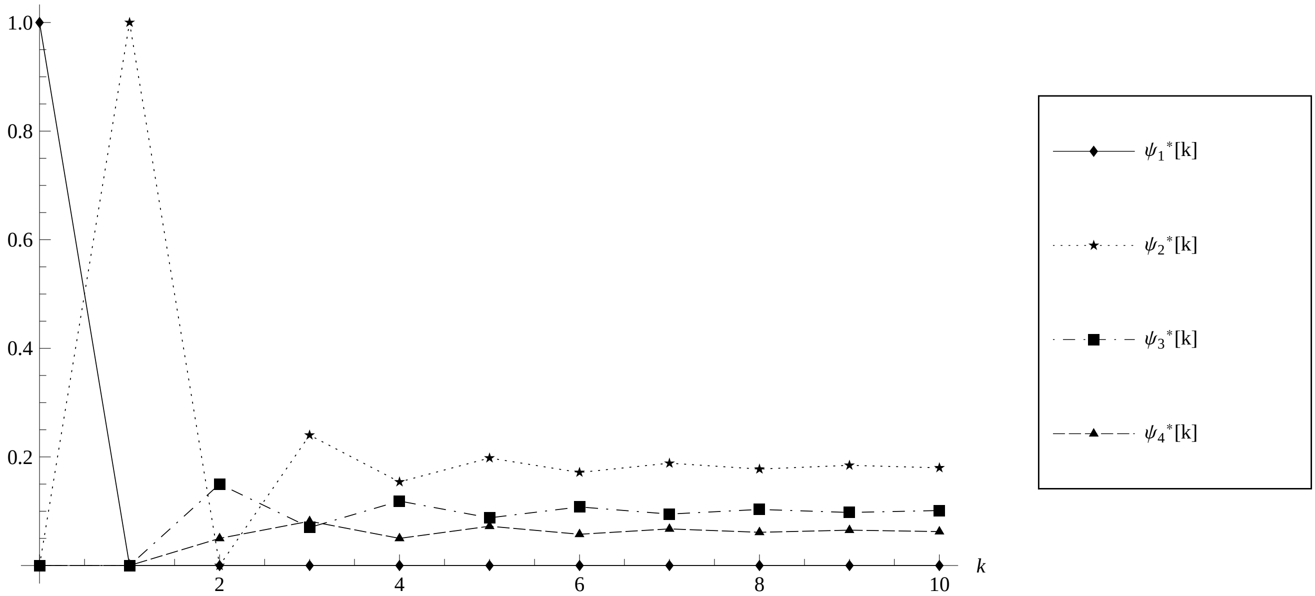
$$\mathbf{P}^* = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} & \frac{1}{20} & \frac{3}{20} & \frac{3}{20} & \frac{1}{20} \\
0 & \frac{3}{11} & 0 & \frac{3}{11} & \frac{3}{11} & \frac{1}{11} & \frac{1}{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{7} & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 & 0 \\
0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 & 0 & \frac{3}{7} & 0 \\
0 & \frac{3}{11} & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & 0 & 0 & \frac{1}{11} & \frac{3}{11} \\
0 & \frac{3}{11} & 0 & 0 & 0 & 0 & \frac{1}{11} & 0 & \frac{3}{11} & \frac{1}{11} & 0 & \frac{3}{11} \\
0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{7} & \frac{3}{7} & 0
\end{pmatrix}.$$

The steady-state PMF for $DTMC^*(\overline{E})$ is

$$\psi^* = \left(0, \frac{2}{11}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}, \frac{7}{110}, \frac{1}{10}, \frac{1}{10}, \frac{7}{110}\right).$$

Transient and steady-state probabilities of the dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
$\psi_3^*[k]$	0	0	0.1500	0.0701	0.1189	0.0878	0.1079	0.0949	0.1033	0.0979	0.1014	0.1000
$\psi_4^*[k]$	0	0	0.0500	0.0818	0.0503	0.0726	0.0578	0.0674	0.0612	0.0652	0.0626	0.0636



Transient probabilities alteration diagram of the dining philosophers system

We depict the probabilities for the states $s_1, \ldots, s_4$ only, since the corresponding values coincide for $s_3, s_6, s_7, s_{10}, s_{11}$ as well as for $s_4, s_5, s_8, s_9, s_{12}$.

Performance indices

- The average recurrence time in the state s_2, where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2^*} = \frac{11}{2} = 5\frac{1}{2}$.
- Nobody eats in the state s_2. The *fraction of time when no philosophers dine* is $\psi_2^* = \frac{2}{11}$.

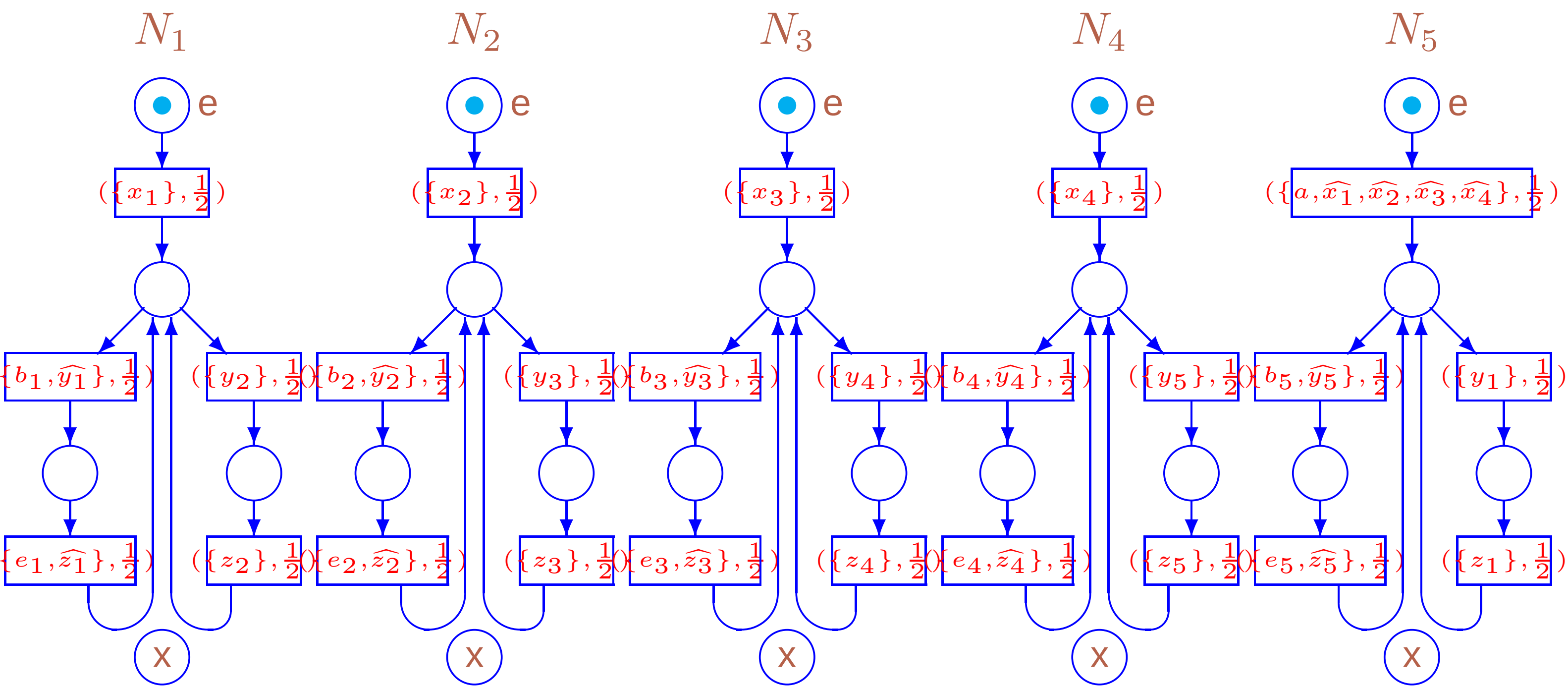
  Only one philosopher eats in the states $s_3, s_6, s_7, s_{10}, s_{11}$. The *fraction of time when only one philosopher dines* is
  $\psi_3^* + \psi_6^* + \psi_7^* + \psi_{10}^* + \psi_{11}^* = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}$.

  Two philosophers eat together in the states $s_4, s_5, s_8, s_9, s_{12}$. The *fraction of time when two philosophers dine* is
  $\psi_4^* + \psi_5^* + \psi_8^* + \psi_9^* + \psi_{12}^* = \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} + \frac{7}{110} = \frac{7}{22}$.

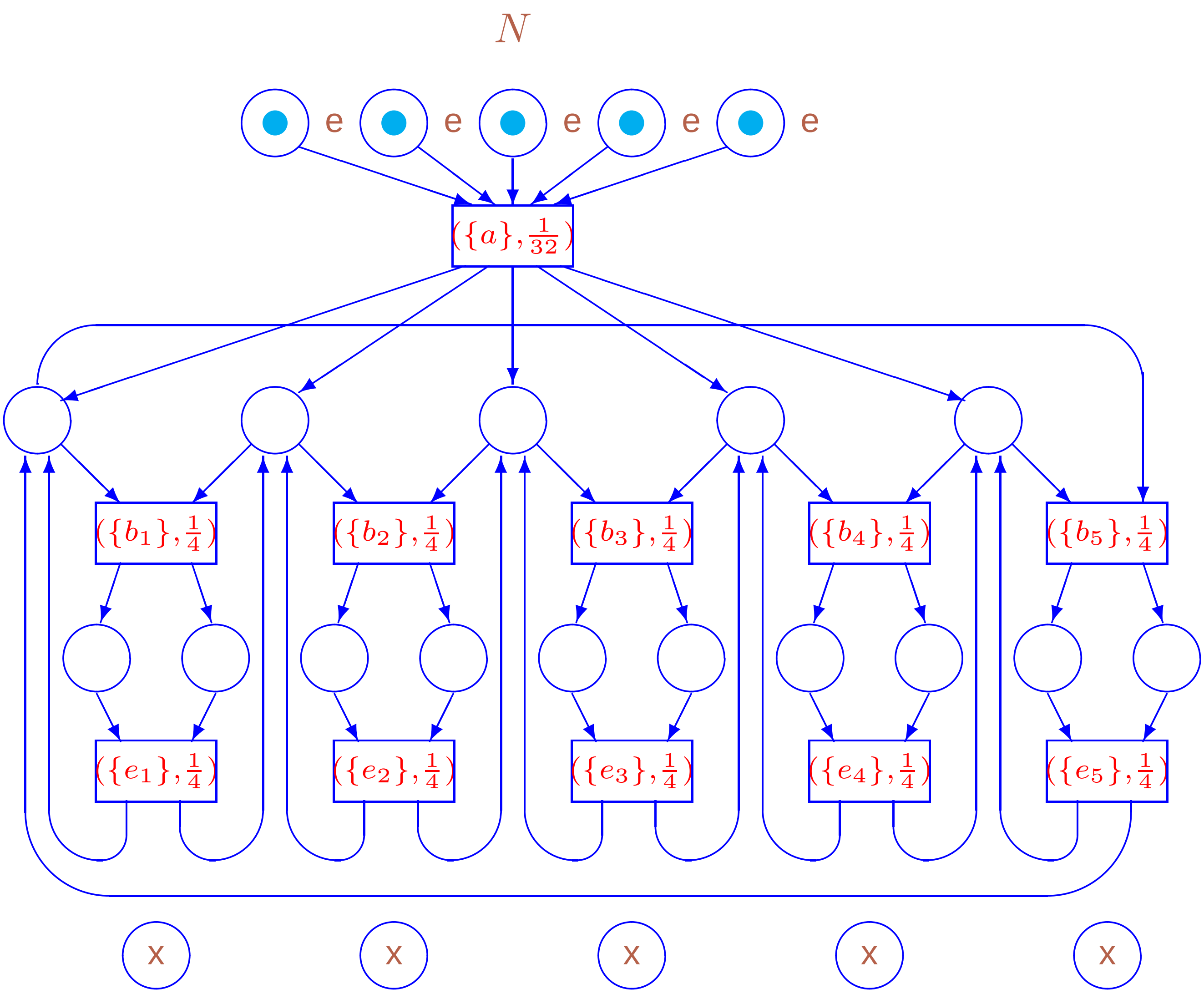
  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.
- The beginning of eating of first philosopher $(\{b_1\}, \frac{1}{4})$ is only possible from the states s_2, s_6, s_7.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b_1\}, \frac{1}{4})$.

  The *steady-state probability of the beginning of eating of first philosopher* is
  $\psi_2^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) +$
  $\psi_6^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) +$
  $\psi_7^* \sum_{\{\Gamma|(\{b_1\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) =$
  $\frac{2}{11}\left(\frac{3}{20} + \frac{1}{20} + \frac{1}{20}\right) + \frac{1}{10}\left(\frac{3}{11} + \frac{1}{11}\right) + \frac{1}{10}\left(\frac{3}{11} + \frac{1}{11}\right) = \frac{13}{110}$.



The marked dts-boxes of the dining philosophers



The marked dts-box of the dining philosophers system

The abstract system

The static expression of the philosopher i $(1 \leq i \leq 4)$ is
$F_i = [(\{x_i\}, \frac{1}{2}) * ((((\{b, \widehat{y_i}\}, \frac{1}{2}); (\{e, \widehat{z_i}\}, \frac{1}{2}))[]((\{y_{i+1}\}, \frac{1}{2}); (\{z_{i+1}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is
$F_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \frac{1}{2}) * ((((\{b, \widehat{y_5}\}, \frac{1}{2}); (\{e, \widehat{z_5}\}, \frac{1}{2}))[]((\{y_1\}, \frac{1}{2}); (\{z_1\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract dining philosophers system is
$F = (F_1 \| F_2 \| F_3 \| F_4 \| F_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS^*(\overline{F})$ is similar to $TS^*(\overline{E})$.

$DTMC^*(\overline{F}) \simeq DTMC^*(\overline{E})$, thus, TPM and the steady-state PMF for $DTMC^*(\overline{F})$ and $DTMC^*(\overline{E})$ coincide.

Performance indices

The first performance index and the second group of the indices are the same for the standard and abstract systems.

The following performance index: non-personalized viewpoint to the philosophers.

- The beginning of eating of a philosopher $(\{b\}, \frac{1}{4})$ is only possible from the states $s_2, s_3, s_6, s_7, s_{10}, s_{11}$.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{1}{4})$.

  The *steady-state probability of the beginning of eating of a philosopher* is

$$\psi_2^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_2) + \psi_3^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_3) +$$
$$\psi_6^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_6) + \psi_7^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_7) +$$
$$\psi_{10}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{10}) +$$
$$\psi_{11}^* \sum_{\{\Gamma|(\{b\},\frac{1}{4})\in\Gamma\}} PT^*(\Gamma, s_{11}) =$$
$$\frac{2}{11}\left(\frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20} + \frac{3}{20} + \frac{1}{20}\right) +$$
$$\frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) +$$
$$\frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) + \frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) +$$
$$\frac{1}{4}\left(\frac{3}{11} + \frac{1}{11} + \frac{3}{11} + \frac{1}{11}\right) = \frac{6}{11}.$$

The reduction of the abstract system

The static expression of the philosopher 1 is

$F_1' = [(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the philosopher 2 is

$F_2' = [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5}); (\{e\}, \frac{1}{4})) * \mathsf{Stop}]$.

The static expression of the reduced abstract dining philosophers system is

$F' = (F_1' \| F_2')$ sy x rs x.

$DR(\overline{F'})$ consists of

$s_1' = [([(\overline{(\{x\}, \frac{1}{2})} * ((\{b\}, \frac{2}{5})_1; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \| [\overline{(\{a, \hat{x}\}, \frac{1}{16})} * ((\{b\}, \frac{2}{5})_2; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])$ sy x rs $x]_\approx$,

$s_2' = [([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1}; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * (\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])$ sy x rs $x]_\approx$,

$s_3' = [([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * (\overline{(\{b\}, \frac{2}{5})_2}; (\{e\}, \frac{1}{4})_2) * \mathsf{Stop}])$ sy x rs $x]_\approx$,

$s_4' = [([(\{x\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{2}{5})_1}; (\{e\}, \frac{1}{4})_1) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}])$ sy x rs $x]_\approx$,

$s_5' = [([(\{x\}, \frac{1}{2}) * ((\{b\}, \frac{2}{5})_1; \overline{(\{e\}, \frac{1}{4})_1}) * \mathsf{Stop}] \| [(\{a, \hat{x}\}, \frac{1}{16}) * ((\{b\}, \frac{2}{5})_2; \overline{(\{e\}, \frac{1}{4})_2}) * \mathsf{Stop}])$ sy x rs $x]_\approx$.

Interpretation of the states

s'_1: the initial state,

s'_2: the system is activated and no philosophers dine,

s'_3, s'_4: one philosopher dines,

s'_5: two philosophers dine.

Consider $\mathcal{R} : \overline{F} \underline{\leftrightarrow}_{ss} \overline{F'}$ such that
$(DR(\overline{F}) \cup DR(\overline{F'}))/_{\mathcal{R}} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$, where

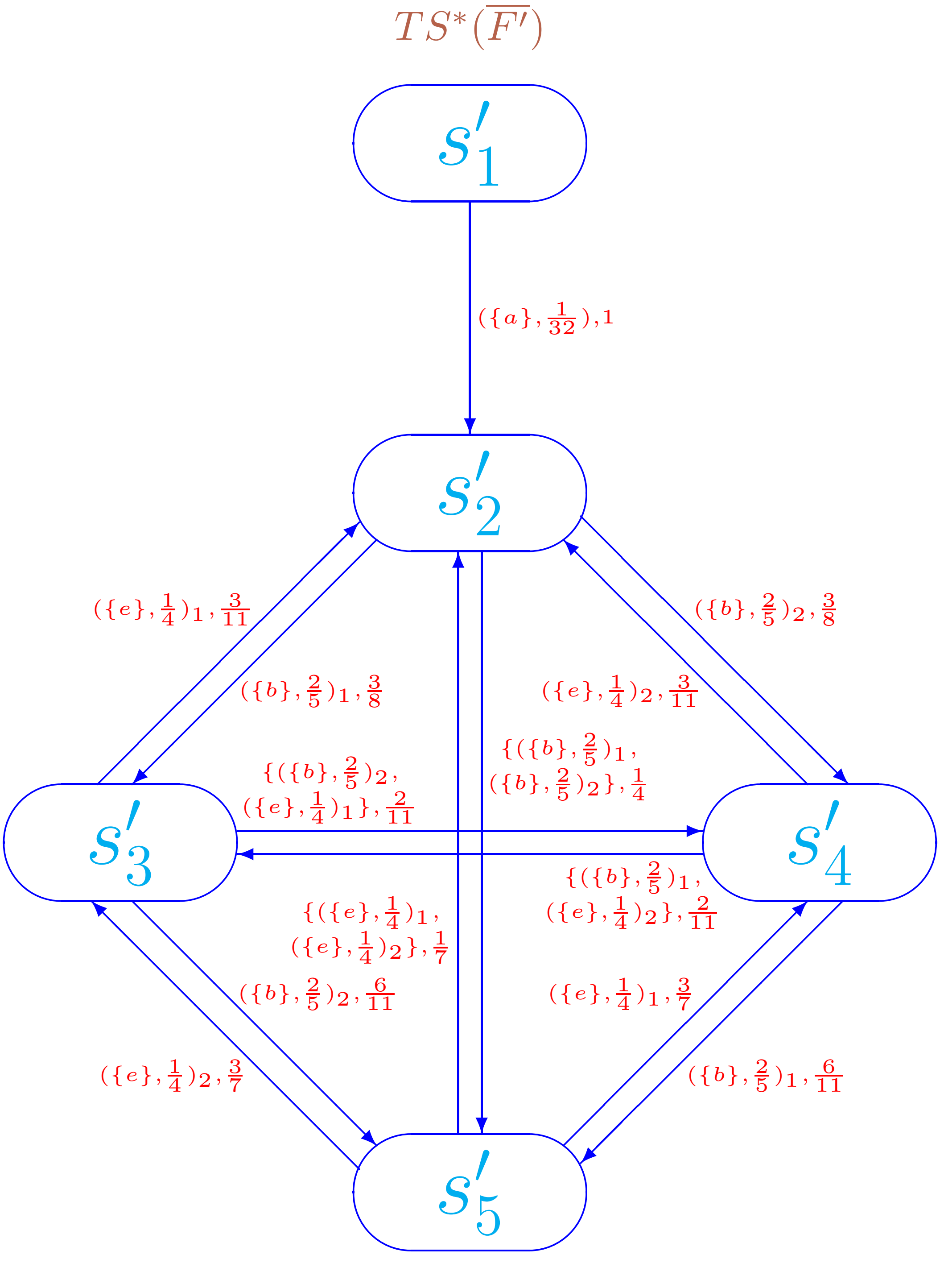
$\mathcal{H}_1 = \{s_1, s'_1\}$ (the initial state),

$\mathcal{H}_2 = \{s_2, s'_2\}$ (the system is activated and no philosophers dine),

$\mathcal{H}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}, s'_3, s'_4\}$ (one philosopher dines),

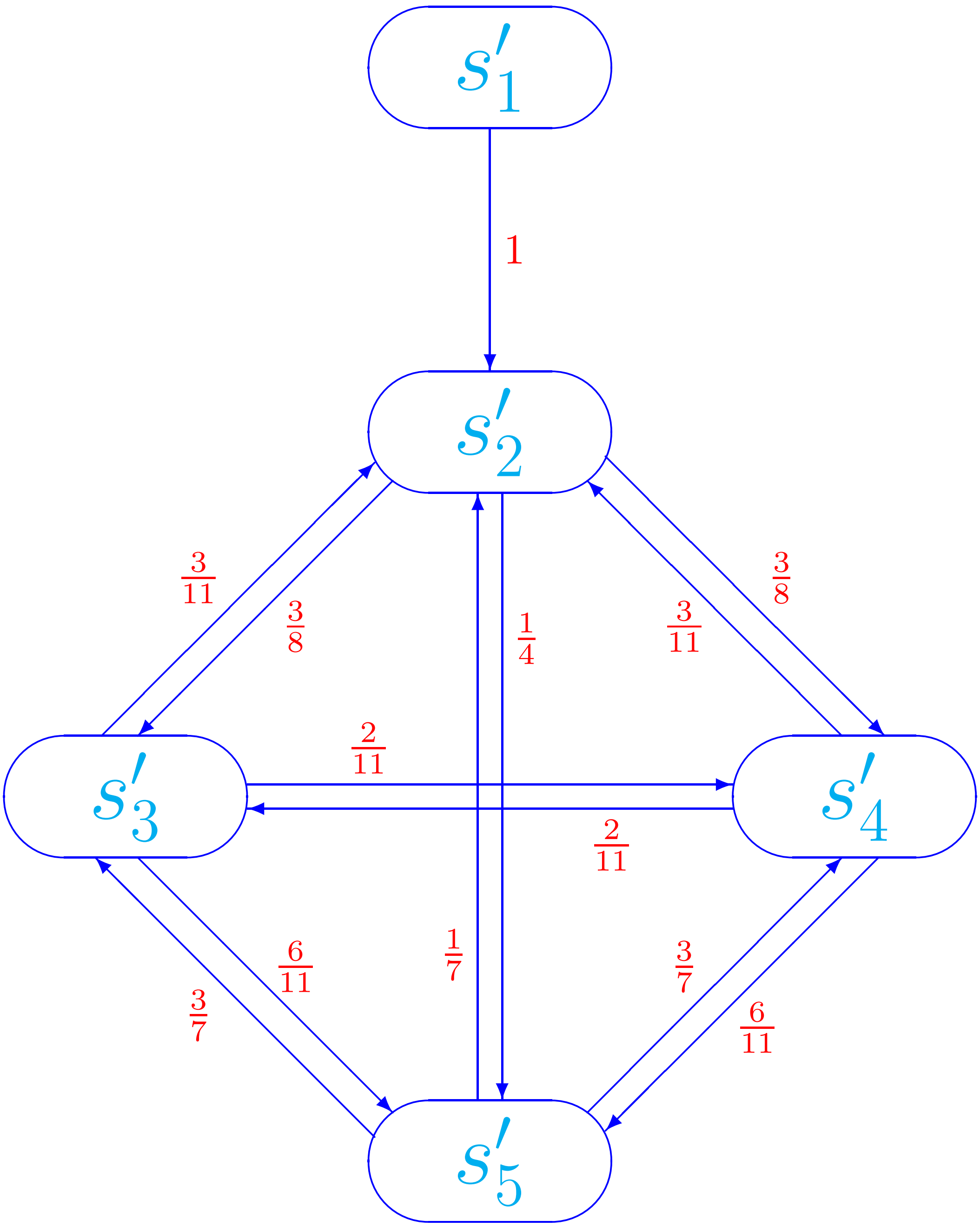
$\mathcal{H}_4 = \{s_4, s_5, s_8, s_9, s_{12}, s'_5\}$ (two philosophers dine).

F' is a reduction of F w.r.t. $\underline{\leftrightarrow}_{ss}$.



The transition system without empty loops of the reduced abstract dining philosophers system

$DTMC^*(\overline{F'})$



The underlying DTMC without empty loops of the reduced abstract dining philosophers system

The TPM for $DTMC^*(\overline{F'})$ is

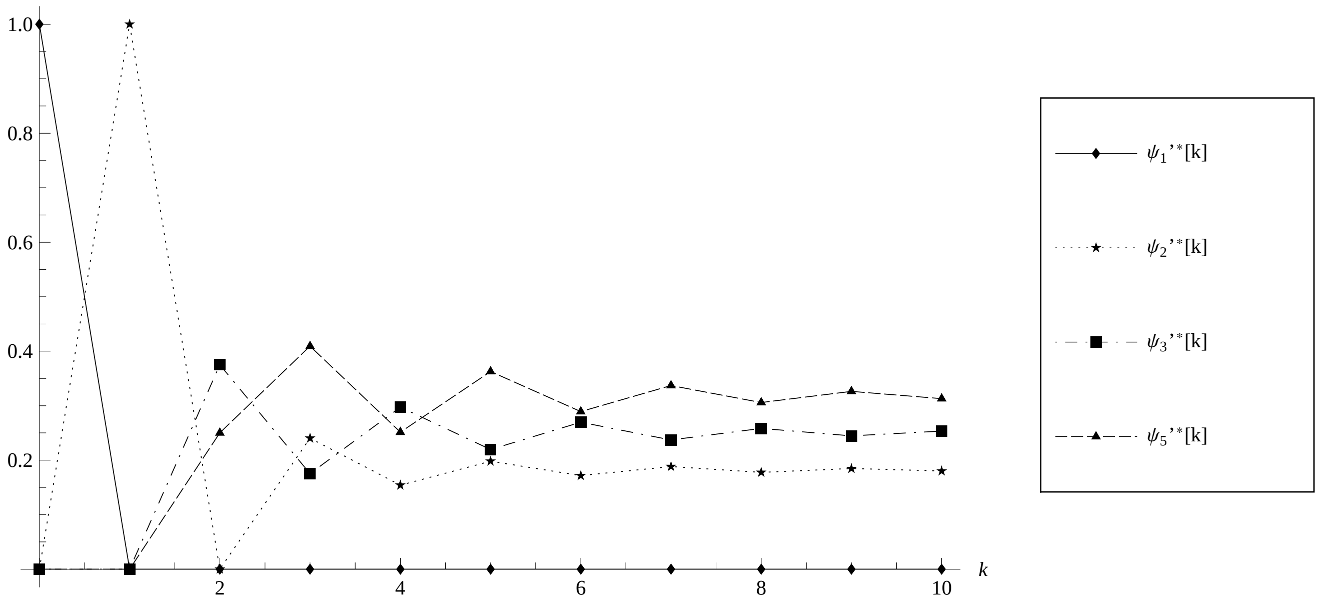
$$\mathbf{P}'^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ 0 & \frac{3}{11} & 0 & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{3}{11} & \frac{2}{11} & 0 & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{3}{7} & \frac{3}{7} & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*(\overline{F'})$ is

$$\psi'^* = \left(0, \frac{2}{11}, \frac{1}{4}, \frac{1}{4}, \frac{7}{22}\right).$$

Transient and steady-state probabilities of the reduced abstract dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1'^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2'^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
$\psi_3'^*[k]$	0	0	0.3750	0.1753	0.2973	0.2195	0.2697	0.2372	0.2583	0.2446	0.2535	0.2500
$\psi_5'^*[k]$	0	0	0.2500	0.4091	0.2513	0.3628	0.2890	0.3371	0.3059	0.3261	0.3130	0.3182



Transient probabilities alteration diagram of the reduced abstract dining philosophers system

We depict the probabilities for the states s'_1, s'_2, s'_3, s'_5 only, since the corresponding values coincide for s'_3, s'_4.

Performance indices

- The average recurrence time in the state s'_2, where all the forks are available, the *average system run-through*, is $\frac{1}{{\psi'_2}^*} = \frac{11}{2} = 5\frac{1}{2}$.
- Nobody eats in the state s'_2. The *fraction of time when no philosophers dine* is ${\psi'_2}^* = \frac{2}{11}$.

  Only one philosopher eats in the states s'_3, s'_4. The *fraction of time when only one philosopher dines* is ${\psi'_3}^* + {\psi'_4}^* = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

  Two philosophers eat together in the state s'_5. The *fraction of time when two philosophers dine* is ${\psi'_5}^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.
- The beginning of eating of a philosopher $(\{b\}, \frac{2}{5})$ is only possible from the states s'_2, s'_3, s'_4.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b\}, \frac{2}{5})$.

  The *steady-state probability of the beginning of eating of a philosopher* is
  ${\psi'_2}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_2) + {\psi'_3}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_3) + {\psi'_4}^* \sum_{\{\Gamma|(\{b\},\frac{2}{5})\in\Gamma\}} PT^*(\Gamma, s'_4) =$
  $\frac{2}{11}\left(\frac{3}{8} + \frac{3}{8} + \frac{1}{4}\right) + \frac{1}{4}\left(\frac{6}{11} + \frac{2}{11}\right) + \frac{1}{4}\left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}$.

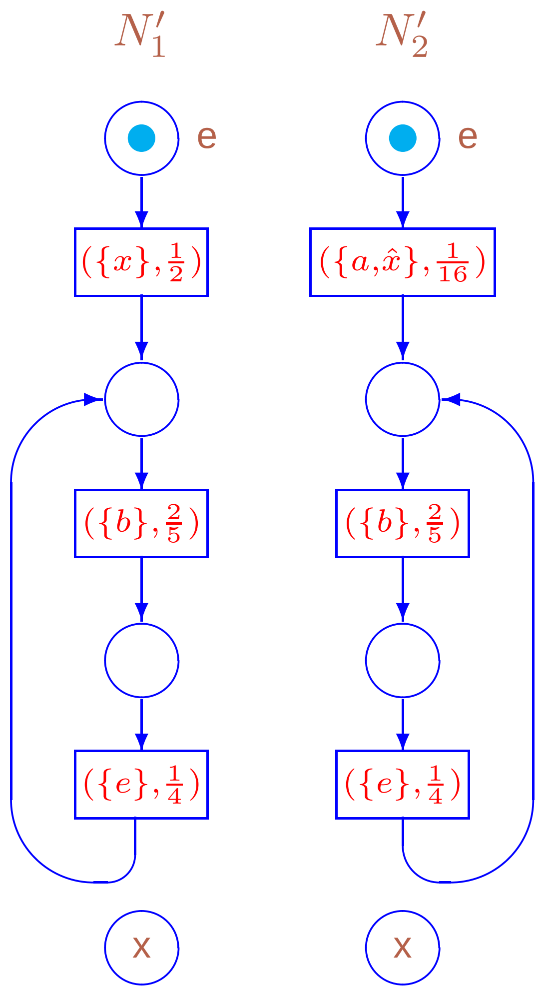
The performance indices are the same for the complete and the reduced abstract dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates the proposition about steady-state probabilities.

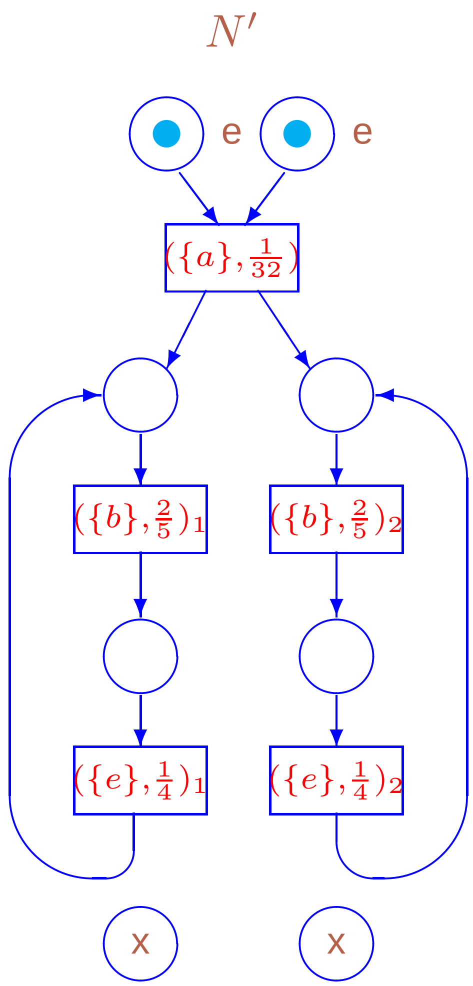
The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces
$\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{F}$ and $\overline{F'}$,

and sum the left and right parts of the three resulting equalities.



The marked dts-boxes of the reduced abstract dining philosophers



The marked dts-box of the reduced abstract dining philosophers system

The quotient of the abstract system

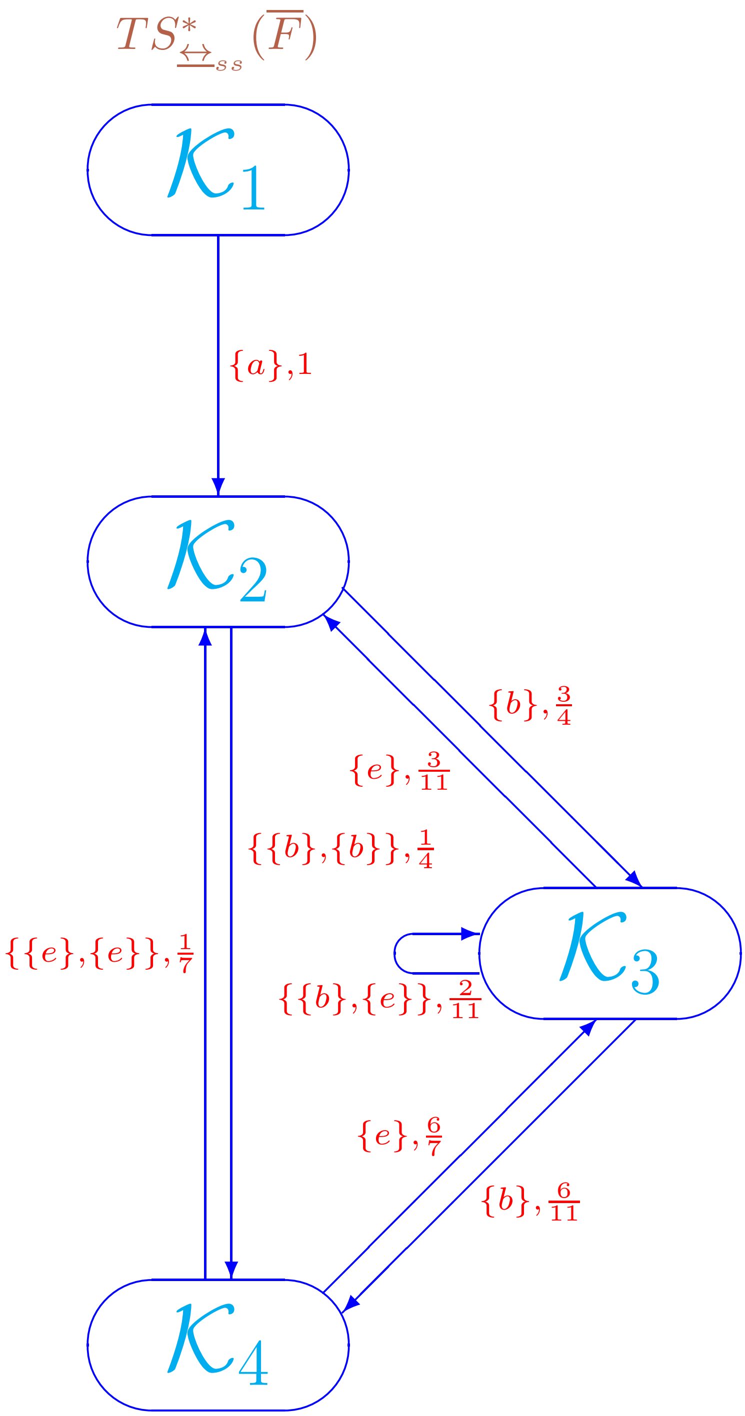
$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$, where

$\mathcal{K}_1 = \{s_1\}$ (the initial state),

$\mathcal{K}_2 = \{s_2\}$ (the system is activated and no philosophers dine),

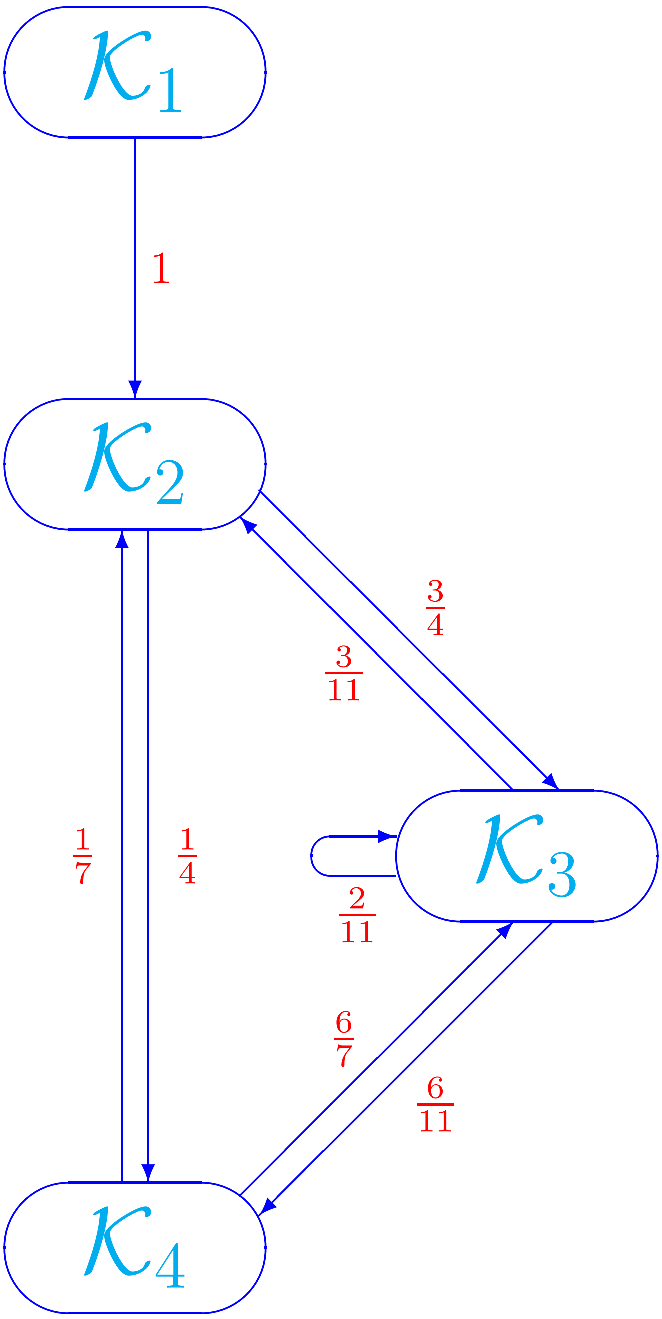
$\mathcal{K}_3 = \{s_3, s_6, s_7, s_{10}, s_{11}\}$ (one philosopher dines),

$\mathcal{K}_4 = \{s_4, s_5, s_8, s_9, s_{12}\}$ (two philosophers dine).



The quotient transition system without empty loops of the abstract dining philosophers system

$DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



The quotient underlying DTMC without empty loops of the abstract dining philosophers system

The TPM for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

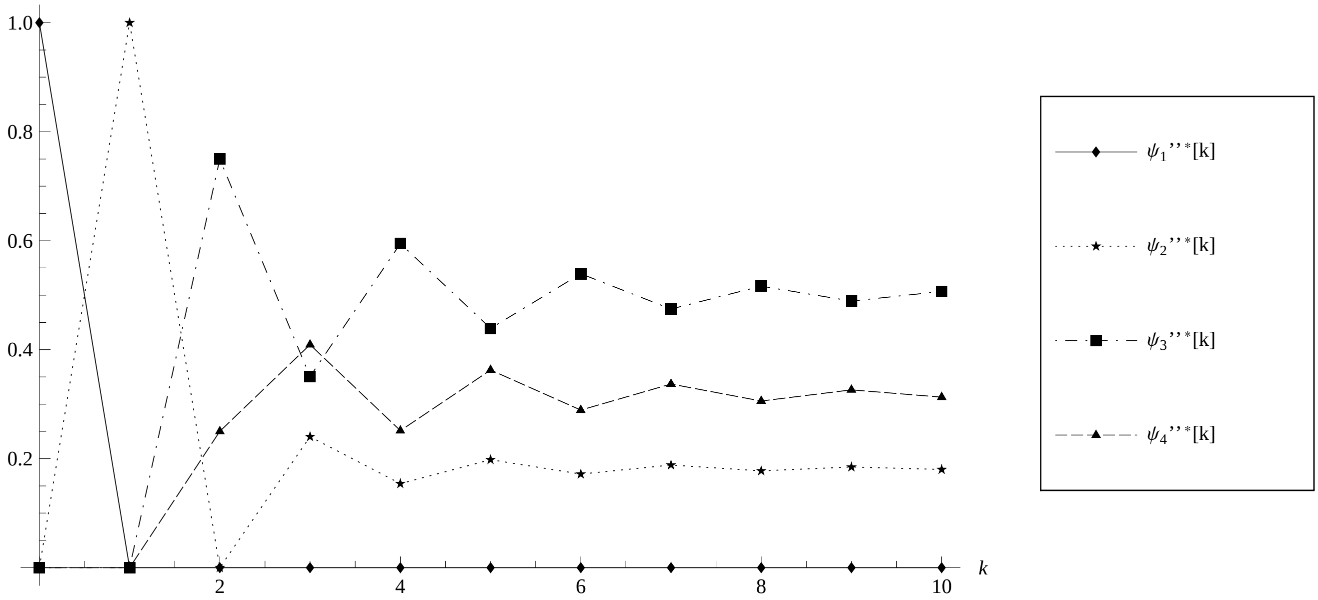
$$\mathbf{P}''^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{11} & \frac{2}{11} & \frac{6}{11} \\ 0 & \frac{1}{7} & \frac{6}{7} & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ is

$$\psi''^* = \left(0, \frac{2}{11}, \frac{1}{2}, \frac{7}{22}\right).$$

Transient and steady-state probabilities of the quotient abstract dining philosophers system

k	0	1	2	3	4	5	6	7	8	9	10	∞
$\psi_1''^*[k]$	1	0	0	0	0	0	0	0	0	0	0	0
$\psi_2''^*[k]$	0	1	0	0.2403	0.1541	0.1981	0.1716	0.1884	0.1776	0.1846	0.1800	0.1818
$\psi_3''^*[k]$	0	0	0.7500	0.3506	0.5946	0.4391	0.5394	0.4745	0.5165	0.4893	0.5069	0.5000
$\psi_4''^*[k]$	0	0	0.2500	0.4091	0.2513	0.3628	0.2890	0.3371	0.3059	0.3261	0.3130	0.3182



Transient probabilities alteration diagram of the quotient abstract dining philosophers system

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where all the forks are available, the *average system run-through*, is $\frac{1}{\psi_2''^*} = \frac{11}{2} = 5\frac{1}{2}$.
- Nobody eats in the state $\mathcal{K}_2$. The *fraction of time when no philosophers dine* is $\psi_2''^* = \frac{2}{11}$.

  Only one philosopher eats in the state $\mathcal{K}_3$. The *fraction of time when only one philosopher dines* is $\psi_3''^* = \frac{1}{2}$.

  Two philosophers eat together in the state $\mathcal{K}_4$. The *fraction of time when two philosophers dine* is $\psi_4''^* = \frac{7}{22}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{7}{22} \cdot \frac{2}{1} = \frac{7}{11}$.
- The beginning of eating of a philosopher $\{b\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_3$.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of multiactions containing $\{b\}$.

  The *steady-state probability of the beginning of eating of a philosopher* is

  $$\psi_2''^* \sum_{\{A,\mathcal{K} \mid \{b\} \in A,\ \mathcal{K}_2 \stackrel{A}{\twoheadrightarrow} \mathcal{K}\}} PM_A^*(\mathcal{K}_2, \mathcal{K}) + \psi_3''^* \sum_{\{A,\mathcal{K} \mid \{b\} \in A,\ \mathcal{K}_3 \stackrel{A}{\twoheadrightarrow} \mathcal{K}\}} PM_A^*(\mathcal{K}_3, \mathcal{K}) = \frac{2}{11}\left(\frac{3}{4} + \frac{1}{4}\right) + \frac{1}{2}\left(\frac{6}{11} + \frac{2}{11}\right) = \frac{6}{11}.$$

The performance indices are the same for the complete and quotient abstract dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates the proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces $\{\{b\}\}, \{\{b\},\{b\}\}, \{\{b\},\{e\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

The generalized system

The static expression of the philosopher i $(1 \leq i \leq 4)$ is
$K_i = [(\{x_i\}, \rho) * (((\{b_i, \widehat{y_i}\}, \rho); (\{e_i, \widehat{z_i}\}, \rho))[]((\{y_{i+1}\}, \rho);$
$(\{z_{i+1}\}, \rho))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is
$K_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \rho) * (((\{b_5, \widehat{y_5}\}, \rho); (\{e_5, \widehat{z_5}\}, \rho))[]((\{y_1\}, \rho);$
$(\{z_1\}, \rho))) * \mathsf{Stop}]$.

The static expression of the generalized dining philosophers system is
$K = (K_1 \| K_2 \| K_3 \| K_4 \| K_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

Interpretation of the states

$\tilde{s}_1$: the initial state,

$\tilde{s}_2$: the system is activated and no philosophers dine,

$\tilde{s}_3$: philosopher 1 dines,

$\tilde{s}_4$: philosophers 1 and 4 dine,

$\tilde{s}_5$: philosophers 1 and 3 dine,

$\tilde{s}_6$: philosopher 4 dines,

$\tilde{s}_7$: philosopher 3 dines,

$\tilde{s}_8$: philosophers 2 and 4 dine,

$\tilde{s}_9$: philosophers 3 and 5 dine,

$\tilde{s}_{10}$: philosopher 2 dines,

$\tilde{s}_{11}$: philosopher 5 dine,

$\tilde{s}_{12}$: philosophers 2 and 5 dine.

The TPM for $DTMC^*(\overline{K})$ is $\widetilde{\mathbf{P}}^* =$

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1-\rho^2}{5} & \frac{\rho^2}{5} & \frac{\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{\rho^2}{5} & \frac{\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{1-\rho^2}{5} & \frac{\rho^2}{5} \\
0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{\rho^2}{3-\rho^2} & 0 & 0 \\
0 & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{\rho^2}{3-\rho^2} & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 \\
0 & \frac{\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & 0 \\
0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & \frac{\rho^2}{3-\rho^2} & \frac{1-\rho^2}{3-\rho^2} \\
0 & \frac{1-\rho^2}{3-\rho^2} & 0 & 0 & 0 & 0 & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{\rho^2}{3-\rho^2} & 0 & \frac{1-\rho^2}{3-\rho^2} \\
0 & \frac{\rho^2}{2-\rho^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0
\end{pmatrix}
$$

The steady-state PMF for $DTMC^*(\overline{K})$ is

$$\tilde{\psi}^* = \left(0, \frac{1}{2(3-\rho^2)}, \frac{1}{10}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{1}{10}, \frac{1}{10}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{2-\rho^2}{10(3-\rho^2)}, \frac{1}{10}, \frac{1}{10}, \frac{2-\rho^2}{10(3-\rho^2)}\right).$$

Performance indices

- The average recurrence time in the state s_2, where all the forks are available, the *average system run-through*, is $\frac{1}{\tilde{\psi}_2^*} = 2(3-\rho^2)$.

- Nobody eats in the state s_2. The *fraction of time when no philosophers dine* is $\tilde{\psi}_2^* = \frac{1}{2(3-\rho^2)}$.

  Only one philosopher eats in the states $s_3, s_6, s_7, s_{10}, s_{11}$. The *fraction of time when only one philosopher dines* is $\tilde{\psi}_3^* + \tilde{\psi}_6^* + \tilde{\psi}_7^* + \tilde{\psi}_{10}^* + \tilde{\psi}_{11}^* = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}$.

  Two philosophers eat together in the states $s_4, s_5, s_8, s_9, s_{12}$. The *fraction of time when two philosophers dine* is $\tilde{\psi}_4^* + \tilde{\psi}_5^* + \tilde{\psi}_8^* + \tilde{\psi}_9^* + \tilde{\psi}_{12}^* = \frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} + \frac{2-\rho^2}{10(3-\rho^2)} = \frac{2-\rho^2}{2(3-\rho^2)}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{2-\rho^2}{2(3-\rho^2)} \cdot \frac{2}{1} = \frac{2-\rho^2}{3-\rho^2}$.

- The beginning of eating of first philosopher $(\{b_1\}, \rho^2)$ is only possible from the states s_2, s_6, s_7.

  The beginning of eating probability in each of the states is a sum of execution probabilities for all multisets of activities containing $(\{b_1\}, \rho^2)$.

  The *steady-state probability of the beginning of eating of first philosopher* is $\tilde{\psi}_2^* \sum_{\{\Gamma | (\{b_1\},\rho^2) \in \Gamma\}} PT^*(\Gamma, s_2) + \tilde{\psi}_6^* \sum_{\{\Gamma | (\{b_1\},\rho^2) \in \Gamma\}} PT^*(\Gamma, s_6) + \tilde{\psi}_7^* \sum_{\{\Gamma | (\{b_1\},\rho^2) \in \Gamma\}} PT^*(\Gamma, s_7) = \frac{1}{2(3-\rho^2)} \left(\frac{1-\rho^2}{5} + \frac{\rho^2}{5} + \frac{\rho^2}{5} \right) + \frac{1}{10} \left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2} \right) + \frac{1}{10} \left(\frac{1-\rho^2}{3-\rho^2} + \frac{\rho^2}{3-\rho^2} \right) = \frac{3+\rho^2}{10(3-\rho^2)}$.

The abstract generalized system

The static expression of the philosopher $i\ (1 \leq i \leq 4)$ is
$L_i = [(\{x_i\}, \rho) * (((\{b, \widehat{y_i}\}, \rho); (\{e, \widehat{z_i}\}, \rho))[]$
$((\{y_{i+1}\}, \rho); (\{z_{i+1}\}, \rho))) * \mathsf{Stop}]$.

The static expression of the philosopher 5 is
$L_5 = [(\{a, \widehat{x_1}, \widehat{x_2}, \widehat{x_2}, \widehat{x_4}\}, \rho) * (((\{b, \widehat{y_5}\}, \rho); (\{e, \widehat{z_5}\}, \rho))[]((\{y_1\}, \rho);$
$(\{z_1\}, \rho))) * \mathsf{Stop}]$.

The static expression of the abstract generalized dining philosophers system is
$L = (L_1 \| L_2 \| L_3 \| L_4 \| L_5)$ sy x_1 sy x_2 sy x_3 sy x_4 sy y_1 sy y_2 sy y_3 sy y_4 sy y_5 sy z_1 sy z_2 sy z_3 sy z_4 sy z_5 rs x_1 rs x_2 rs x_3 rs x_4 rs y_1 rs y_2 rs y_3 rs y_4 rs y_5 rs z_1 rs z_2 rs z_3 rs z_4 rs z_5.

$DR(\overline{L})$ resembles $DR(\overline{K})$, and $TS^*(\overline{L})$ is similar to $TS^*(\overline{K})$.

$DTMC^*(\overline{L}) \simeq DTMC^*(\overline{K})$, thus, TPM and the steady-state PMF for $DTMC^*(\overline{L})$ and $DTMC^*(\overline{K})$ coincide.

Performance indices

The first performance index and the second group of the indices are the same for the generalized system and its abstract modification.

The following performance index: non-personalized viewpoint to the philosophers.

- The beginning of eating of a philosopher $(\{b\},\rho^2)$ is only possible from the states $\tilde{s}_2,\tilde{s}_3,\tilde{s}_6,\tilde{s}_7,\tilde{s}_{10},\tilde{s}_{11}$.

  The beginning of eating probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{b\},\rho^2)$.

  The *steady-state probability of the beginning of eating of a philosopher* is

$$\tilde{\psi}_2^*\sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_2)+\tilde{\psi}_3^*\sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_3)+\tilde{\psi}_6^*\sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_6)+\tilde{\psi}_7^*\sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_7)+\tilde{\psi}_{10}^*\sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_{10})+\tilde{\psi}_{11}^*\sum_{\{\Gamma|(\{b\},\rho^2)\in\Gamma\}} PT^*(\Gamma,\tilde{s}_{11})=$$

$$\frac{1}{2(3-\rho^2)}\left(\frac{1-\rho^2}{5}+\frac{\rho^2}{5}+\frac{1-\rho^2}{5}+\frac{\rho^2}{5}+\frac{1-\rho^2}{5}+\frac{\rho^2}{5}+\frac{1-\rho^2}{5}+\frac{\rho^2}{5}+\frac{1-\rho^2}{5}+\frac{\rho^2}{5}\right)+$$
$$\frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}+\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}\right)+$$
$$\frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}+\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}\right)+$$
$$\frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}+\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}\right)+$$
$$\frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}+\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}\right)+$$
$$\frac{1}{10}\left(\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}+\frac{1-\rho^2}{3-\rho^2}+\frac{\rho^2}{3-\rho^2}\right)=\frac{3}{2(3-\rho^2)}.$$

The reduction of the abstract generalized system

The static expression of the philosopher 1 is
$L_1' = [(\{x\}, \rho) * ((\{b\}, \frac{2\rho^2}{1+\rho^2}); (\{e\}, \rho^2)) * \mathsf{Stop}]$.

The static expression of the philosopher 2 is
$L_2' = [(\{a, \hat{x}\}, \rho^4) * ((\{b\}, \frac{2\rho^2}{1+\rho^2}); (\{e\}, \rho^2)) * \mathsf{Stop}]$.

The static expression of the reduced abstract generalized dining philosophers system is $L' = (L_1' \| L_2')$ sy x rs x.

Consider $\mathcal{R} : \overline{L} \underline{\leftrightarrow}_{ss} \overline{L'}$ such that
$(DR(\overline{L}) \cup DR(\overline{L'}))/_{\mathcal{R}} = \{\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2, \widetilde{\mathcal{H}}_3, \widetilde{\mathcal{H}}_4\}$, where

$\widetilde{\mathcal{H}}_1 = \{\tilde{s}_1, \tilde{s}_1'\}$ (the initial state),

$\widetilde{\mathcal{H}}_2 = \{\tilde{s}_2, \tilde{s}_2'\}$ (the system is activated and no philosophers dine),

$\widetilde{\mathcal{H}}_3 = \{\tilde{s}_3, \tilde{s}_6, \tilde{s}_7, \tilde{s}_{10}, \tilde{s}_{11}, \tilde{s}_3', \tilde{s}_4'\}$ (one philosopher dines),

$\widetilde{\mathcal{H}}_4 = \{\tilde{s}_4, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9, \tilde{s}_{12}, \tilde{s}_5'\}$ (two philosophers dine).

L' is a reduction of L w.r.t. $\underline{\leftrightarrow}_{ss}$.

The TPM for $DTMC^*(\overline{L'})$ is

$$\widetilde{\mathbf{P}}'^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\rho^2}{2} & \frac{1-\rho^2}{2} & \rho^2 \\ 0 & \frac{1-\rho^2}{3-\rho^2} & 0 & \frac{2\rho^2}{3-\rho^2} & \frac{2(1-\rho^2)}{3-\rho^2} \\ 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{2\rho^2}{3-\rho^2} & 0 & \frac{2(1-\rho^2)}{3-\rho^2} \\ 0 & \frac{\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & \frac{1-\rho^2}{2-\rho^2} & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*(\overline{L'})$ is

$$\tilde{\psi}'^* = \left(0, \frac{1}{2(3-\rho^2)}, \frac{1}{4}, \frac{1}{4}, \frac{2-\rho^2}{2(3-\rho^2)}\right).$$

Performance indices

- The average recurrence time in the state $\tilde{s}'_2$, where all the forks are available, *average system run-through*, is $\frac{1}{\tilde{\psi}'^*_2} = 2(3-\rho^2)$.
- Nobody eats in the state $\tilde{s}'_2$. The *fraction of time when no philosophers dine* is $\tilde{\psi}'^*_2 = \frac{1}{2(3-\rho^2)}$.

  Only one philosopher eats in the states $\tilde{s}'_3, \tilde{s}'_4$. The *fraction of time when only one philosopher dines* is $\tilde{\psi}'^*_3 + \tilde{\psi}'^*_4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

  Two philosophers eat together in the state $\tilde{s}'_5$. The *fraction of time when two philosophers dine* is $\tilde{\psi}'^*_5 = \frac{2-\rho^2}{2(3-\rho^2)}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{2-\rho^2}{2(3-\rho^2)} \cdot \frac{2}{1} = \frac{2-\rho^2}{3-\rho^2}$.

- The beginning of eating of a philosopher $(\{b\}, \frac{2\rho^2}{1+\rho^2})$ is only possible from the states $\tilde{s}'_2, \tilde{s}'_3, \tilde{s}'_4$.

  The beginning of eating probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{b\}, \frac{2\rho^2}{1+\rho^2})$.

  The *steady-state probability of the beginning of eating of a philosopher* is

  $$\tilde{\psi}'^*_2 \sum_{\{\Gamma|(\{b\},\frac{2\rho^2}{1+\rho^2})\in\Gamma\}} PT^*(\Gamma, \tilde{s}'_2) + \tilde{\psi}'^*_3 \sum_{\{\Gamma|(\{b\},\frac{2\rho^2}{1+\rho^2})\in\Gamma\}} PT^*(\Gamma, \tilde{s}'_3) + \tilde{\psi}'^*_4 \sum_{\{\Gamma|(\{b\},\frac{2\rho^2}{1+\rho^2})\in\Gamma\}} PT^*(\Gamma, \tilde{s}'_4) = \frac{1}{2(3-\rho^2)}\left(\frac{1-\rho^2}{2} + \frac{1-\rho^2}{2} + \rho^2\right) + \frac{1}{4}\left(\frac{2(1-\rho^2)}{3-\rho^2} + \frac{2\rho^2}{3-\rho^2}\right) + \frac{1}{4}\left(\frac{2(1-\rho^2)}{3-\rho^2} + \frac{2\rho^2}{3-\rho^2}\right) = \frac{3}{2(3-\rho^2)}.$$

The performance indices are the same for the complete and the reduced abstract generalized dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates the proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces $\{\{b\}\}$, $\{\{b\},\{b\}\}$, $\{\{b\},\{e\}\}$ of $\overline{L}$ and $\overline{L'}$,

and sum the left and right parts of the three resulting equalities.

The quotients for the abstract generalized system

$DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4\}$, where

$\widetilde{\mathcal{K}}_1 = \{\tilde{s}_1\}$ (the initial state),

$\widetilde{\mathcal{K}}_2 = \{\tilde{s}_2\}$ (the system is activated and no philosophers dine),

$\widetilde{\mathcal{K}}_3 = \{\tilde{s}_3, \tilde{s}_6, \tilde{s}_7, \tilde{s}_{10}, \tilde{s}_{11}\}$ (one philosopher dines),

$\widetilde{\mathcal{K}}_4 = \{\tilde{s}_4, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9, \tilde{s}_{12}\}$ (two philosophers dine).

The TPM for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ is

$$\widetilde{\mathbf{P}}''^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\rho^2 & \rho^2 \\ 0 & \frac{1-\rho^2}{3-\rho^2} & \frac{2\rho^2}{3-\rho^2} & \frac{2(1-\rho^2)}{3-\rho^2} \\ 0 & \frac{\rho^2}{2-\rho^2} & \frac{2(1-\rho^2)}{2-\rho^2} & 0 \end{pmatrix}.$$

The steady-state PMF for $DTMC^*_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ is

$$\tilde{\psi}''^* = \left(0, \frac{1}{2(3-\rho^2)}, \frac{1}{2}, \frac{2-\rho^2}{2(3-\rho^2)}\right).$$

Performance indices

- The average recurrence time in the state $\widetilde{\mathcal{K}}_2$, where all the forks are available, the *average system run-through*, is $\frac{1}{\tilde{\psi}_2^{\prime\prime *}} = 2(3-\rho^2)$.

- Nobody eats in the state $\widetilde{\mathcal{K}}_2$. The *fraction of time when no philosophers dine* is $\tilde{\psi}_2^{\prime\prime *} = \frac{1}{2(3-\rho^2)}$.

  Only one philosopher eats in the state $\widetilde{\mathcal{K}}_3$. The *fraction of time when only one philosopher dines* is $\tilde{\psi}_3^{\prime\prime *} = \frac{1}{2}$.

  Two philosophers eat together in the state $\widetilde{\mathcal{K}}_4$. The *fraction of time when two philosophers dine* is $\tilde{\psi}_4^{\prime\prime *} = \frac{2-\rho^2}{2(3-\rho^2)}$.

  The *relative fraction of time when two philosophers dine w.r.t. when only one philosopher dines* is $\frac{2-\rho^2}{2(3-\rho^2)} \cdot \frac{2}{1} = \frac{2-\rho^2}{3-\rho^2}$.

- The beginning of eating of a philosopher $\{b\}$ is only possible from the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3$.

  The beginning of eating probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{b\}$.

  The *steady-state probability of the beginning of eating of a philosopher* is

$$\tilde{\psi}_2^{\prime\prime *} \sum_{\{A,\widetilde{\mathcal{K}} \mid \{b\}\in A,\ \widetilde{\mathcal{K}}_2 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}) +$$
$$\tilde{\psi}_3^{\prime\prime *} \sum_{\{A,\widetilde{\mathcal{K}} \mid \{b\}\in A,\ \widetilde{\mathcal{K}}_3 \stackrel{A}{\twoheadrightarrow} \widetilde{\mathcal{K}}\}} PM_A^*(\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}) =$$
$$\frac{1}{2(3-\rho^2)}((1-\rho^2)+\rho^2) + \frac{1}{2}\left(\frac{2(1-\rho^2)}{3-\rho^2} + \frac{2\rho^2}{3-\rho^2}\right) = \frac{3}{2(3-\rho^2)}.$$

The performance indices are the same for the complete and quotient abstract generalized dining philosophers systems.

The coincidence of the first performance index as well as the second group of indices illustrates the proposition about steady-state probabilities.

The coincidence of the third performance index is by the theorem about derived step traces from steady states:

one should apply its result to the derived step traces $\{\{b\}\},\ \{\{b\},\{b\}\},\ \{\{b\},\{e\}\}$ of $\overline{L}$ and itself,

and sum the left and right parts of the three resulting equalities.

Effect of quantitative changes of ρ to performance of the quotient abstract generalized dining philosophers system in its steady state

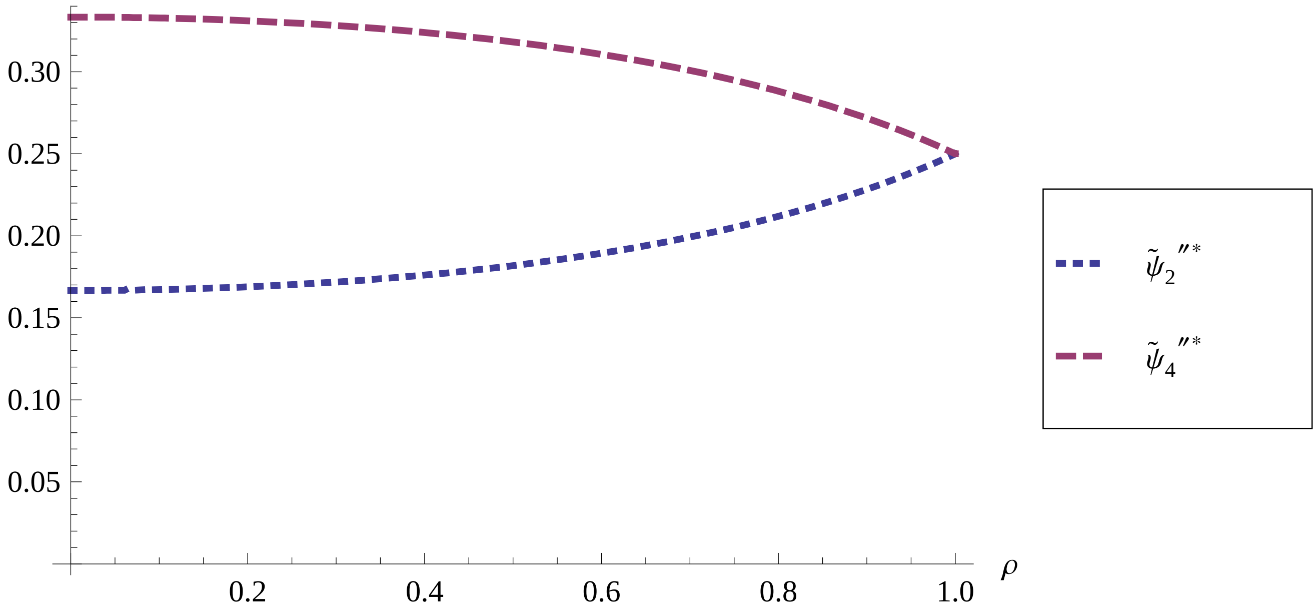
$\rho \in (0;1)$ is the probability of every multiaction of the system.

$\tilde{\psi}_1''^* = 0$ and $\tilde{\psi}_3''^* = \frac{1}{2}$ are constants, and they do not depend on ρ.

$\tilde{\psi}_2''^* = \frac{1}{2(3-\rho^2)}$ and $\tilde{\psi}_4''^* = \frac{2-\rho^2}{2(3-\rho^2)}$ depend on ρ.

$\tilde{\psi}_2''^* + \tilde{\psi}_4''^* = \frac{1}{2(3-\rho^2)} + \frac{2-\rho^2}{2(3-\rho^2)} = \frac{1}{2}$, hence,
the sum of these steady-state probabilities does not depend on ρ.

Interpretation: the fraction of time when no or two philosophers dine coincides with that when only one philosopher dines,
and both fractions are equal to $\frac{1}{2}$.



Steady-state probabilities $\tilde{\psi}_2''^*$ and $\tilde{\psi}_4''^*$ as functions of the parameter ρ

The diagrams in figure above are symmetric w.r.t. the probability $\frac{1}{4}$.

The more is value of ρ, the less is the difference
$\tilde{\psi}_4''^* - \tilde{\psi}_2''^* = \frac{2-\rho^2}{2(3-\rho^2)} - \frac{1}{2(3-\rho^2)} = \frac{1-\rho^2}{2(3-\rho^2)}$.

The difference tends to $\frac{1}{6}$ when ρ approaches 0.

The difference tends to 0 when ρ approaches 1.

Note that $\rho \neq 0$ and $\rho \neq 1$.

Interpretation: the difference between the fractions of time when two and when no philosophers dine.

More interesting value: $\tilde{\psi}_3''^* + \tilde{\psi}_4''^* - \tilde{\psi}_2''^* = \frac{1}{2} + \frac{2-\rho^2}{2(3-\rho^2)} - \frac{1}{2(3-\rho^2)} = \frac{2-\rho^2}{3-\rho^2}$.

The value tends to $\frac{2}{3}$ when ρ approaches 0.

The value tends to $\frac{1}{2}$ when ρ approaches 1.

Interpretation: the difference between the fractions of time when some (one or two) and when no philosophers dine.

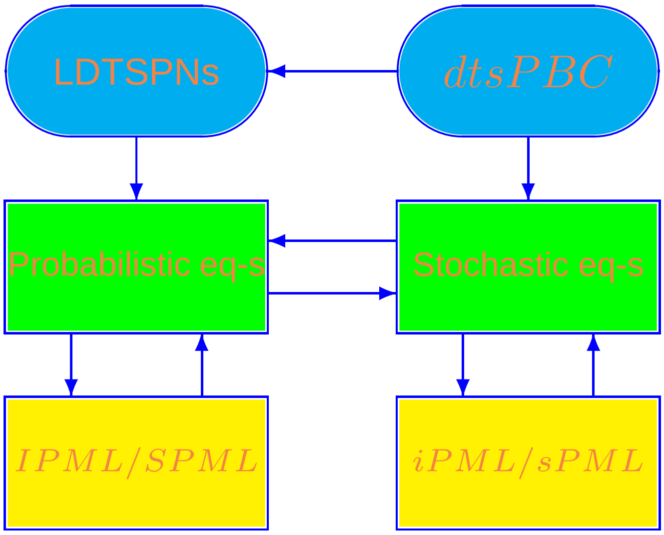
When ρ is closer to 0, more time is spent for eating and less time remains for thinking: *dining is preferred*.

When ρ is closer to 1, the situation is symmetric: *thinking is preferred*.

The influence of ρ to the performance indices presented before: similarly.

Overview and open questions

The results obtained



Stochastic formalisms and equivalences

- A discrete time stochastic extension $dtsPBC$ of finite PBC enriched with iteration.
- The step operational semantics based on labeled probabilistic transition systems.
- The denotational semantics in terms of a subclass of LDTSPNs.
- The stochastic algebraic equivalences which have natural net analogues on LDTSPNs.
- The transition systems and DTMCs reduction modulo stochastic equivalences.
- A logical characterization of stochastic bisimulation equivalences via probabilistic modal logics.

- An application of the equivalences to comparison of stationary behaviour.
- A preservation w.r.t. algebraic operations and the congruence relation.
- The case studies of performance analysis.

Further research

- Abstracting from silent activities

  in definitions of the equivalences.

- Introducing the immediate multiactions

  with zero delay.

- Extending the syntax

  with recursion operator.

Discrete time stochastic Petri box calculus with immediate multiactions[a]

Abstract: In [MVF01], a continuous time stochastic extension $sPBC$ of finite Petri box calculus PBC [BDH92] was proposed. In [MVCC03], iteration operator was added to $sPBC$.

Algebra $sPBC$ has an interleaving semantics, but PBC has a step one.

We constructed a discrete time stochastic extension $dtsPBC$ of finite PBC [Tar05] and enriched it with iteration [Tar06].

We present the extension $dtsiPBC$ of $dtsPBC$ with immediate multiactions [TMV10,TMV13]. $dtsiPBC$ is a discrete time analog of $sPBC$ with immediate multiactions.

The step operational semantics is defined in terms of labeled probabilistic transition systems.

The denotational semantics is defined in terms of a subclass of labeled DTSPNs with immediate transitions (LDTSIPNs), called discrete time stochastic and immediate Petri boxes (dtsi-boxes).

The corresponding semi-Markov chain and (reduced) discrete time Markov chain are analyzed to evaluate performance.

We propose step stochastic bisimulation equivalence and investigate its interrelations with others.

We explain how to use this equivalence for reduction of transition systems and semi-Markov chains.

We demonstrate how to apply this equivalence to compare stationary behaviour and simplify performance analysis.

The case study of performance evaluation is presented: the shared memory system.

[a] The joint work with Hermenegilda Macià S. and Valentín Valero R., High School of Computer Science Engineering, University of Castilla - La Mancha, Albacete, Spain.

Keywords: stochastic Petri net, stochastic process algebra, Petri box calculus, discrete time, immediate multiaction, transition system, operational semantics, immediate transition, dtsi-box, denotational semantics, Markov chain, performance evaluation, stochastic equivalence, reduction, shared memory system.

Contents

- **Reduction modulo equivalences**
- **Stationary behaviour**
  - **Steady state and equivalences**
  - **Simplification of performance analysis**
- **Shared memory system**
  - **The standard system**
  - **The abstract system and its reduction**
  - **The generalized system**
  - **The abstract generalized system and its reduction**
- **Overview and open questions**
  - **Concurrency interpretation**
  - **The results obtained**
  - **Further research**

Introduction

Previous work

- Continuous time (subsets of $\mathbb{R}_+$): interleaving semantics
  - *Continuous time stochastic Petri nets (CTSPNs)* [Mol82,FN85]:
    exponential transition firing delays,
    Continuous time Markov chain (CTMC).
  - *Generalized stochastic Petri nets (GSPNs)* [MCB84,CMBC93]:
    exponential and zero transition firing delays,
    Semi-Markov chain (SMC).
  - *Extended generalized stochastic Petri nets (EGSPNs)* [HS89,MBBCCC89]:
    hyper-exponential or Erlang or phase and zero transition firing delays.
  - *Deterministic stochastic Petri nets (DSPNs)* [MC87,MCF90]:
    exponential and deterministic transition firing delays,
    Semi-Markov process (SMP), if no two deterministic transitions are enabled in any marking.
  - *Extended deterministic stochastic Petri nets (EDSPNs)* [GL94]:
    non-exponential and deterministic transition firing delays.
  - *Extended stochastic Petri nets (ESPNs)* [DTGN85]:
    arbitrary transition firing delays.

- Discrete time (subsets of $\mathbb{N}$): step semantics
  - *Discrete time stochastic Petri nets (DTSPNs)* [Mol85,ZG94]:
    geometric transition firing delays,
    Discrete time Markov chain (DTMC).
  - *Discrete time deterministic and stochastic Petri nets (DTDSPNs)* [ZFH01]:
    geometric and fixed transition firing delays,
    Semi-Markov chain (SMC).
  - *Discrete deterministic and stochastic Petri nets (DDSPNs)* [ZCH97]:
    phase and fixed transition firing delays,
    Semi-Markov chain (SMC).

Stochastic process algebras

- $MTIPP$ [HR94]
- $GSPA$ [BKLL95]
- $PEPA$ [Hil96]
- $S\pi$ [Pri96]
- $EMPA$ [BGo98]
- $GSMPA$ [BBGo98]
- $sACP$ [AHR00]
- TCP^{dst} [MVi08]

More stochastic process calculi

- $TIPP$ [GHR93]
- $WSCCS$ [Tof94]
- $PM-TIPP$ [Ret95]
- $SPADES$ [AKB98]
- $NMSPA$ [LN00]
- $SM-PEPA$ [Brad05]
- $iPEPA$ [HBC13]
- $mCCS$ [DH13]
- $PHASE$ [CR14]

Algebra PBC and its extensions

- Petri box calculus PBC [BDH92]
- Time Petri box calculus $tPBC$ [Kou00]
- Timed Petri box calculus $TPBC$ [MF00]
- Stochastic Petri box calculus $sPBC$ [MVF01,MVCC03]
- Ambient Petri box calculus $APBC$ [FM03]
- Arc time Petri box calculus $atPBC$ [Nia05]
- Generalized stochastic Petri box calculus $gsPBC$ [MVCR08]
- Discrete time stochastic Petri box calculus $dtsPBC$ [Tar05,Tar06]
- Discrete time stochastic and immediate Petri box calculus $dtsiPBC$ [TMV10,TMV13]

SPACLS: Classification of stochastic process algebras

Time	Immediate (multi)actions	Interleaving semantics	Non-interleaving semantics
Continuous	No	$MTIPP$ (CTMC), $PEPA$ (CTMP) $sPBC$ (CTMC)	$GSPA$ (GSMP), $S\pi$, $GSMPA$ (GSMP)
	Yes	$EMPA$ (SMC, CTMC) $gsPBC$ (SMC)	—
Discrete	No	—	$dtsPBC$ (DTMC)
	Yes	TCP^{dst} (DTMRC)	$sACP$, $dtsiPBC$ (SMC, DTMC)

The SPNs-based denotational semantics: orange SPA names.

The underlying stochastic process: in parentheses near the SPA names.

Transition labeling

- CTSPNs [Buc95]
- GSPNs [Buc98]
- DTSPNs [BT00]

Stochastic equivalences

- Probabilistic transition systems (PTSs) [BM89,Chr90,LS91,BHe97,KN98]
- SPAs [HR94,Hil94,BGo98]
- Markov process algebras (MPAs) [Buc94,BKe01]
- CTSPNs [Buc95]
- GSPNs [Buc98]
- Stochastic automata (SAs) [Buc99]
- Stochastic event structures (SESs) [MCW03]

Syntax

The *set of all finite multisets* over X is $\mathbb{N}_{fin}^{X}$.

The *set of all subsets (powerset)* of X is 2^X.

$Act = \{a, b, \ldots\}$ is the set of *elementary actions*.

$\widehat{Act} = \{\hat{a}, \hat{b}, \ldots\}$ is the set of *conjugated actions (conjugates)* s.t. $\hat{a} \neq a$ and $\hat{\hat{a}} = a$.

$\mathcal{A} = Act \cup \widehat{Act}$ is the set of *all actions*.

$\mathcal{L} = \mathbb{N}_{fin}^{\mathcal{A}}$ is the set of *all multiactions*.

The *alphabet* of $\alpha \in \mathcal{L}$ is $\mathcal{A}(\alpha) = \{x \in \mathcal{A} \mid \alpha(x) > 0\}$.

A *stochastic multiaction* is a pair (α, ρ), where
$\alpha \in \mathcal{L}$ and $\rho \in (0; 1)$ is the *probability* of the multiaction α.

$\mathcal{SL}$ is the set of *all stochastic multiactions*.

An *immediate multiaction* is a pair (α, l), where
$\alpha \in \mathcal{L}$ and $l \in \mathbb{N}_{\geq 1}$ is the *weight* of the multiaction α.

$\mathcal{IL}$ is the set of *all immediate multiactions*.

$\mathcal{SIL} = \mathcal{SL} \cup \mathcal{IL}$ is the set of *all activities*.

The *alphabet* of $(\alpha, \kappa) \in \mathcal{SIL}$ is $\mathcal{A}(\alpha, \kappa) = \mathcal{A}(\alpha)$.

The *alphabet* of $\Upsilon \in \mathbb{N}_{fin}^{\mathcal{SIL}}$ is $\mathcal{A}(\Upsilon) = \cup_{(\alpha,\kappa)\in\Upsilon}\mathcal{A}(\alpha)$.

For $(\alpha, \kappa) \in \mathcal{SIL}$, its *multiaction part* is $\mathcal{L}(\alpha, \kappa) = \alpha$ and
its *probability* or *weight part* is $\Omega(\alpha, \kappa) = \kappa$.

The *multiaction part* of $\Upsilon \in \mathbb{N}_{fin}^{\mathcal{SIL}}$ is $\mathcal{L}(\Upsilon) = \sum_{(\alpha,\kappa)\in\Upsilon} \alpha$.

The operations: *sequential execution* ;, *choice* $[]$, *parallelism* $\|$, *relabeling* $[f]$, *restriction* rs, *synchronization* sy and *iteration* $[**]$.

Sequential execution and choice have the standard interpretation.

Parallelism does not include synchronization unlike that in standard process algebras.

Relabeling functions $f : \mathcal{A} \to \mathcal{A}$ are bijections preserving conjugates: $\forall x \in \mathcal{A}\ f(\hat{x}) = \widehat{f(x)}$.

For $\alpha \in \mathcal{L}$, let $f(\alpha) = \sum_{x\in\alpha} f(x)$.

For $\Upsilon \in I\!N_{fin}^{\mathcal{SIL}}$, let $f(\Upsilon) = \sum_{(\alpha,\kappa)\in\Upsilon}(f(\alpha), \kappa)$.

Restriction over $a \in Act$: any process behaviour containing a or its conjugate $\hat{a}$ is not allowed.

Let $\alpha, \beta \in \mathcal{L}$ be two multiactions s.t. for $a \in Act$ we have $a \in \alpha$ and $\hat{a} \in \beta$, or $\hat{a} \in \alpha$ and $a \in \beta$. Synchronization of α and β by a is $\alpha \oplus_a \beta = \gamma$:

$$\gamma(x) = \begin{cases} \alpha(x) + \beta(x) - 1, & x = a \text{ or } x = \hat{a}; \\ \alpha(x) + \beta(x), & \text{otherwise}. \end{cases}$$

In the iteration, the initialization subprocess is executed first,
then the body one is performed zero or more times,
finally, the termination one is executed.

Static expressions specify the structure of processes.

Definition 151 *Let* $(\alpha, \kappa) \in \mathcal{SIL}$ *and* $a \in Act$. *A* static expression *of* $dtsiPBC$ *is*

$$E ::= \ (\alpha, \kappa) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}E{*}E].$$

$StatExpr$ is the set of *all static expressions* of $dtsiPBC$.

Definition 152 *Let* $(\alpha, \kappa) \in \mathcal{SIL}$ *and* $a \in Act$. *A* regular static expression *of* $dtsiPBC$ *is*

$$E ::= \ (\alpha, \kappa) \mid E;E \mid E[]E \mid E\|E \mid E[f] \mid E \text{ rs } a \mid E \text{ sy } a \mid [E{*}D{*}E],$$

where $D ::= \ (\alpha, \kappa) \mid D;E \mid D[]D \mid D[f] \mid D \text{ rs } a \mid D \text{ sy } a \mid [D{*}D{*}E].$

$RegStatExpr$ is the set of *all regular static expressions* of $dtsiPBC$.

Dynamic expressions specify the states of processes.

Dynamic expressions are obtained from static ones annotated with upper or lower bars.

The *underlying static expression* of a dynamic one:
removing all upper and lower bars.

Definition 153 *Let $a \in Act$ and $E \in StatExpr$. A* dynamic expression *of $dtsiPBC$ is*

$$G ::= \overline{E} \mid \underline{E} \mid G;E \mid E;G \mid G[]E \mid E[]G \mid G\|G \mid G[f] \mid G \text{ rs } a \mid G \text{ sy } a \mid [G*E*E] \mid [E*G*E] \mid [E*E*G].$$

$DynExpr$ is the set of *all dynamic expressions* of $dtsiPBC$.

A

Definition 154 *A dynamic expression is* regular *if its underlying static expression is regular.*

$RegDynExpr$ is the set of *all regular dynamic expressions* of $dtsiPBC$.

We shall consider regular expressions only and omit the word “regular”.

Operational semantics

Inaction rules

Inaction rules: instantaneous structural transformations. Let $E, F, K \in RegStatExpr$ and $a \in Act$.

IRULES1: Inaction rules for overlined and underlined regular static expressions

$\overline{E;F} \Rightarrow \overline{E};F$	$\underline{E};F \Rightarrow E;\overline{F}$	$E;\underline{F} \Rightarrow \underline{E;F}$
$\overline{E[]F} \Rightarrow \overline{E}[]F$	$\overline{E[]F} \Rightarrow E[]\overline{F}$	$\underline{E}[]F \Rightarrow \underline{E[]F}$
$E[]\underline{F} \Rightarrow \underline{E[]F}$	$\overline{E \parallel F} \Rightarrow \overline{E} \parallel \overline{F}$	$\underline{E} \parallel \underline{F} \Rightarrow \underline{E \parallel F}$
$\overline{E[f]} \Rightarrow \overline{E}[f]$	$\underline{E}[f] \Rightarrow \underline{E[f]}$	$\overline{E \text{ rs } a} \Rightarrow \overline{E} \text{ rs } a$
$\underline{E} \text{ rs } a \Rightarrow \underline{E \text{ rs } a}$	$\overline{E \text{ sy } a} \Rightarrow \overline{E} \text{ sy } a$	$\underline{E} \text{ sy } a \Rightarrow \underline{E \text{ sy } a}$
$\overline{[E*F*K]} \Rightarrow [\overline{E}*F*K]$	$[\underline{E}*F*K] \Rightarrow [E*\overline{F}*K]$	$[E*\underline{F}*K] \Rightarrow [E*\overline{F}*K]$
$[E*\underline{F}*K] \Rightarrow [E*F*\overline{K}]$	$[E*F*\underline{K}] \Rightarrow \underline{[E*F*K]}$	

Let $E, F \in RegStatExpr,\ G, H, \widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

IRULES2: Inaction rules for arbitrary regular dynamic expressions

$$\frac{G \Rightarrow \widetilde{G},\ \circ \in \{;, []\}}{G \circ E \Rightarrow \widetilde{G} \circ E} \qquad \frac{G \Rightarrow \widetilde{G},\ \circ \in \{;, []\}}{E \circ G \Rightarrow E \circ \widetilde{G}} \qquad \frac{G \Rightarrow \widetilde{G}}{G \| H \Rightarrow \widetilde{G} \| H} \qquad \frac{H \Rightarrow \widetilde{H}}{G \| H \Rightarrow G \| \widetilde{H}}$$

$$\frac{G \Rightarrow \widetilde{G}}{G[f] \Rightarrow \widetilde{G}[f]} \qquad \frac{G \Rightarrow \widetilde{G},\ \circ \in \{\text{rs}, \text{sy}\}}{G \circ a \Rightarrow \widetilde{G} \circ a} \qquad \frac{G \Rightarrow \widetilde{G}}{[G * E * F] \Rightarrow [\widetilde{G} * E * F]} \qquad \frac{G \Rightarrow \widetilde{G}}{[E * G * F] \Rightarrow [E * \widetilde{G} * F]}$$

$$\frac{G \Rightarrow \widetilde{G}}{[E * F * G] \Rightarrow [E * F * \widetilde{G}]}$$

Definition 155 *A regular dynamic expression is operative if no inaction rule can be applied to it.*

$OpRegDynExpr$ is the set of *all operative regular dynamic expressions* of $dtsiPBC$.

We shall consider regular expressions only and omit the word “regular”.

Definition 156 $\approx\ = (\Rightarrow \cup \Leftarrow)^*$ *is the structural equivalence of dynamic expressions in* $dtsiPBC$*.*

G *and* G' *are structurally equivalent,* $G \approx G'$*, if they can be reached each from other by applying inaction rules in a forward or backward direction.*

Action and empty loop rules

Action rules with stochastic multiactions: execution of non-empty multisets of stochastic multiactions.

Action rules with immediate multiactions: execution of non-empty multisets of immediate multiactions.

Empty loop rule: execution of the empty multiset of activities at a time step.

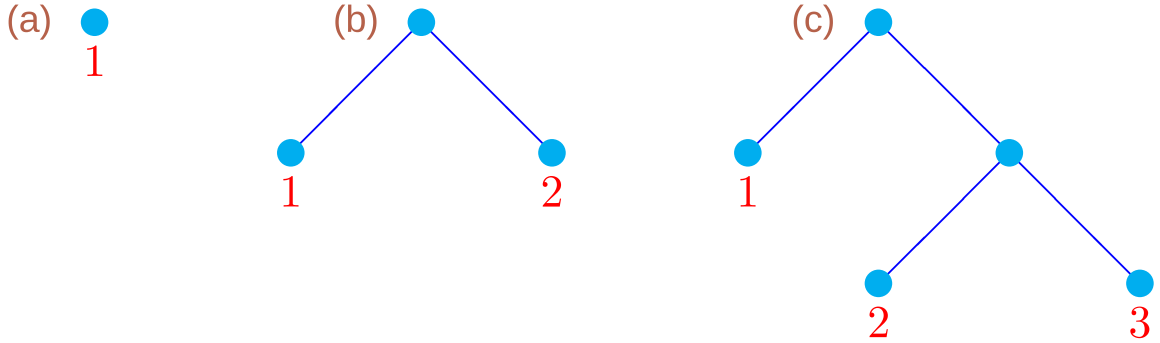
Definition 157 *Let* $n \in I\!N$*. The* numbering *of expressions is*

$$\iota ::= n \mid (\iota)(\iota).$$

Num is the set of *all numberings* of expressions.

The *content* of a numbering $\iota \in Num$ is

$$Cont(\iota) = \begin{cases} \{\iota\}, & \iota \in I\!N; \\ Cont(\iota_1) \cup Cont(\iota_2), & \iota = (\iota_1)(\iota_2). \end{cases}$$



BTRNUM: The binary trees encoded with the numberings 1, $(1)(2)$ and $(1)((2)(3))$

$[G]_\approx = \{H \mid G \approx H\}$ is the equivalence class of $G \in RegDynExpr$ w.r.t. structural equivalence.

G is an *initial* dynamic expression, $init(G)$, if $\exists E \in RegStatExpr\ G \in [\overline{E}]_\approx$.

G is a *final* dynamic expression, $final(G)$, if $\exists E \in RegStatExpr\ G \in [\underline{E}]_\approx$.

Definition 158 *Let $G \in OpRegDynExpr$. The set of all non-empty multisets of activities which can be potentially executed from G is $Can(G)$. Let $(\alpha, \kappa) \in \mathcal{SIL}$, $E, F \in RegStatExpr$, $H \in OpRegDynExpr$ and $a \in Act$.*

1. *If $final(G)$ then $Can(G) = \emptyset$.*
2. *If $G = \overline{(\alpha, \kappa)}$ then $Can(G) = \{\{(\alpha, \kappa)\}\}$.*
3. *If $\Upsilon \in Can(G)$ then $\Upsilon \in Can(G \circ E)$, $\Upsilon \in Can(E \circ G)$ $(\circ \in \{;, [], \|\})$, $f(\Upsilon) \in Can(G[f])$, $\Upsilon \in Can(G \text{ rs } a)$ (when $a, \hat{a} \notin \mathcal{A}(\Upsilon)$), $\Upsilon \in Can(G \text{ sy } a)$, $\Upsilon \in Can([G * E * F])$, $\Upsilon \in Can([E * G * F])$, $\Upsilon \in Can([E * F * G])$.*
4. *If $\Upsilon \in Can(G)$ and $\Xi \in Can(H)$ then $\Upsilon + \Xi \in Can(G \| H)$.*
5. *If $\Upsilon \in Can(G \text{ sy } a)$ and $(\alpha, \kappa), (\beta, \lambda) \in \Upsilon$ are different activities such that $a \in \alpha$, $\hat{a} \in \beta$, then*
   (a) $(\Upsilon + \{(\alpha \oplus_a \beta, \kappa \cdot \lambda)\}) \setminus \{(\alpha, \kappa), (\beta, \lambda)\} \in Can(G \text{ sy } a)$, if $\kappa, \lambda \in (0; 1)$;
   (b) $(\Upsilon + \{(\alpha \oplus_a \beta, \kappa + \lambda)\}) \setminus \{(\alpha, \kappa), (\beta, \lambda)\} \in Can(G \text{ sy } a)$, if $\kappa, \lambda \in \mathbb{N}_{\geq 1}$.

If $\Upsilon \in Can(G)$ then by definition of $Can(G)$ $\forall \Xi \subseteq \Upsilon$, $\Xi \neq \emptyset$ we have $\Xi \in Can(G)$.

If there are only stochastic (or only immediate) multiactions in the multisets from $Can(G) \neq \emptyset$: these stochastic (or immediate) multiactions can be executed from G.

Otherwise, besides stochastic ones, there are immediate multiactions in the multisets from $Can(G)$.

By the note above, there are non-empty multisets of immediate multiactions in $Can(G)$ as well: $\exists \Upsilon \in Can(G)$ $\Upsilon \in \mathbb{N}_{fin}^{\mathcal{IL}} \setminus \{\emptyset\}$.

Then no stochastic multiactions can be executed from G, even if $Can(G)$ contains non-empty multisets of stochastic multiactions: immediate multiactions have a priority over stochastic ones, and should be executed first.

Definition 159 *Let* $G \in OpRegDynExpr$. *The* set of all non-empty multisets of activities which can be executed from G *is*

$$Now(G) = \begin{cases} Can(G), & (Can(G) \subseteq \mathbb{N}_{fin}^{\mathcal{SL}} \setminus \{\emptyset\}) \vee \\ & (Can(G) \subseteq \mathbb{N}_{fin}^{\mathcal{IL}} \setminus \{\emptyset\}); \\ Can(G) \cap \mathbb{N}_{fin}^{\mathcal{IL}}, & \textit{otherwise}. \end{cases}$$

G is *tangible*, $tang(G)$, if $Now(G) \subseteq \mathbb{N}_{fin}^{\mathcal{SL}} \setminus \{\emptyset\}$. We have $tang(G)$, if $Now(G) = \emptyset$.

G is *vanishing*, $vanish(G)$, if $\emptyset \neq Now(G) \subseteq \mathbb{N}_{fin}^{\mathcal{IL}} \setminus \{\emptyset\}$.

Let $G = (\overline{(\{a\},1)}[](\{b\},2))\|\overline{(\{c\},\frac{1}{2})}$ and $G' = ((\{a\},1)[]\overline{(\{b\},2)})\|\overline{(\{c\},\frac{1}{2})}$.

We have $G \approx G'$, since $G \Leftarrow G'' \Rightarrow G'$ for $G'' = \overline{((\{a\},\natural_1)[](\{b\},\natural_2))}\|\overline{(\{c\},\frac{1}{2})}$, but
$Can(G) = \{\{(\{a\},1)\},\{(\{c\},\frac{1}{2})\},\{(\{a\},1),(\{c\},\frac{1}{2})\}\}$,
$Can(G') = \{\{(\{b\},2)\},\{(\{c\},\frac{1}{2})\},\{(\{b\},2),(\{c\},\frac{1}{2})\}\}$ and
$Now(G) = \{\{(\{a\},1)\}\},\ Now(G') = \{\{(\{b\},2)\}\}$.

Clearly, $vanish(G)$ and $vanish(G')$.

The executions like that of $\{(\{c\},\frac{1}{2})\}$ (and all multisets including it) from G and G' must be disabled using pre-conditions in the action rules.

Immediate multiactions have a priority over stochastic ones:
the former are always executed first.

Let $H = \overline{(\{a\}, 1)}[](\{b\}, \frac{1}{2})$ and $H' = (\{a\}, 1)[]\overline{(\{b\}, \frac{1}{2})}$.

Then $H \approx H'$, since $H \Leftarrow H'' \Rightarrow H'$ for $H'' = \overline{(\{a\}, \natural_1)[](\{b\}, \frac{1}{2})}$, but
$Can(H) = Now(H) = \{\{(\{a\}, 1)\}\}$ and
$Can(H') = Now(H') = \{\{(\{b\}, \frac{1}{2})\}\}$.

We have $vanish(H)$, but $tang(H')$.

To make the action rules correct under structural equivalence: the executions like that of $\{(\{b\}, \frac{1}{2})\}$ from H' must be disabled using the pre-conditions.

Immediate multiactions have a priority over stochastic ones:
the choices between them are always resolved in favour of the former.

Let $(\alpha, \rho), (\beta, \chi) \in \mathcal{SL}$, $(\alpha, l), (\beta, m) \in \mathcal{IL}$ and $(\alpha, \kappa) \in \mathcal{SIL}$.

Further, $E, F \in RegStatExpr$, $G, H \in OpRegDynExpr$,
$\widetilde{G}, \widetilde{H} \in RegDynExpr$ and $a \in Act$.

Moreover, $\Gamma, \Delta \in I\!N_{fin}^{\mathcal{SL}} \setminus \{\emptyset\}$, $\Gamma' \in I\!N_{fin}^{\mathcal{SL}}$, $I, J \in I\!N_{fin}^{\mathcal{IL}} \setminus \{\emptyset\}$,
$I' \in I\!N_{fin}^{\mathcal{IL}}$ and $\Upsilon \in I\!N_{fin}^{\mathcal{SIL}} \setminus \{\emptyset\}$.

The names of the action rules with immediate multiactions have a suffix '$\mathbf{i}$'.

ARULES: Action and empty loop rules

El $\dfrac{tang(G)}{G \xrightarrow{\emptyset} G}$

B $\overline{(\alpha,\kappa)} \xrightarrow{\{(\alpha,\kappa)\}} \underline{(\alpha,\kappa)}$

S $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{G;E \xrightarrow{\Upsilon} \widetilde{G};E \quad E;G \xrightarrow{\Upsilon} E;\widetilde{G}}$

C $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ \neg init(G) \vee (init(G) \wedge tang(\overline{E}))}{G[]E \xrightarrow{\Gamma} \widetilde{G}[]E \quad E[]G \xrightarrow{\Gamma} E[]\widetilde{G}}$

Ci $\dfrac{G \xrightarrow{I} \widetilde{G}}{G[]E \xrightarrow{I} \widetilde{G}[]E \quad E[]G \xrightarrow{I} E[]\widetilde{G}}$

P1 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ tang(H)}{G\|H \xrightarrow{\Gamma} \widetilde{G}\|H \quad H\|G \xrightarrow{\Gamma} H\|\widetilde{G}}$

P1i $\dfrac{G \xrightarrow{I} \widetilde{G}}{G\|H \xrightarrow{I} \widetilde{G}\|H \quad H\|G \xrightarrow{I} H\|\widetilde{G}}$

P2 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ H \xrightarrow{\Delta} \widetilde{H}}{G\|H \xrightarrow{\Gamma+\Delta} \widetilde{G}\|\widetilde{H}}$

P2i $\dfrac{G \xrightarrow{I} \widetilde{G},\ H \xrightarrow{J} \widetilde{H}}{G\|H \xrightarrow{I+J} \widetilde{G}\|\widetilde{H}}$

L $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{G[f] \xrightarrow{f(\Upsilon)} \widetilde{G}[f]}$

Rs $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G},\ a,\hat{a} \notin \mathcal{A}(\Upsilon)}{G \text{ rs } a \xrightarrow{\Upsilon} \widetilde{G} \text{ rs } a}$

I1 $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{[G*E*F] \xrightarrow{\Upsilon} [\widetilde{G}*E*F]}$

I2 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ \neg init(G) \vee (init(G) \wedge tang(\overline{F}))}{[E*G*F] \xrightarrow{\Gamma} [E*\widetilde{G}*F]}$

I2i $\dfrac{G \xrightarrow{I} \widetilde{G}}{[E*G*F] \xrightarrow{I} [E*\widetilde{G}*F]}$

I3 $\dfrac{G \xrightarrow{\Gamma} \widetilde{G},\ \neg init(G) \vee (init(G) \wedge tang(\overline{F}))}{[E*F*G] \xrightarrow{\Gamma} [E*F*\widetilde{G}]}$

I3i $\dfrac{G \xrightarrow{I} \widetilde{G}}{[E*F*G] \xrightarrow{I} [E*F*\widetilde{G}]}$

Sy1 $\dfrac{G \xrightarrow{\Upsilon} \widetilde{G}}{G \text{ sy } a \xrightarrow{\Upsilon} \widetilde{G} \text{ sy } a}$

Sy2 $\dfrac{G \text{ sy } a \xrightarrow{\Gamma'+\{(\alpha,\rho)\}+\{(\beta,\chi)\}} \widetilde{G} \text{ sy } a,\ a \in \alpha,\ \hat{a} \in \beta}{G \text{ sy } a \xrightarrow{\Gamma'+\{(\alpha \oplus_a \beta, \rho\cdot\chi)\}} \widetilde{G} \text{ sy } a}$

Sy2i $\dfrac{G \text{ sy } a \xrightarrow{I'+\{(\alpha,l)\}+\{(\beta,m)\}} \widetilde{G} \text{ sy } a,\ a \in \alpha,\ \hat{a} \in \beta}{G \text{ sy } a \xrightarrow{I'+\{(\alpha \oplus_a \beta, l+m)\}} \widetilde{G} \text{ sy } a}$

RULECMP: Comparison of inaction, action and empty loop rules

Rules	State change	Time progress	Activities execution
Inaction rules	−	−	−
Action rules (stochastic multiactions)	±	+	+
Action rules (immediate multiactions)	±	−	+
Empty loop rule	−	+	−

Transition systems

Definition 160 *The derivation set $DR(G)$ of a dynamic expression G is the minimal set:*

- $[G]_\approx \in DR(G)$;
- *if* $[H]_\approx \in DR(G)$ *and* $\exists \Upsilon\ H \stackrel{\Upsilon}{\to} \widetilde{H}$ *then* $[\widetilde{H}]_\approx \in DR(G)$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

The set of *all multisets of activities executable from s* is
$Exec(s) = \{\Upsilon \mid \exists H \in s\ \exists \widetilde{H}\ H \stackrel{\Upsilon}{\to} \widetilde{H}\}$.

The state s is *tangible*, if $Exec(s) \subseteq I\!N_{fin}^{\mathcal{SL}}$.
For tangible states we may have $Exec(s) = \{\emptyset\}$.

The state s is *vanishing*, if $Exec(s) \subseteq I\!N_{fin}^{\mathcal{IL}} \setminus \{\emptyset\}$.

The set of *all tangible states from $DR(G)$* is $DR_T(G)$.

The set of *all vanishing states from $DR(G)$* is $DR_V(G)$.

Obviously, $DR(G) = DR_T(G) \uplus DR_V(G)$.

Let $\Upsilon \in Exec(s) \setminus \{\emptyset\}$. The *probability of the multiset of stochastic multiactions* or the *weight of the multiset of immediate multiactions* Υ *which is ready for execution in* s:

$$PF(\Upsilon, s) = \begin{cases} \prod_{(\alpha,\rho)\in\Upsilon} \rho \cdot \\ \prod_{\{\{(\beta,\chi)\}\in Exec(s) \mid (\beta,\chi)\not\in\Upsilon\}} (1-\chi), & s \in DR_T(G); \\ \sum_{(\alpha,l)\in\Upsilon} l, & s \in DR_V(G). \end{cases}$$

In the case $\Upsilon = \emptyset$ and $s \in DR_T(G)$ we define

$$PF(\emptyset, s) = \begin{cases} \prod_{\{(\beta,\chi)\}\in Exec(s)} (1-\chi), & Exec(s) \neq \{\emptyset\}; \\ 1, & Exec(s) = \{\emptyset\}. \end{cases}$$

Let $\Upsilon \in Exec(s)$. The *probability to execute the multiset of activities* Υ *in* s:

$$PT(\Upsilon, s) = \frac{PF(\Upsilon, s)}{\sum_{\Xi\in Exec(s)} PF(\Xi, s)}.$$

If s is tangible, then $PT(\emptyset, s) \in (0; 1]$: the residence time in s is ≥ 1.

The *probability to move from* s *to* $\tilde{s}$ *by executing any multiset of activities*:

$$PM(s, \tilde{s}) = \sum_{\{\Upsilon \mid \exists H\in s\ \exists \widetilde{H}\in\tilde{s}\ H \xrightarrow{\Upsilon} \widetilde{H}\}} PT(\Upsilon, s).$$

Definition 161 *The (labeled probabilistic) transition system of a dynamic expression* G *is* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, *where*

- *the set of states is* $S_G = DR(G)$;
- *the set of labels is* $L_G = \mathbb{N}_{fin}^{\mathcal{SIL}} \times (0; 1]$;
- *the set of transitions is* $\mathcal{T}_G = \{(s, (\Upsilon, PT(\Upsilon, s)), \tilde{s}) \mid s, \tilde{s} \in DR(G),\ \exists H \in s\ \exists \widetilde{H} \in \tilde{s}\ H \stackrel{\Upsilon}{\to} \widetilde{H}\}$;
- *the initial state is* $s_G = [G]_{\approx}$.

A transition $(s, (\Upsilon, \mathcal{P}), \tilde{s}) \in \mathcal{T}_G$ is written as $s \stackrel{\Upsilon}{\to}_{\mathcal{P}} \tilde{s}$.

We write $s \stackrel{\Upsilon}{\to} \tilde{s}$ if $\exists \mathcal{P}\ s \stackrel{\Upsilon}{\to}_{\mathcal{P}} \tilde{s}$ and $s \to \tilde{s}$ if $\exists \Upsilon\ s \stackrel{\Upsilon}{\to} \tilde{s}$.

Definition 162 *Let* G, G' *be dynamic expressions and* $TS(G) = (S_G, L_G, \mathcal{T}_G, s_G)$, $TS(G') = (S_{G'}, L_{G'}, \mathcal{T}_{G'}, s_{G'})$ *be their transition systems. A mapping* $\beta : S_G \to S_{G'}$ *is an isomorphism between* $TS(G)$ *and* $TS(G')$, $\beta : TS(G) \simeq TS(G')$, *if*

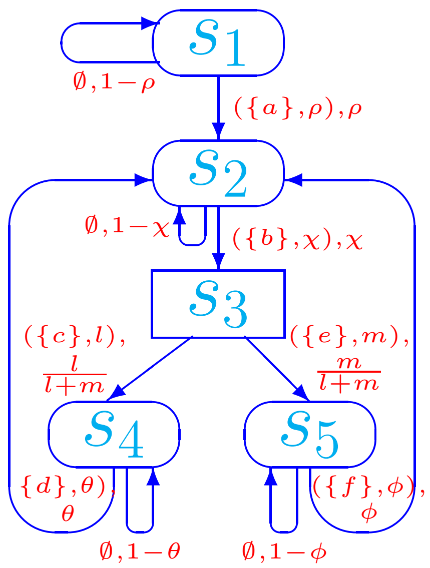
1. β *is a bijection s.t.* $\beta(s_G) = s_{G'}$;
2. $\forall s, \tilde{s} \in S_G\ \forall \Upsilon\ s \stackrel{\Upsilon}{\to}_{\mathcal{P}} \tilde{s} \Leftrightarrow \beta(s) \stackrel{\Upsilon}{\to}_{\mathcal{P}} \beta(\tilde{s})$.

$TS(G)$ *and* $TS(G')$ *are isomorphic,* $TS(G) \simeq TS(G')$, *if* $\exists \beta : TS(G) \simeq TS(G')$.

For $E \in RegStatExpr$, let $TS(E) = TS(\overline{E})$.

Definition 163 G *and* G' *are equivalent w.r.t. transition systems,* $G =_{ts} G'$, *if* $TS(G) \simeq TS(G')$.

$TS(\overline{E})$



TS: The transition system of $\overline{E}$ for

$$E = [(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \text{Stop}]$$

$\text{Stop} = (\{c\},\frac{1}{2})$ rs c is the process that performs empty loops with probability 1 and never terminates.

$DR(\overline{E})$ consists of:

$s_1 = [[\overline{(\{a\},\rho)} * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \text{Stop}]]_\approx$,

$s_2 = [[(\{a\},\rho) * (\overline{(\{b\},\chi)};(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \text{Stop}]]_\approx$,

$s_3 = [[(\{a\},\rho) * ((\{b\},\chi);\overline{(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))}) * \text{Stop}]]_\approx$,

$s_4 = [[(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);\overline{(\{d\},\theta)})[]((\{e\},m);(\{f\},\phi)))) * \text{Stop}]]_\approx$,

$s_5 = [[(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);\overline{(\{f\},\phi)}))) * \text{Stop}]]_\approx$.

$DR_T(\overline{E}) = \{s_1, s_2, s_4, s_5\}$ and $DR_V(\overline{E}) = \{s_3\}$.

Denotational semantics

Labeled DTSIPNs

Definition 164 *A* labeled discrete time stochastic and immediate Petri net (LDTSIPN) *is* $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$, *where*

- P_N *and* $T_N = Ts_N \uplus Ti_N$ *are finite sets of* places *and* stochastic and immediate transitions, *s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$*;*
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbb{N}$ *is the* arc weight *function;*
- Ω_N *is the* transition probability and weight *function s.t.*
  - $\Omega_N|_{Ts_N} : Ts_N \to (0; 1)$ *(it associates stochastic transitions with* probabilities*);*
  - $\Omega_N|_{Ti_N} : Ti_N \to \mathbb{N}_{\geq 1}$ *(it associates immediate transitions with* weights*);*
- $L_N : T_N \to \mathcal{L}$ *is the* transition labeling *function;*
- $M_N \in \mathbb{N}_{fin}^{P_N}$ *is the* initial marking*.*

Concurrent transition firings at discrete time moments.

LDTSIPNs have *step* semantics.

Let N be an LDTSIPN and $M, \widetilde{M} \in \mathbb{N}_{fin}^{P_N}$.

Immediate transitions have a priority over stochastic ones:
immediate transitions always fire first, if they can.

A transition $t \in T_N$ is *enabled* in M if ${}^\bullet t \subseteq M$ and one of the following holds:

1. $t \in Ti_N$ or
2. $\forall u \in T_N \ {}^\bullet u \subseteq M \ \Rightarrow \ u \in Ts_N$.

$Ena(M)$ is the set of *all transitions enabled in* M.

$Ena(M) \subseteq Ti_N$ or $Ena(M) \subseteq Ts_N$

A set of transitions $U \subseteq Ena(M)$ is *enabled* in M if ${}^\bullet U \subseteq M$.

The marking M is *tangible*, $tang(M)$, if $Ena(M) \subseteq Ts_N$, in particular, if $Ena(M) = \emptyset$.

The marking M is *vanishing*, $vanish(M)$, if $Ena(M) \subseteq Ti_N$ and $Ena(M) \neq \emptyset$.

If $tang(M)$ then a stochastic transition $t \in Ena(M)$ fires in the next time moment with probability $\Omega_N(t)$, if no other conflicting stochastic transition is enabled in M.

Let $U \subseteq Ena(M)$, $U \neq \emptyset$ and ${}^{\bullet}U \subseteq M$. The *probability of the set of stochastic transitions* or the *weight of the set of immediate transitions U which is ready for firing in M* is

$$PF(U, M) = \begin{cases} \prod_{t \in U} \Omega_N(t) \cdot \prod_{u \in Ena(M) \setminus U} (1 - \Omega_N(u)), & tang(M); \\ \sum_{t \in U} \Omega_N(t), & vanish(M). \end{cases}$$

In the case $U = \emptyset$ and $tang(M)$ we define

$$PF(\emptyset, M) = \begin{cases} \prod_{u \in Ena(M)} (1 - \Omega_N(u)), & Ena(M) \neq \emptyset; \\ 1, & Ena(M) = \emptyset. \end{cases}$$

Let $U \subseteq Ena(M)$ and ${}^\bullet U \subseteq M$. The *probability that the set of transitions U fires in M*:

$$PT(U, M) = \frac{PF(U, M)}{\sum_{\{V \mid {}^\bullet V \subseteq M\}} PF(V, M)}.$$

If $U = \emptyset$ and $tang(M)$ then $M = \widetilde{M}$.

If $tang(M)$ then $PT(\emptyset, M) \in (0; 1]$: the residence time in M is ≥ 1.

Firing of U changes M to $\widetilde{M} = M - {}^\bullet U + U^\bullet$, $M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$, where $\mathcal{P} = PT(U, M)$.

We write $M \xrightarrow{U} \widetilde{M}$ if $\exists \mathcal{P}\ M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$ and $M \rightarrow \widetilde{M}$ if $\exists U\ M \xrightarrow{U} \widetilde{M}$.

The *probability to move from M to $\widetilde{M}$ by firing any set of transitions*:

$$PM(M, \widetilde{M}) = \sum_{\{U \mid M \xrightarrow{U} \widetilde{M}\}} PT(U, M).$$

We write $M \xrightarrow{U} \widetilde{M}$ if $\exists \mathcal{P}\ M \xrightarrow{U}_{\mathcal{P}} \widetilde{M}$ and $M \rightarrow \widetilde{M}$ if $\exists U\ M \xrightarrow{U} \widetilde{M}$.

Definition 165 *Let N be an LDTSIPN.*

- *The* reachability set $RS(N)$ *is the minimal set of markings s.t.*
  - $M_N \in RS(N)$;
  - *if* $M \in RS(N)$ *and* $M \rightarrow \widetilde{M}$ *then* $\widetilde{M} \in RS(N)$.
- *The* reachability graph $RG(N)$ *is a directed labeled graph with*
  - *the set of nodes* $RS(N)$;
  - *an arc labeled by* $(U, \mathcal{P})$ *from node* M *to* $\widetilde{M}$ *if* $M \stackrel{U}{\rightarrow}_{\mathcal{P}} \widetilde{M}$.

The set of *all tangible markings from* $RS(N)$ is $RS_T(N)$.

The set of *all vanishing markings from* $RS(N)$ is $RS_V(N)$.

$RS(N) = RS_T(N) \cup RS_V(N)$.

Algebra of dtsi-boxes

Definition 166 *A* discrete time stochastic and immediate Petri box (dtsi-box) *is* $N = (P_N, T_N, W_N, \Lambda_N)$, *where:*

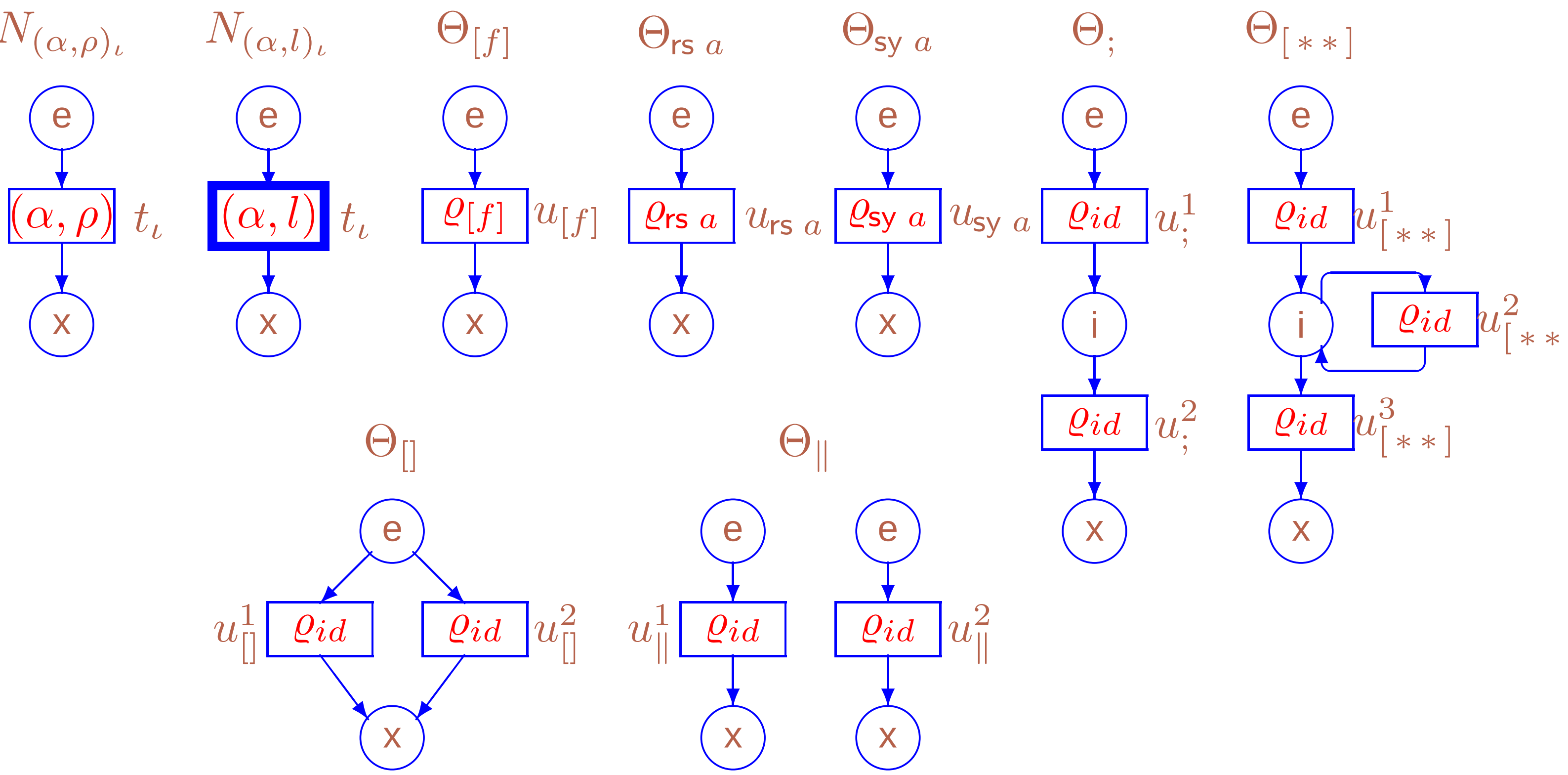
- P_N *and* T_N *are finite sets of* places *and* transitions, *s.t.* $P_N \cup T_N \neq \emptyset$ *and* $P_N \cap T_N = \emptyset$;
- $W_N : (P_N \times T_N) \cup (T_N \times P_N) \to \mathbb{N}$ *is a function of the* weights of arcs *between places and transitions and vice versa;*
- Λ_N *is the* place and transition labeling *function s.t.*
  - $\Lambda_N|_{P_N} : P_N \to \{\mathrm{e}, \mathrm{i}, \mathrm{x}\}$ *(it specifies* entry, internal *and* exit *places);*
  - $\Lambda_N|_{T_N} : T_N \to \{\varrho \mid \varrho \subseteq \mathbb{N}_{fin}^{\mathcal{SL}} \times \mathcal{SL}\}$ *(it associates transitions with the* relabeling relations*).*

Moreover, $\forall t \in T_N\ {}^\bullet t \neq \emptyset \neq t^\bullet$.

For the set of entry *places of* N, ${}^\circ N = \{p \in P_N \mid \Lambda_N(p) = \mathrm{e}\}$, *and the set of* exit *places of* N, $N^\circ = \{p \in P_N \mid \Lambda_N(p) = \mathrm{x}\}$, *it holds:* ${}^\circ N \neq \emptyset \neq N^\circ$ *and* ${}^\bullet({}^\circ N) = \emptyset = (N^\circ)^\bullet$.

A dtsi-box is *plain* if $\forall t \in T_N\ \Lambda_N(t) = \varrho_{(\alpha,\kappa)}$, where $\varrho_{(\alpha,\kappa)} = \{(\emptyset, (\alpha, \kappa))\}$ is a *constant relabeling*, identified with (α, κ).

A *marked plain dtsi-box* is a pair (N, M_N), where N is a plain dtsi-box and $M_N \in \mathbb{N}_{fin}^{P_N}$ is its marking. Let $\overline{N} = (N, {}^\circ N)$ and $\underline{N} = (N, N^\circ)$.



BOXOPS: The plain and operator dtsi-boxes

Definition 167 *Let* $(\alpha, \kappa) \in \mathcal{SIL}$, $a \in Act$ *and* $E, F, K \in RegStatExpr$. *The* denotational semantics *of* $dtsiPBC$ *is a mapping* Box_{dtsi} *from* $RegStatExpr$ *into plain dtsi-boxes:*

1. $Box_{dtsi}((\alpha, \kappa)_\iota) = N_{(\alpha,\kappa)_\iota}$;
2. $Box_{dtsi}(E \circ F) = \Theta_\circ(Box_{dtsi}(E), Box_{dtsi}(F)),\ \circ \in \{;, [], \|\}$;
3. $Box_{dtsi}(E[f]) = \Theta_{[f]}(Box_{dtsi}(E))$;
4. $Box_{dtsi}(E \circ a) = \Theta_{\circ a}(Box_{dtsi}(E)),\ \circ \in \{\text{rs}, \text{sy}\}$;
5. $Box_{dtsi}([E * F * K]) =$
   $\Theta_{[**]}(Box_{dtsi}(E), Box_{dtsi}(F), Box_{dtsi}(K))$.

For $E \in RegStatExpr$, let $Box_{dtsi}(\overline{E}) = \overline{Box_{dtsi}(E)}$ and $Box_{dtsi}(\underline{E}) = \underline{Box_{dtsi}(E)}$.

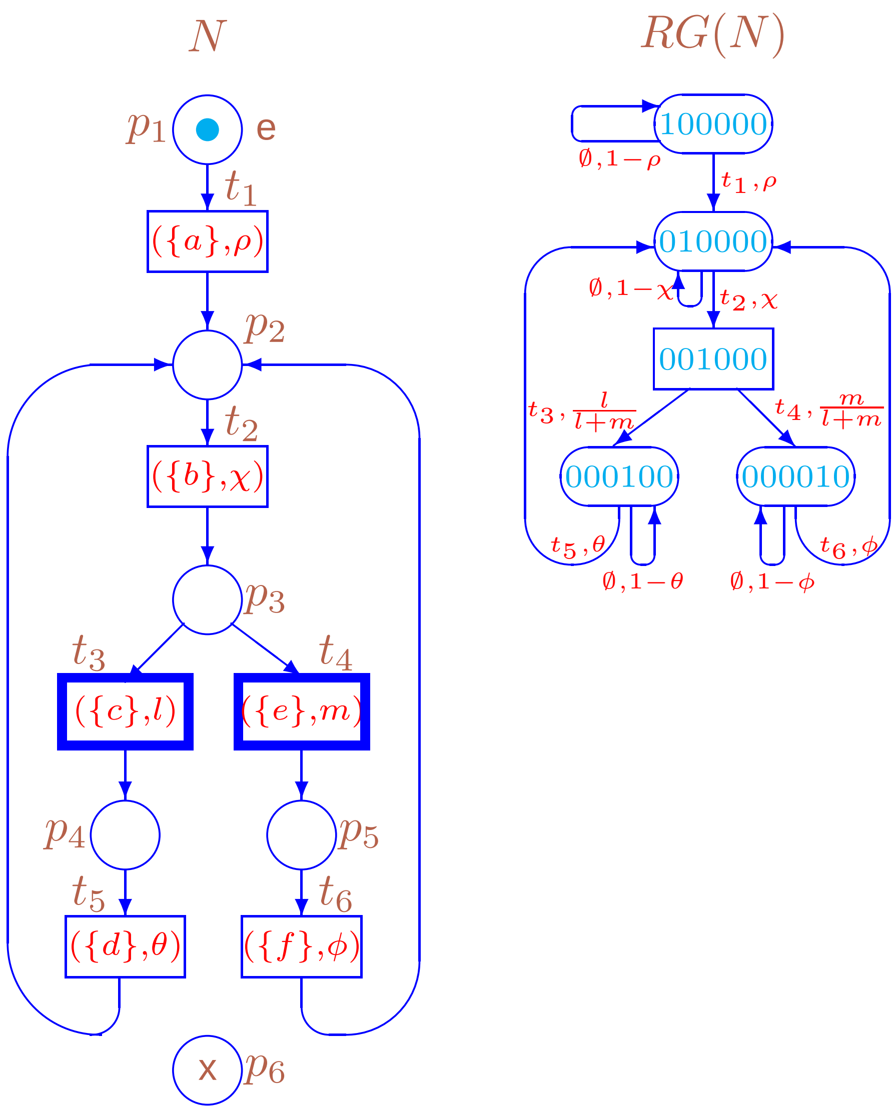
We denote isomorphism of transition systems by $\simeq$,

and the same symbol denotes isomorphism of reachability graphs and DTMCs

as well as isomorphism between transition systems and reachability graphs.

Theorem 43 (*OPDNSEM*) *For any static expression* E

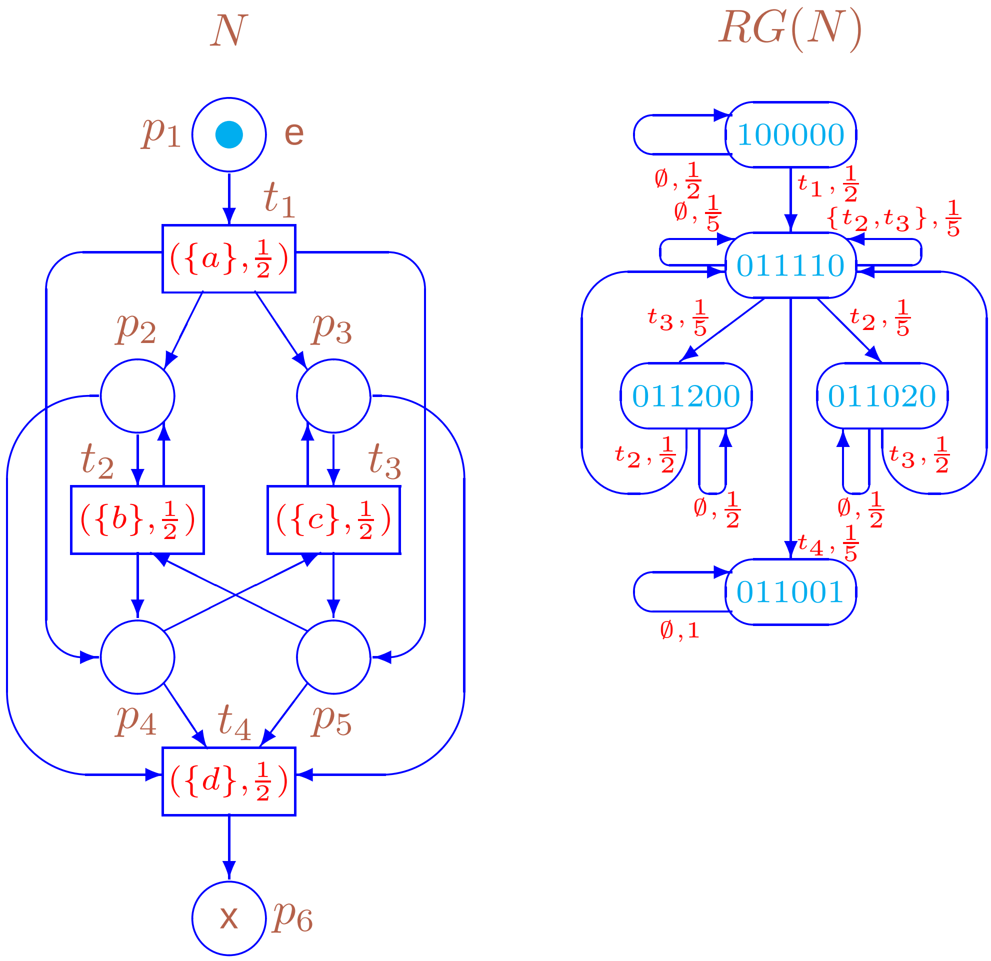
$$TS(\overline{E}) \simeq RG(Box_{dtsi}(\overline{E})).$$



BOXRG: The marked dtsi-box $N = Box_{dtsi}(\overline{E})$ for

$E = [(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \mathsf{Stop}]$

and its reachability graph



NRBOXRG: The marked dtsi-box $N = Box_{dtsi}(\overline{E})$ for $E = [((\{a\},\frac{1}{2}) * ((\{b\},\frac{1}{2})\|(\{c\},\frac{1}{2})) * (\{d\},\frac{1}{2})]$ and its reachability graph

$M_1 = (1, 0, 0, 0, 0, 0)$ is the initial marking.

$M_2 = (0, 1, 1, 1, 1, 0)$ is obtained from M_1 by firing t_1.

$M_3 = (0, 1, 1, 2, 0, 0)$ is obtained from M_2 by firing t_2 and has 2 tokens in the place p_4.

$M_4 = (0, 1, 1, 0, 2, 0)$ is obtained from M_2 by firing t_3 and has 2 tokens in the place p_5.

Concurrency in the second argument of iteration in $\overline{E}$ can lead to non-safeness of the corresponding marked dtsi-box N, but it is 2-bounded in the worst case.

The origin of the problem: N has as a self-loop with two subnets which can function independently.

Performance evaluation

Analysis of the underlying SMC

For a dynamic expression G, a discrete random variable is associated with every tangible state from $DR_T(G)$.

The random variables (residence time in the tangible states) are geometrically distributed: the probability to stay in the tangible state $s \in DR_T(G)$ for $k-1$ moments
and leave it at the moment $k \geq 1$ is $PM(s,s)^{k-1}(1-PM(s,s))$.

The mean value formula: the *average sojourn time in the tangible state* s is $\frac{1}{1-PM(s,s)}$.

The *average sojourn time in the vanishing state* s is 0.

The *average sojourn time in the state* s is

$$SJ(s) = \begin{cases} \frac{1}{1-PM(s,s)}, & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

The *average sojourn time vector* SJ of G has the elements $SJ(s),\ s \in DR(G)$.

The *sojourn time variance in the state* s is

$$VAR(s) = \begin{cases} \frac{PM(s,s)}{(1-PM(s,s))^2}, & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

The *sojourn time variance vector* VAR of G has the elements $VAR(s),\ s \in DR(G)$.

The stochastic process associated with a dynamic expression G: the *underlying semi-Markov chain (SMC)* of G, $SMC(G)$.

$SMC(G)$ is analyzed by extracting the *embedded (absorbing) discrete time Markov chain (EDTMC)* of G, $EDTMC(G)$.

Let G be a dynamic expression and $s, \tilde{s} \in DR(G)$.

Let $s \to s$. The *probability to stay in s due to k $(k \geq 1)$ self-loops* is $PM(s, s)^k$.

Let $s \to \tilde{s}$ and $s \neq \tilde{s}$. The *probability to move from s to $\tilde{s}$ by executing any multiset of activities after possible self-loops* is

$$PM^*(s, \tilde{s}) = \left\{ \begin{array}{ll} PM(s, \tilde{s}) \sum_{k=0}^{\infty} PM(s, s)^k = \frac{PM(s,\tilde{s})}{1-PM(s,s)}, & s \to s; \\ PM(s, \tilde{s}), & \text{otherwise}; \end{array} \right\}$$

$$= SL(s)PM(s, \tilde{s}), \text{ where } SL(s) = \left\{ \begin{array}{ll} \frac{1}{1-PM(s,s)}, & s \to s; \\ 1, & \text{otherwise}; \end{array} \right.$$

is the *self-loops abstraction factor in the state s*.

The *self-loops abstraction vector SL* of G has the elements $SL(s),\ s \in DR(G)$.

We have $\forall s \in DR_T(G)\ SL(s) = \frac{1}{1-PM(s,s)} = SJ(s)$, hence, $\forall s \in DR_T(G)\ PM^*(s, \tilde{s}) = SJ(s)PM(s, \tilde{s})$.

Definition 168 *Let G be a dynamic expression. The* embedded (absorbing) discrete time Markov chain (EDTMC) *of G, $EDTMC(G)$, has the state space $DR(G)$, the initial state $[G]_{\approx}$ and the transitions $s \twoheadrightarrow_{\mathcal{P}} \tilde{s}$, if $s \to \tilde{s}$ and $s \neq \tilde{s}$, where $\mathcal{P} = PM^*(s, \tilde{s})$.*

The underlying SMC *of G, $SMC(G)$, has the EDTMC $EDTMC(G)$ and the sojourn time in every $s \in DR_T(G)$ is geometrically distributed with the parameter $1 - PM(s, s)$ while the sojourn time in every $s \in DR_V(G)$ is equal to zero.*

For $E \in RegStatExpr$, let $EDTMC(E) = EDTMC(\overline{E})$ and $SMC(E) = SMC(\overline{E})$.

Let G be a dynamic expression. The elements $\mathcal{P}^*_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of *(one-step) transition probability matrix (TPM)* $\mathbf{P}^*$ *for* $EDTMC(G)$:

$$\mathcal{P}^*_{ij} = \begin{cases} PM^*(s_i, s_j), & s_i \to s_j,\ s_i \neq s_j; \\ 0, & \text{otherwise}. \end{cases}$$

The *transient (k-step, $k \in \mathbb{N}$) probability mass function (PMF)* $\psi^*[k] = (\psi^*[k](s_1), \ldots, \psi^*[k](s_n))$ *for* $EDTMC(G)$ is calculated as

$$\psi^*[k] = \psi^*[0](\mathbf{P}^*)^k,$$

where $\psi^*[0] = (\psi^*[0](s_1), \ldots, \psi^*[0](s_n))$ is the *initial PMF*:

$$\psi^*[0](s_i) = \begin{cases} 1, & s_i = [G]_\approx; \\ 0, & \text{otherwise}. \end{cases}$$

We have $\psi^*[k+1] = \psi^*[k]\mathbf{P}^*$ $(k \in \mathbb{N})$.

The *steady-state PMF* $\psi^* = (\psi^*(s_1), \ldots, \psi^*(s_n))$ *for* $EDTMC(G)$ is a solution of

$$\begin{cases} \psi^*(\mathbf{P}^* - \mathbf{I}) = \mathbf{0} \\ \psi^* \mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of order n and $\mathbf{0}$ is a row vector of n values $0,$ $\mathbf{1}$ is that of n values 1.

When $EDTMC(G)$ has the single steady state, $\psi^* = \lim_{k\to\infty} \psi^*[k]$.

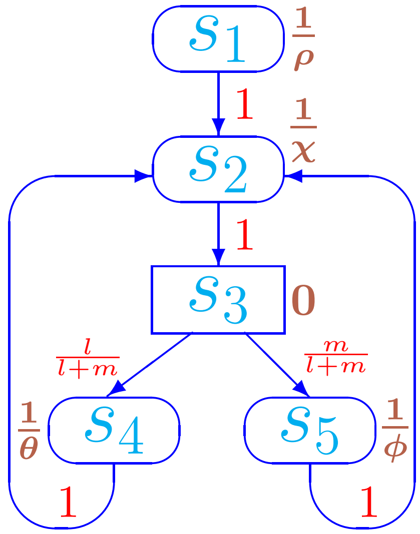
The *steady-state PMF* $\varphi = (\varphi(s_1), \ldots, \varphi(s_n))$ *for* $SMC(G)$:

$$\varphi(s_i) = \begin{cases} \frac{\psi^*(s_i)SJ(s_i)}{\sum_{j=1}^{n} \psi^*(s_j)SJ(s_j)}, & s_i \in DR_T(G); \\ 0, & s_i \in DR_V(G). \end{cases}$$

To calculate φ, we apply abstracting from self-loops to get $\mathbf{P}^*$ and ψ^*, followed by weighting by SJ and normalization.

$EDTMC(G)$ has no self-loops, unlike $SMC(G)$, hence, the behaviour of $EDTMC(G)$ stabilizes quicker than that of $SMC(G)$, since $\mathbf{P}^*$ has only zero elements at the main diagonal.

$SMC(\overline{E})$



EXPRSMC: The underlying SMC of $\overline{E}$ for

$E = [(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \mathsf{Stop}]$

The average sojourn time vector of $\overline{E}$:

$$SJ = \left(\frac{1}{\rho}, \frac{1}{\chi}, 0, \frac{1}{\theta}, \frac{1}{\phi}\right).$$

The sojourn time variance vector of $\overline{E}$:

$$VAR = \left(\frac{1-\rho}{\rho^2}, \frac{1-\chi}{\chi^2}, 0, \frac{1-\theta}{\theta^2}, \frac{1-\phi}{\phi^2}\right).$$

The TPM for $EDTMC(\overline{E})$:

$$\mathbf{P}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC(\overline{E})$:

$$\psi^* = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{l}{3(l+m)}, \frac{m}{3(l+m)}\right).$$

The steady-state PMF ψ^* weighted by SJ:

$$\left(0, \frac{1}{3\chi}, 0, \frac{l}{3\theta(l+m)}, \frac{m}{3\phi(l+m)}\right).$$

We normalize the steady-state weighted PMF dividing it by the sum of its components:

$$\psi^* SJ^T = \frac{\theta\phi(l+m) + \chi(\phi l + \theta m)}{3\chi\theta\phi(l+m)}.$$

Thus, the steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \frac{1}{\theta\phi(l+m) + \chi(\phi l + \theta m)}(0, \theta\phi(l+m), 0, \chi\phi l, \chi\theta m).$$

The case $l = m$ and $\theta = \phi$:

$$\varphi = \frac{1}{2(\chi + \theta)}(0, 2\theta, 0, \chi, \chi).$$

Let G be a dynamic expression and $s, \tilde{s} \in DR(G),\ S, \widetilde{S} \subseteq DR(G)$.
The following performance indices (measures) are based on the steady-state PMF for $SMC(G)$.

- The *average recurrence (return) time in the state s* (the number of discrete time units or steps required for this) is $\frac{1}{\varphi(s)}$.
- The *fraction of residence time in the state s* is $\varphi(s)$.
- The *fraction of residence time in the set of states $S \subseteq DR(G)$* or the *probability of the event determined by a condition that is true for all states from S* is $\sum_{s \in S} \varphi(s)$.
- The *relative fraction of residence time in the set of states S w.r.t. that in $\widetilde{S}$* is $\frac{\sum_{s \in S} \varphi(s)}{\sum_{\tilde{s} \in \widetilde{S}} \varphi(\tilde{s})}$.
- The *rate of leaving the state s* is $\frac{\varphi(s)}{SJ(s)}$.
- The *steady-state probability to perform a step with a multiset of activities Ξ* is $\sum_{s \in DR(G)} \varphi(s) \sum_{\{\Upsilon \mid \Xi \subseteq \Upsilon\}} PT(\Upsilon, s)$.
- The *probability of the event determined by a reward function r on the states* is $\sum_{s \in DR(G)} \varphi(s) r(s)$, where $\forall s \in DR(G)\ 0 \leq r(s) \leq 1$.

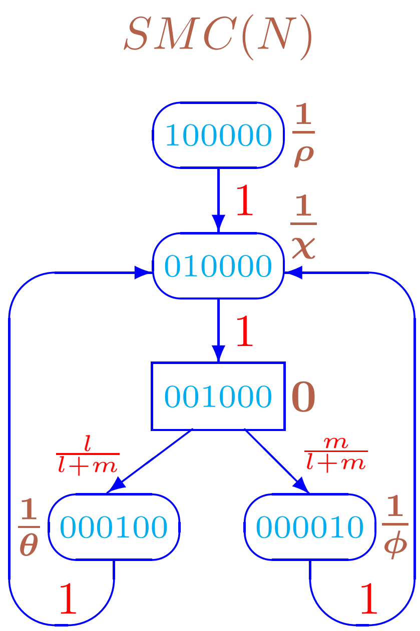
Let $N = (P_N, T_N, W_N, \Omega_N, L_N, M_N)$ be a LDTSIPN and $M, \widetilde{M} \in \mathbb{N}_{fin}^{P_N}$.

The average sojourn time $SJ(M)$, the sojourn time variance $VAR(M)$, the probabilities $PM^*(M, \widetilde{M})$, the transition relation $M \twoheadrightarrow_{\mathcal{P}} \widetilde{M}$, the *EDTMC* $EDTMC(N)$, the *underlying SMC* $SMC(N)$ and the steady-state PMF for it are defined like for dynamic expressions.

We denote isomorphism of SMCs by $\simeq$.

Proposition 32 *(SMCS) For any static expression* E

$$SMC(\overline{E}) \simeq SMC(Box_{dtsi}(\overline{E})).$$



BOXSMC: The underlying SMC of $N = Box_{dtsi}(\overline{E})$ for

$E = [(\{a\}, \rho) * ((\{b\}, \chi); (((\{c\}, l); (\{d\}, \theta))[]((\{e\}, m); (\{f\}, \phi)))) * \mathsf{Stop}]$

Analysis of the DTMC

Definition 169 *Let G be a dynamic expression. The discrete time Markov chain (DTMC) of G, $DTMC(G)$, has the state space $DR(G)$, the initial state $[G]_\approx$ and the transitions $s \to_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = PM(s, \tilde{s})$.*

For $E \in RegStatExpr$, let $DTMC(E) = DTMC(\overline{E})$.

Let G be a dynamic expression. The elements $\mathcal{P}_{ij}$ $(1 \leq i, j \leq n = |DR(G)|)$ of (one-step) transition probability matrix (TPM) $\mathbf{P}$ for $DTMC(G)$ are

$$\mathcal{P}_{ij} = \begin{cases} PM(s_i, s_j), & s_i \to s_j; \\ 0, & \text{otherwise}. \end{cases}$$

The steady-state PMF ψ for $DTMC(G)$ is defined like that for $EDTMC(G)$.

Theorem 44 *(PMFS) Let G be a dynamic expression and SL be its self-loops abstraction vector. Then the steady-state PMFs ψ for $DTMC(G)$ and ψ^* for $EDTMC(G)$ are related as: $\forall s \in DR(G)$*

$$\psi(s) = \frac{\psi^*(s) SL(s)}{\sum_{\tilde{s} \in DR(G)} \psi^*(\tilde{s}) SL(\tilde{s})}.$$

Proposition 33 *(PMFSMC) Let G be a dynamic expression, φ be the steady-state PMF for $SMC(G)$ and ψ be the steady-state PMF for $DTMC(G)$. Then $\forall s \in DR(G)$*

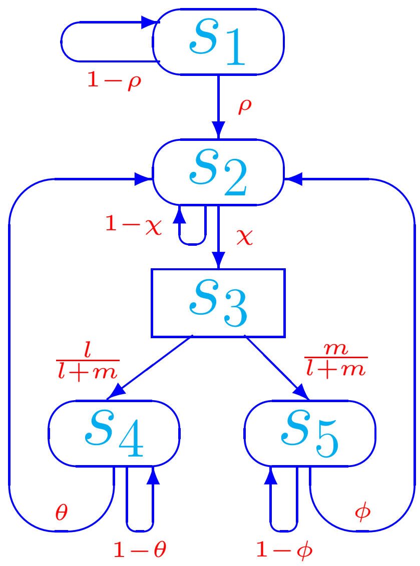
$$\varphi(s) = \begin{cases} \frac{\psi(s)}{\sum_{\tilde{s} \in DR_T(G)} \psi(\tilde{s})}, & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

To calculate φ, we apply normalization to some elements of ψ (corresponding to the tangible states), instead of abstracting from self-loops to get $\mathbf{P}^*$ and ψ^*, followed by weighting by SJ and normalization.

Using $DTMC(G)$ instead of $EDTMC(G)$ allows one to avoid multistage analysis.

$DTMC(G)$ has self-loops, unlike $EDTMC(G)$, hence, the behaviour of $DTMC(G)$ stabilizes slower than that of $EDTMC(G)$ and $\mathbf{P}$ is denser matrix than $\mathbf{P}^*$, since $\mathbf{P}$ may have non-zero elements at the main diagonal.

$DTMC(\overline{E})$



EXPRDTMC: The DTMC of $\overline{E}$ for

$E = [(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \mathsf{Stop}]$

The TPM for $DTMC(\overline{E})$:

$$\mathbf{P} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 & 0 \\ 0 & 1-\chi & \chi & 0 & 0 \\ 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & \theta & 0 & 1-\theta & 0 \\ 0 & \phi & 0 & 0 & 1-\phi \end{pmatrix}.$$

The steady-state PMF for $DTMC(\overline{E})$:

$$\psi = \frac{1}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}(0, \theta\phi(l+m), \chi\theta\phi(l+m), \chi\phi l, \chi\theta m).$$

Since $DR_T(\overline{E}) = \{s_1, s_2, s_4, s_5\}$, $DR_V(\overline{E}) = \{s_3\}$ and by Proposition PMFSMC:

$$\sum_{\tilde{s}\in DR_T(\overline{E})} \psi(\tilde{s}) = \psi(s_1) + \psi(s_2) + \psi(s_4) + \psi(s_5) = \frac{\theta\phi(l+m)+\chi(\phi l+\theta m)}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}.$$

$$\varphi(s_1) = 0 \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = 0,$$

$$\varphi(s_2) = \frac{\theta\phi(l+m)}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)} \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = \frac{\theta\phi(l+m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)},$$

$$\varphi(s_3) = 0,$$

$$\varphi(s_4) = \frac{\chi\phi l}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)} \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = \frac{\chi\phi l}{\theta\phi(l+m)+\chi(\phi l+\theta m)},$$

$$\varphi(s_5) = \frac{\chi\theta m}{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)} \cdot \frac{\theta\phi(1+\chi)(l+m)+\chi(\phi l+\theta m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)} = \frac{\chi\theta m}{\theta\phi(l+m)+\chi(\phi l+\theta m)}.$$

The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \frac{1}{\theta\phi(l+m)+\chi(\phi l+\theta m)}(0, \theta\phi(l+m), 0, \chi\phi l, \chi\theta m).$$

This coincides with the result obtained with the use of ψ^* and SJ.

Analysis of the reduced DTMC

Let G be a dynamic expression and $\mathbf{P}$ be the TPM for $DTMC(G)$.

Reordering the states from $DR(G)$: the first rows and columns of $\mathbf{P}$ correspond to the states from $DR_V(G)$ and the last ones correspond to the states from $DR_T(G)$.

Let $|DR(G)| = n$ and $|DR_T(G)| = m$. The resulting matrix is decomposed as:

$$\mathbf{P} = \begin{pmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{pmatrix}.$$

The elements of the $(n-m) \times (n-m)$ submatrix $\mathbf{C}$: the probabilities to move from vanishing to vanishing states.

The elements of the $(n-m) \times m$ submatrix $\mathbf{D}$: the probabilities to move from vanishing to tangible states.

The elements of the $m \times (n-m)$ submatrix $\mathbf{E}$: the probabilities to move from tangible to vanishing states.

The elements of the $m \times m$ submatrix $\mathbf{F}$: the probabilities to move from tangible to tangible states.

The TPM $\mathbf{P}^{\diamond}$ for $RDTMC(G)$ is the $m \times m$ matrix:

$$\mathbf{P}^{\diamond} = \mathbf{F} + \mathbf{EGD},$$

where the elements of the matrix $\mathbf{G}$ are the probabilities to move from vanishing to vanishing states in any number of state transitions, without traversal of the tangible states:

$$\mathbf{G} = \sum_{k=0}^{\infty} \mathbf{C}^k = \begin{cases} \sum_{k=0}^{l} \mathbf{C}^k, & \exists l \in I\!N\ \forall k > l\ \mathbf{C}^k = \mathbf{0}, \\ & \text{no loops among vanishing states;} \\ (\mathbf{I} - \mathbf{C})^{-1}, & \lim_{k \to \infty} \mathbf{C}^k = \mathbf{0}, \\ & \text{loops among vanishing states;} \end{cases}$$

where $\mathbf{0}$ is the square matrix consisting only of zeros and $\mathbf{I}$ is the identity matrix, both of size $n - m$.

For $1 \le i, j \le m$ and $1 \le k, l \le n - m$, let $\mathcal{F}_{ij}$ be the elements of the matrix $\mathbf{F}$, $\mathcal{E}_{ik}$ be those of $\mathbf{E}$, $\mathcal{G}_{kl}$ be those of $\mathbf{G}$ and $\mathcal{D}_{lj}$ be those of $\mathbf{D}$.

The elements $\mathcal{P}^{\diamond}_{ij}$ of the matrix $\mathbf{P}^{\diamond}$ are

$$\mathcal{P}^{\diamond}_{ij} = \mathcal{F}_{ij} + \sum_{k=1}^{n-m} \sum_{l=1}^{n-m} \mathcal{E}_{ik}\mathcal{G}_{kl}\mathcal{D}_{lj} =$$

$$\mathcal{F}_{ij} + \sum_{k=1}^{n-m} \mathcal{E}_{ik} \sum_{l=1}^{n-m} \mathcal{G}_{kl}\mathcal{D}_{lj} = \mathcal{F}_{ij} + \sum_{l=1}^{n-m} \mathcal{D}_{lj} \sum_{k=1}^{n-m} \mathcal{E}_{ik}\mathcal{G}_{kl},$$

i.e. $\mathcal{P}^{\diamond}_{ij}\ (1 \le i, j \le m)$ is the total probability to move from the tangible state s_i to the tangible state s_j in any number of steps, without traversal of tangible states, but possibly going through vanishing states.

Let $s, \tilde{s} \in DR_T(G)$ such that $s = s_i,\ \tilde{s} = s_j$.

The *probability to move from s to $\tilde{s}$ in any number of steps, without traversal of tangible states* is

$$PM^\diamond(s, \tilde{s}) = \mathcal{P}^\diamond_{ij}.$$

Definition 170 *Let G be a dynamic expression and $[G]_\approx \in DR_T(G)$.*

The reduced discrete time Markov chain (RDTMC) of G, denoted by $RDTMC(G)$, has the state space $DR_T(G)$, the initial state $[G]_\approx$ and the transitions $s \hookrightarrow_{\mathcal{P}} \tilde{s}$, where $\mathcal{P} = PM^\diamond(s, \tilde{s})$.

RDTMCs of static expressions can be defined as well. For $E \in RegStatExpr$, let $RDTMC(E) = RDTMC(\overline{E})$.

Let $DR_T(G) = \{s_1, \ldots, s_m\}$ and $[G]_\approx \in DR_T(G)$. The transient (k-step, $k \in I\!N$) probability mass function (PMF) $\psi^\diamond[k] = (\psi^\diamond[k](s_1), \ldots, \psi^\diamond[k](s_m))$ for $RDTMC(G)$ is calculated as

$$\psi^\diamond[k] = \psi^\diamond[0](\mathbf{P}^\diamond)^k,$$

where $\psi^\diamond[0] = (\psi^\diamond[0](s_1), \ldots, \psi^\diamond[0](s_m))$ is the initial PMF:

$$\psi^\diamond[0](s_i) = \begin{cases} 1, & s_i = [G]_\approx; \\ 0, & \text{otherwise}. \end{cases}$$

$\psi^\diamond[k+1] = \psi^\diamond[k]\mathbf{P}^\diamond \ (k \in I\!N)$.

The steady-state PMF $\psi^\diamond = (\psi^\diamond(s_1), \ldots, \psi^\diamond(s_m))$ for $RDTMC(G)$ is a solution of:

$$\begin{cases} \psi^\diamond(\mathbf{P}^\diamond - \mathbf{I}) = \mathbf{0} \\ \psi^\diamond \mathbf{1}^T = 1 \end{cases},$$

where $\mathbf{I}$ is the identity matrix of size m and $\mathbf{0}$ is a row vector of m values 0, $\mathbf{1}$ is that of m values 1.

When $RDTMC(G)$ has the single steady state, $\psi^\diamond = \lim_{k\to\infty} \psi^\diamond[k]$.

Proposition 34 *(PMFSMCT) Let G be a dynamic expression, φ be the steady-state PMF for $SMC(G)$ and $\psi^\diamond$ be the steady-state PMF for $RDTMC(G)$. Then $\forall s \in DR(G)$*

$$\varphi(s) = \begin{cases} \psi^\diamond(s), & s \in DR_T(G); \\ 0, & s \in DR_V(G). \end{cases}$$

To calculate φ, we take all the elements of $\psi^\diamond$ as the steady-state probabilities of the tangible states, instead of abstracting from self-loops to get $\mathbf{P}^*$ and ψ^*, followed by weighting by SJ and normalization.

Using $RDTMC(G)$ instead of $EDTMC(G)$ allows one to avoid multistage analysis. Constructing $\mathbf{P}^\diamond$ requires calculating matrix powers or inverse matrices.

$RDTMC(G)$ has self-loops, unlike $EDTMC(G)$, hence, the behaviour of $RDTMC(G)$ may stabilize slower than that of $EDTMC(G)$. $\mathbf{P}^\diamond$ is smaller and denser matrix than $\mathbf{P}^*$, since $\mathbf{P}^\diamond$ has non-zero elements at the main diagonal and many of them outside it.

The complexity of the analytical calculation of $\psi^\diamond$ w.r.t. ψ^* depends on the model structure: the number of vanishing states and loops among them. Usually it is lower, since the matrix size reduction plays an important role.

The elimination of vanishing states.

- The system models with many immediate activities:

  significant simplification of the solution.

- The abstraction level of SMCs:

  decreases their impact to the solution complexity.

- The abstraction level of transition systems:

  allows immediate activities to specify logical structure.

$E =$
$[(\{a\},\rho) * ((\{b\},\chi); (((\{c\},l); (\{d\},\theta))[]((\{e\},m); (\{f\},\phi)))) * \mathsf{Stop}]$.

$DR_T(\overline{E}) = \{s_1, s_2, s_4, s_5\}$ and $DR_V(\overline{E}) = \{s_3\}$.

We reorder the states from $DR(\overline{E})$, by moving the vanishing states to the first positions: s_3, s_1, s_2, s_4, s_5.

The reordered TPM for $DTMC(\overline{E})$:

$$\mathbf{P}_r = \begin{pmatrix} 0 & 0 & 0 & \frac{l}{l+m} & \frac{m}{l+m} \\ 0 & 1-\rho & \rho & 0 & 0 \\ \chi & 0 & 1-\chi & 0 & 0 \\ 0 & 0 & \theta & 1-\theta & 0 \\ 0 & 0 & \phi & 0 & 1-\phi \end{pmatrix}.$$

The result of the decomposing $\mathbf{P}_r$:

$$\mathbf{C} = 0, \qquad \mathbf{D} = \left(0, 0, \frac{l}{l+m}, \frac{m}{l+m}\right),$$

$$\mathbf{E} = \begin{pmatrix} 0 \\ \chi \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & 0 & 0 \\ 0 & \theta & 1-\theta & 0 \\ 0 & \phi & 0 & 1-\phi \end{pmatrix}.$$

Since $\mathbf{C}^1 = \mathbf{0}$, we have $\forall k > 0\ \mathbf{C}^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\mathbf{G} = \sum_{k=0}^{l} \mathbf{C}^k = \mathbf{C}^0 = \mathbf{I}.$$

The TPM for $RDTMC(\overline{E})$:

$$\mathbf{P}^{\diamond} = \mathbf{F}{+}\mathbf{EGD} = \mathbf{F}{+}\mathbf{EID} = \mathbf{F}{+}\mathbf{ED} = \begin{pmatrix} 1-\rho & \rho & 0 & 0 \\ 0 & 1-\chi & \frac{\chi l}{l+m} & \frac{\chi m}{l+m} \\ 0 & \theta & 1-\theta & 0 \\ 0 & \phi & 0 & 1-\phi \end{pmatrix}.$$

The steady-state PMF for $RDTMC(\overline{E})$:

$$\psi^{\diamond} = \frac{1}{\theta\phi(l+m)+\chi(\phi l+\theta m)}(0, \theta\phi(l+m), \chi\phi l, \chi\theta m).$$

Note that $\psi^{\diamond} = (\psi^{\diamond}(s_1), \psi^{\diamond}(s_2), \psi^{\diamond}(s_4), \psi^{\diamond}(s_5))$.

By Proposition PMFSMCT,

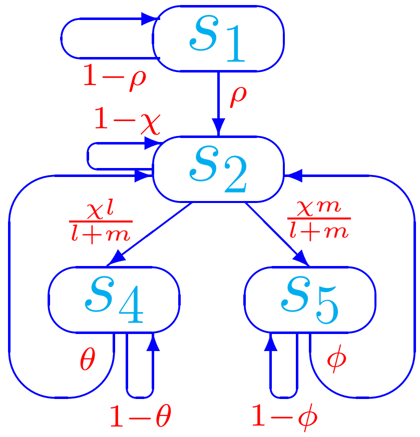
$$\varphi(s_1) = 0,$$
$$\varphi(s_2) = \frac{\theta\phi(l+m)}{\theta\phi(l+m)+\chi(\phi l+\theta m)},$$
$$\varphi(s_3) = 0,$$
$$\varphi(s_4) = \frac{\chi\phi l}{\theta\phi(l+m)+\chi(\phi l+\theta m)},$$
$$\varphi(s_5) = \frac{\chi\theta m}{\theta\phi(l+m)+\chi(\phi l+\theta m)}.$$

The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \frac{1}{\theta\phi(l+m)+\chi(\phi l+\theta m)}(0, \theta\phi(l+m), 0, \chi\phi l, \chi\theta m).$$

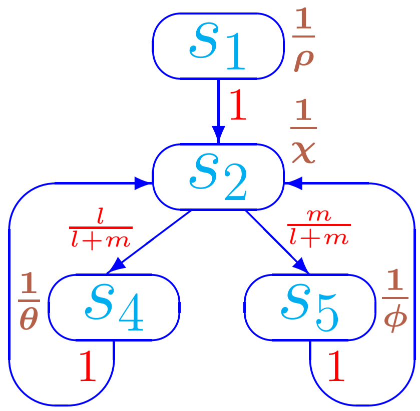
This coincides with the result obtained with the use of ψ^* and SJ.

$RDTMC(\overline{E})$



EXPRRDTMC: The reduced DTMC of $\overline{E}$ for $E =$
$[(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \mathsf{Stop}]$

$RSMC(\overline{E})$



EXPRRSMC: The reduced SMC of $\overline{E}$ for $E =$
$[(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{e\},m);(\{f\},\phi)))) * \mathsf{Stop}]$

Stochastic equivalences

Step stochastic bisimulation equivalence

For $\Upsilon \in I\!N_{fin}^{\mathcal{SIL}}$, we consider $\mathcal{L}(\Upsilon) \in I\!N_{fin}^{\mathcal{L}}$, i.e. (possibly empty) multisets of multiactions.

Let G be a dynamic expression and $\mathcal{H} \subseteq DR(G)$. For $s \in DR(G)$ and $A \in I\!N_{fin}^{\mathcal{L}}$ we write $s \xrightarrow{A}_{\mathcal{P}} \mathcal{H}$, where $\mathcal{P} = PM_A(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via steps with the multiaction part A*:

$$PM_A(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{\Gamma} \tilde{s},\ \mathcal{L}(\Gamma) = A\}} PT(\Gamma, s).$$

We write $s \xrightarrow{A} \mathcal{H}$ if $\exists \mathcal{P}\ s \xrightarrow{A}_{\mathcal{P}} \mathcal{H}$.

We write $s \rightarrow_{\mathcal{P}} \mathcal{H}$ if $\exists A\ s \xrightarrow{A} \mathcal{H}$, where $\mathcal{P} = PM(s, \mathcal{H})$ is the *overall probability to move from s into the set of states $\mathcal{H}$ via any steps*:

$$PM(s, \mathcal{H}) = \sum_{\{\Gamma \mid \exists \tilde{s} \in \mathcal{H}\ s \xrightarrow{\Gamma} \tilde{s}\}} PT(\Gamma, s).$$

Definition 171 *Let G and G' be dynamic expressions. An equivalence relation $\mathcal{R} \subseteq (DR(G) \cup DR(G'))^2$ is a step stochastic bisimulation between G and G', $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, if:*

1. $([G]_\approx, [G']_\approx) \in \mathcal{R}$.
2. $(s_1, s_2) \in \mathcal{R} \Rightarrow \forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}\ \forall A \in I\!N_{fin}^{\mathcal{L}}$

$$s_1 \xrightarrow{A}_{\mathcal{P}} \mathcal{H} \Leftrightarrow s_2 \xrightarrow{A}_{\mathcal{P}} \mathcal{H}.$$

Two dynamic expressions G and G' are step stochastic bisimulation equivalent, $G \underline{\leftrightarrow}_{ss} G'$, if $\exists \mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$.

Proposition 35 *(BISSPL) Let G and G' be dynamic expressions and $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$. Then*

$$\mathcal{R} \subseteq (DR_T(G) \cup DR_T(G'))^2 \uplus (DR_V(G) \cup DR_V(G'))^2,$$

where $\uplus$ is disjoint union.

$\mathcal{R}_{ss}(G, G') = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G'\}$ is the *union of all step stochastic bisimulations* between G and G'.

Proposition 36 *(LARBIS) Let G and G' be dynamic expressions and $G \underline{\leftrightarrow}_{ss} G'$. Then $\mathcal{R}_{ss}(G, G')$ is the largest step stochastic bisimulation between G and G'.*

Interrelations of the stochastic equivalences

$$\underline{\leftrightarrow}_{ss} \longleftarrow =_{ts} \longleftarrow \approx$$

INTSTEQ: Interrelations of the stochastic equivalences

Theorem 45 *(INTSTEQ) Let $\leftrightarrow, \twoheadleftrightarrow \in \{\underline{\leftrightarrow}, =, \approx\}$ and $\star, \star\star \in \{\_, ss, ts\}$. For dynamic expressions G and G'*

$$G \leftrightarrow_{\star} G' \Rightarrow G \twoheadleftrightarrow_{\star\star} G'$$

iff in the graph above there exists a directed path from $\leftrightarrow_{\star}$ to $\twoheadleftrightarrow_{\star\star}$.

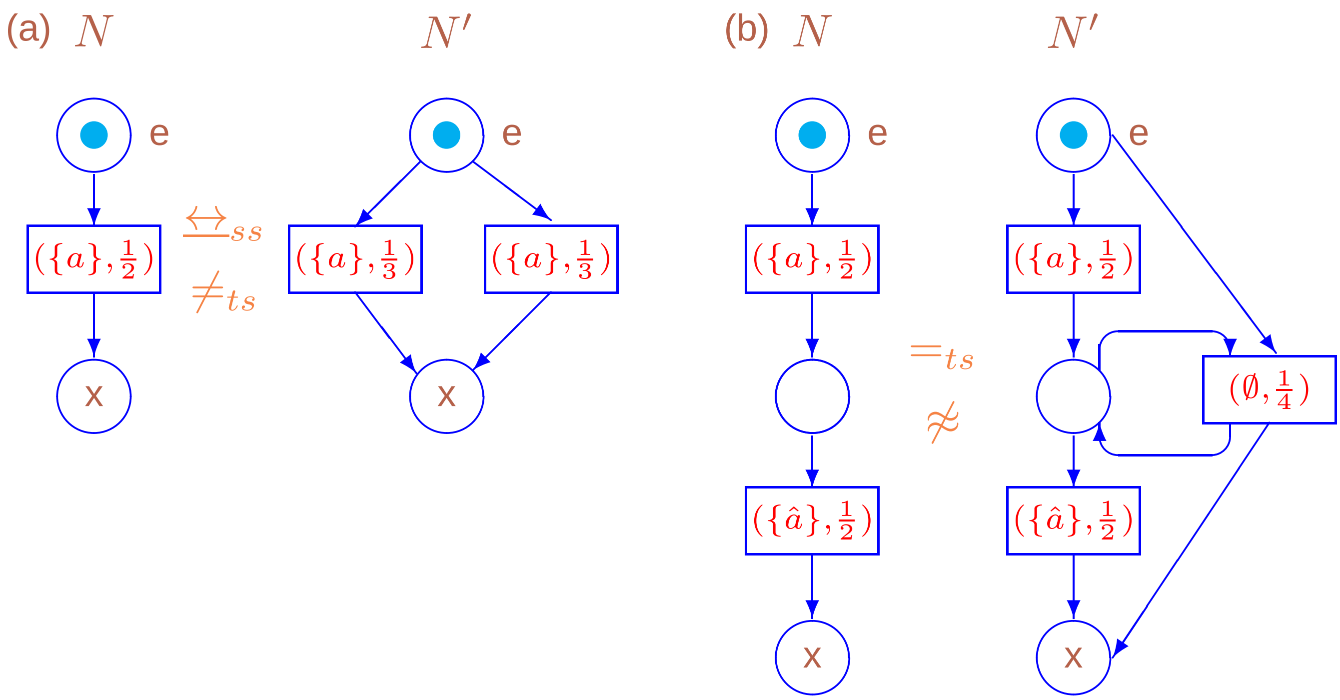
Validity of the implications

- The implication $=_{ts} \to \underline{\leftrightarrow}_{ss}$ is proved as follows. Let $\beta : G =_{ts} G'$. Then $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$, where $\mathcal{R} = \{(s, \beta(s)) \mid s \in DR(G)\}$.
- The implication $\approx \to =_{ts}$ is valid, since the transition system of a dynamic formula is defined based on its structural equivalence class.

Absence of the additional nontrivial arrows

(a) Let $E = (\{a\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{3})_1[](\{a\}, \frac{1}{3})_2$. Then $\overline{E} \underline{\leftrightarrow}_{ss} \overline{E'}$, but $\overline{E} \neq_{ts} \overline{E'}$, since $TS(\overline{E})$ has only one transition from the initial to the final state while $TS(\overline{E'})$ has two such ones.

(b) Let $E = (\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2})$ and $E' = (\{a\}, \frac{1}{2}); (\{\hat{a}\}, \frac{1}{2}))$ sy a. Then $\overline{E} =_{ts} \overline{E'}$, but $\overline{E} \not\approx \overline{E'}$, since $\overline{E}$ and $\overline{E'}$ cannot be reached from each other by applying inaction rules.



EXMSTEQ: Dtsi-boxes of the dynamic expressions from equivalence examples of the Theorem INTSTEQ

In Figure EXMSTEQ, $N = Box_{dtsi}(\overline{E})$ and $N' = Box_{dtsi}(\overline{E'})$ for each picture (a)–(b).

Reduction modulo equivalences

An *autobisimulation* is a bisimulation between an expression and itself.

For a dynamic expression G and a step stochastic autobisimulation $\mathcal{R}: G \underline{\leftrightarrow}_{ss} G$, let $\mathcal{K} \in DR(G)/_{\mathcal{R}}$ and $s_1, s_2 \in \mathcal{K}$.

We have $\forall \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}}\ \forall A \in I\!N_{fin}^{\mathcal{L}}\ s_1 \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}} \Leftrightarrow s_2 \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

The equality is valid for all $s_1, s_2 \in \mathcal{K}$, hence, we can rewrite it as $\mathcal{K} \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM_A(\mathcal{K}, \widetilde{\mathcal{K}}) = PM_A(s_1, \widetilde{\mathcal{K}}) = PM_A(s_2, \widetilde{\mathcal{K}})$.

We write $\mathcal{K} \stackrel{A}{\rightarrow} \widetilde{\mathcal{K}}$ if $\exists \mathcal{P}\ \mathcal{K} \stackrel{A}{\rightarrow}_{\mathcal{P}} \widetilde{\mathcal{K}}$ and $\mathcal{K} \rightarrow \widetilde{\mathcal{K}}$ if $\exists A\ \mathcal{K} \stackrel{A}{\rightarrow} \widetilde{\mathcal{K}}$.

The similar arguments: we write $\mathcal{K} \rightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM(\mathcal{K}, \widetilde{\mathcal{K}}) = PM(s_1, \widetilde{\mathcal{K}}) = PM(s_2, \widetilde{\mathcal{K}})$.

Since $\mathcal{R} \subseteq (DR_T(G))^2 \uplus (DR_V(G))^2$, we have $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}}$, all states from $\mathcal{K}$ are tangible, when $\mathcal{K} \in DR_T(G)/_{\mathcal{R}}$, or all of them are vanishing, when $\mathcal{K} \in DR_V(G)/_{\mathcal{R}}$.

The *average sojourn time in the equivalence class (w.r.t. $\mathcal{R}$) of states $\mathcal{K}$* is

$$SJ_{\mathcal{R}}(\mathcal{K}) = \begin{cases} \frac{1}{1-PM(\mathcal{K},\mathcal{K})}, & \mathcal{K} \in DR_T(G)/_{\mathcal{R}}; \\ 0, & \mathcal{K} \in DR_V(G)/_{\mathcal{R}}. \end{cases}$$

The *average sojourn time vector for the equivalence classes (w.r.t. $\mathcal{R}$) of states* of G, $SJ_{\mathcal{R}}$, has the elements $SJ_{\mathcal{R}}(\mathcal{K}),\ \mathcal{K} \in DR(G)/_{\mathcal{R}}$.

The *sojourn time variance in the equivalence class (w.r.t. $\mathcal{R}$) of states $\mathcal{K}$* is

$$VAR_{\mathcal{R}}(\mathcal{K}) = \begin{cases} \frac{PM(\mathcal{K},\mathcal{K})}{(1-PM(\mathcal{K},\mathcal{K}))^2}, & \mathcal{K} \in DR_T(G)/_{\mathcal{R}}; \\ 0, & \mathcal{K} \in DR_V(G)/_{\mathcal{R}}. \end{cases}$$

The *sojourn time variance vector for the equivalence classes (w.r.t. $\mathcal{R}$) of states* of G, $VAR_{\mathcal{R}}$, has the elements $VAR_{\mathcal{R}}(\mathcal{K}),\ \mathcal{K} \in DR(G)/_{\mathcal{R}}$.

$\mathcal{R}_{ss}(G) = \bigcup\{\mathcal{R} \mid \mathcal{R} : G \underline{\leftrightarrow}_{ss} G\}$ is the *largest step stochastic autobisimulation* on G.

Definition 172 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) (labeled probabilistic) transition system *of a dynamic expression* G *is* $TS_{\underline{\leftrightarrow}_{ss}}(G) = (S_{\underline{\leftrightarrow}_{ss}}, L_{\underline{\leftrightarrow}_{ss}}, \mathcal{T}_{\underline{\leftrightarrow}_{ss}}, s_{\underline{\leftrightarrow}_{ss}})$, *where*

- $S_{\underline{\leftrightarrow}_{ss}} = DR(G)/_{\mathcal{R}_{ss}(G)}$;
- $L_{\underline{\leftrightarrow}_{ss}} \subseteq (\mathbb{N}^{\mathcal{L}}_{fin}) \times (0; 1]$;
- $\mathcal{T}_{\underline{\leftrightarrow}_{ss}} = \{(\mathcal{K}, (A, PM_A(\mathcal{K}, \widetilde{\mathcal{K}})), \widetilde{\mathcal{K}}) \mid \mathcal{K}, \widetilde{\mathcal{K}} \in DR(G)/_{\mathcal{R}_{ss}(G)}, \mathcal{K} \xrightarrow{A} \widetilde{\mathcal{K}}\}$;
- $s_{\underline{\leftrightarrow}_{ss}} = [[G]_\approx]_{\mathcal{R}_{ss}(G)}$.

The transition $(\mathcal{K}, (A, \mathcal{P}), \widetilde{\mathcal{K}}) \in \mathcal{T}_{\underline{\leftrightarrow}_{ss}}$ will be written as $\mathcal{K} \xrightarrow{A}_{\mathcal{P}} \widetilde{\mathcal{K}}$.

For $E \in RegStatExpr$, let $TS_{\underline{\leftrightarrow}_{ss}}(E) = TS_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

Let F be an abstraction of E from the examples above, s.t.
$c = e,\ d = f,\ \theta = \phi$:

$$F = [(\{a\}, \rho) * ((\{b\}, \chi); (((\{c\}, l); (\{d\}, \theta))[]((\{c\}, m); (\{d\}, \theta)))) * \mathsf{Stop}].$$

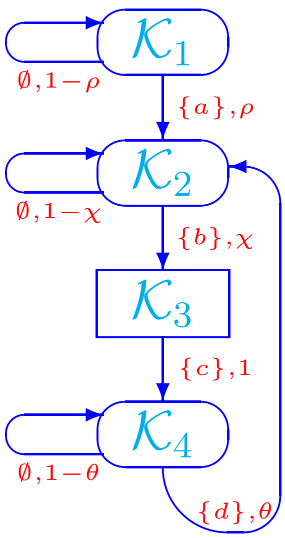
$DR(\overline{F}) = \{s_1, s_2, s_3, s_4, s_5\}$ is obtained from $DR(\overline{E})$ via substitution of $e,\ f,\ \phi$ by $c,\ d,\ \theta$, respectively.

$DR_T(\overline{F}) = \{s_1, s_2, s_4, s_5\}$ and $DR_V(\overline{F}) = \{s_3\}$.

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}$,
where $\mathcal{K}_1 = \{s_1\},\ \mathcal{K}_2 = \{s_2\},\ \mathcal{K}_3 = \{s_3\},\ \mathcal{K}_4 = \{s_4, s_5\}$.

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3\}$.

$TS_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



QTS: The quotient transition system of $\overline{F}$ for $F =$
$[(\{a\}, \rho) * ((\{b\}, \chi); (((\{c\}, l); (\{d\}, \theta))[]((\{c\}, m); (\{d\}, \theta)))) * \mathsf{Stop}]$

The *quotient (by $\underline{\leftrightarrow}_{ss}$) average sojourn time vector* of G is $SJ_{\underline{\leftrightarrow}_{ss}} = SJ_{\mathcal{R}_{ss}(G)}$.

The *quotient (by $\underline{\leftrightarrow}_{ss}$) sojourn time variance vector* of G is $VAR_{\underline{\leftrightarrow}_{ss}} = VAR_{\mathcal{R}_{ss}(G)}$.

Let $\mathcal{K} \to \widetilde{\mathcal{K}}$ and $\mathcal{K} \neq \widetilde{\mathcal{K}}$. The *probability to move from $\mathcal{K}$ to $\widetilde{\mathcal{K}}$ by executing any multiset of activities after possible self-loops* is

$$PM^*(\mathcal{K}, \widetilde{\mathcal{K}}) = \begin{cases} PM(\mathcal{K}, \widetilde{\mathcal{K}}) \sum_{k=0}^{\infty} PM(\mathcal{K}, \mathcal{K})^k = \frac{PM(\mathcal{K}, \widetilde{\mathcal{K}})}{1 - PM(\mathcal{K}, \mathcal{K})}, & \mathcal{K} \to \mathcal{K}; \\ PM(\mathcal{K}, \widetilde{\mathcal{K}}), & \text{otherwise}. \end{cases}$$

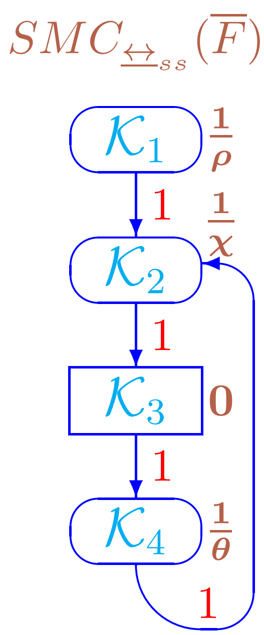
We have $\forall \mathcal{K} \in DR_T(G)/_{\mathcal{R}_{ss}(G)}\ PM^*(\mathcal{K}, \widetilde{\mathcal{K}}) = SJ_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) PM(\mathcal{K}, \widetilde{\mathcal{K}})$.

Definition 173 *The* quotient (by $\underline{\leftrightarrow}_{ss}$) EDTMC *of a dynamic expression* G, $EDTMC_{\underline{\leftrightarrow}_{ss}}(G)$, *has the state space* $DR(G)/_{\underline{\leftrightarrow}_{ss}(G)}$, *the initial state* $[[G]_{\approx}]_{\mathcal{R}_{ss}(G)}$ *and the transitions* $\mathcal{K} \twoheadrightarrow_{\mathcal{P}} \widetilde{\mathcal{K}}$, *if* $\mathcal{K} \to \widetilde{\mathcal{K}}$ *and* $\mathcal{K} \neq \widetilde{\mathcal{K}}$, *where* $\mathcal{P} = PM^*(\mathcal{K}, \widetilde{\mathcal{K}})$.

The quotient (by $\underline{\leftrightarrow}_{ss}$) underlying SMC *of* G, $SMC_{\underline{\leftrightarrow}_{ss}}(G)$, *has the EDTMC* $EDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ *and the sojourn time in every* $\mathcal{K} \in DR_T(G)/_{\mathcal{R}_{ss}(G)}$ *is geometrically distributed with the parameter* $1 - PM(\mathcal{K}, \mathcal{K})$ *while the sojourn time in every* $\mathcal{K} \in DR_V(G)/_{\mathcal{R}_{ss}(G)}$ *is equal to zero.*

For $E \in RegStatExpr$, let $SMC_{\underline{\leftrightarrow}_{ss}}(E) = SMC_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

The steady-state PMFs $\psi^*_{\underline{\leftrightarrow}_{ss}}$ for $EDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ and $\varphi_{\underline{\leftrightarrow}_{ss}}$ for $SMC_{\underline{\leftrightarrow}_{ss}}(G)$ are defined like ψ^* for $EDTMC(G)$ and φ for $SMC(G)$.

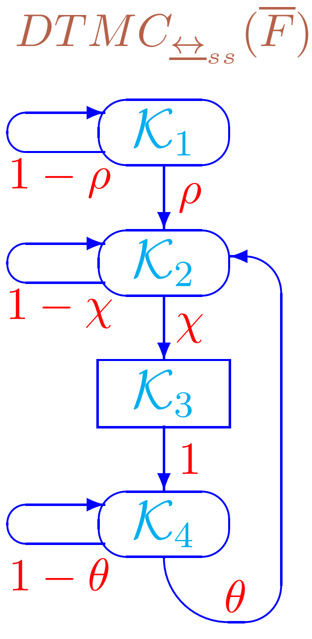


EXPRQSMC: The quotient underlying SMC of $\overline{F}$ for $F = [(\{a\}, \rho) * ((\{b\}, \chi); (((\{c\}, l); (\{d\}, \theta))[]((\{c\}, m); (\{d\}, \theta)))) * \mathsf{Stop}]$

Definition 174 *Let G be a dynamic expression. The quotient (by $\underline{\leftrightarrow}_{ss}$) DTMC of G, $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$, has the state space $DR(G)/_{\mathcal{R}_{ss}(G)}$, the initial state $[[G]_\approx]_{\mathcal{R}_{ss}(G)}$ and the transitions $\mathcal{K} \to_{\mathcal{P}} \widetilde{\mathcal{K}}$, where $\mathcal{P} = PM(\mathcal{K}, \widetilde{\mathcal{K}})$.*

For $E \in RegStatExpr$, let $DTMC_{\underline{\leftrightarrow}_{ss}}(E) = DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

The steady-state PMF $\psi_{\underline{\leftrightarrow}_{ss}}$ for $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$ is defined like ψ for $DTMC(G)$.



EXPRQDTMC: The quotient DTMC of $\overline{F}$ for $F =$ $[(\{a\}, \rho) * ((\{b\}, \chi); (((\{c\}, l); (\{d\}, \theta))[]((\{c\}, m); (\{d\}, \theta)))) * \mathsf{Stop}]$

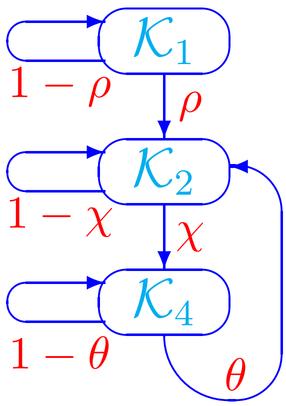
Definition 175 *The reduced quotient (by $\underline{\leftrightarrow}_{ss}$) DTMC of G, denoted by $RDTMC_{\underline{\leftrightarrow}_{ss}}(G)$, is defined like $RDTMC(G)$, but it is constructed from $DTMC_{\underline{\leftrightarrow}_{ss}}(G)$ instead of $DTMC(G)$.*

For $E \in RegStatExpr$, let $RDTMC_{\underline{\leftrightarrow}_{ss}}(E) = RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{E})$.

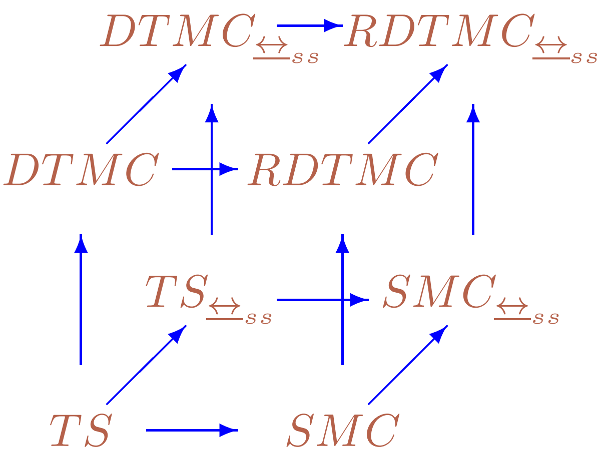
The steady-state PMF $\psi^{\diamond}_{\underline{\leftrightarrow}_{ss}}$ for $RDTMC_{\underline{\leftrightarrow}_{ss}}(G)$ is defined like $\psi^{\diamond}$ for $RDTMC(G)$.

The relationships between the steady-state PMFs $\psi_{\underline{\leftrightarrow}_{ss}}$ and $\psi^{*}_{\underline{\leftrightarrow}_{ss}}$, $\varphi_{\underline{\leftrightarrow}_{ss}}$ and $\psi_{\underline{\leftrightarrow}_{ss}}$, $\varphi_{\underline{\leftrightarrow}_{ss}}$ and $\psi^{\diamond}_{\underline{\leftrightarrow}_{ss}}$ are the same as those between their “non-quotient” versions.

$RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$



EXPRQRDTMC: The reduced quotient DTMC of $\overline{F}$ for $F =$
$[(\{a\},\rho) * ((\{b\},\chi);(((\{c\},l);(\{d\},\theta))[]((\{c\},m);(\{d\},\theta)))) * \mathsf{Stop}]$



CUBTSMCQ: The cube of interrelations for standard and quotient transition systems and Markov chains of expressions

Stationary behaviour

Steady state and equivalences

Proposition 37 *(STPROB) Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ and φ be the steady-state PMF for $SMC(G)$, φ' be the steady-state PMF for $SMC(G')$. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$\sum_{s \in \mathcal{H} \cap DR(G)} \varphi(s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \varphi'(s').$$

Let G be a dynamic expression and φ be the steady-state PMF for $SMC(G)$, $\varphi_{\underline{\leftrightarrow}_{ss}}$ be the steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(G)$.

By Proposition STPROB: $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)}$

$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) = \sum_{s \in \mathcal{K}} \varphi(s).$$

Definition 176 *A derived step trace of a dynamic expression G is $\Sigma = A_1 \cdots A_n \in (\mathbb{N}_{fin}^{\mathcal{L}})^*$, where $\exists s \in DR(G)\ s \stackrel{\Gamma_1}{\to} s_1 \stackrel{\Gamma_2}{\to} \cdots \stackrel{\Gamma_n}{\to} s_n$, $\mathcal{L}(\Gamma_i) = A_i\ (1 \le i \le n)$.*

The probability to execute the derived step trace Σ in s:

$$PT(\Sigma, s) = \sum_{\{\Gamma_1,\ldots,\Gamma_n \mid s=s_0 \stackrel{\Gamma_1}{\to} s_1 \stackrel{\Gamma_2}{\to} \cdots \stackrel{\Gamma_n}{\to} s_n,\ \mathcal{L}(\Gamma_i)=A_i\ (1\le i\le n)\}} \prod_{i=1}^{n} PT(\Gamma_i, s_{i-1}).$$

Theorem 46 *(STTRAC) Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$ and φ be the steady-state PMF for $SMC(G)$, φ' be the steady-state PMF for $SMC(G')$ and Σ be a derived step trace of G and G'. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$\sum_{s \in \mathcal{H} \cap DR(G)} \varphi(s) PT(\Sigma, s) = \sum_{s' \in \mathcal{H} \cap DR(G')} \varphi'(s') PT(\Sigma, s').$$

By Theorem STTRAC: $\forall \mathcal{K} \in DR(G)/_{\mathcal{R}_{ss}(G)}$

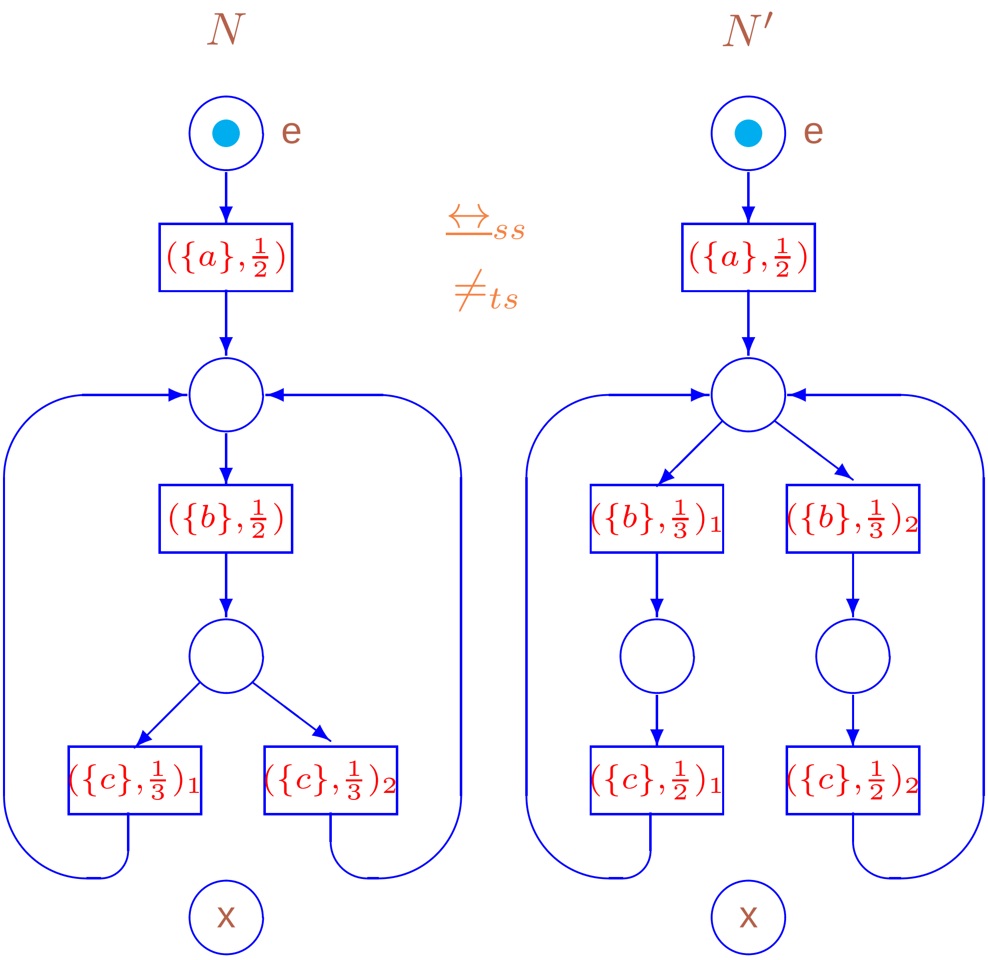
$$\varphi_{\underline{\leftrightarrow}_{ss}}(\mathcal{K}) PT(\Sigma, \mathcal{K}) = \sum_{s \in \mathcal{K}} \varphi(s) PT(\Sigma, s),$$

where $\forall s \in \mathcal{K}\ PT(\Sigma, \mathcal{K}) = PT(\Sigma, s)$.

Proposition 38 *(SJAVVA) Let G, G' be dynamic expressions with $\mathcal{R} : G \underline{\leftrightarrow}_{ss} G'$. Then $\forall \mathcal{H} \in (DR(G) \cup DR(G'))/_{\mathcal{R}}$*

$$SJ_{\mathcal{R} \cap (DR(G))^2}(\mathcal{H} \cap DR(G)) = SJ_{\mathcal{R} \cap (DR(G'))^2}(\mathcal{H} \cap DR(G')),$$

$$VAR_{\mathcal{R} \cap (DR(G))^2}(\mathcal{H} \cap DR(G)) = VAR_{\mathcal{R} \cap (DR(G'))^2}(\mathcal{H} \cap DR(G')).$$



SSBSSP: $\underline{\leftrightarrow}_{ss}$ preserves steady-state behaviour and sojourn time properties in the equivalence classes

Let $E = [(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2)) * \mathsf{Stop}]$ and
$E' = [(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{3})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{3})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]$.

We have $\overline{E}\underline{\leftrightarrow}_{ss}\overline{E'}$.

In Figure SSBSSP, $N = Box_{dtsi}(\overline{E})$ and $N' = Box_{dtsi}(\overline{E'})$.

$DR(\overline{E})$ consists of

$s_1 = [[\overline{(\{a\}, \frac{1}{2})} * ((\{b\}, \frac{1}{2}); ((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2)) * \mathsf{Stop}]]_{\approx}$,

$s_2 = [[(\{a\}, \frac{1}{2}) * (\overline{(\{b\}, \frac{1}{2})}; ((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2)) * \mathsf{Stop}]]_{\approx}$,

$s_3 = [[(\{a\}, \frac{1}{2}) * ((\{b\}, \frac{1}{2}); \overline{((\{c\}, \frac{1}{3})_1[](\{c\}, \frac{1}{3})_2))} * \mathsf{Stop}]]_{\approx}$.

$DR(\overline{E'})$ consists of

$s_1' =$
$[[\overline{(\{a\}, \frac{1}{2})} * (((\{b\}, \frac{1}{3})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{3})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_{\approx}$,

$s_2' =$
$[[(\{a\}, \frac{1}{2}) * \overline{(((\{b\}, \frac{1}{3})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{3})_2; (\{c\}, \frac{1}{2})_2))} * \mathsf{Stop}]]_{\approx}$,

$s_3' =$
$[[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{3})_1; \overline{(\{c\}, \frac{1}{2})_1})[]((\{b\}, \frac{1}{3})_2; (\{c\}, \frac{1}{2})_2)) * \mathsf{Stop}]]_{\approx}$,

$s_4' =$
$[[(\{a\}, \frac{1}{2}) * (((\{b\}, \frac{1}{3})_1; (\{c\}, \frac{1}{2})_1)[]((\{b\}, \frac{1}{3})_2; \overline{(\{c\}, \frac{1}{2})_2})) * \mathsf{Stop}]]_{\approx}$.

The steady-state PMFs φ for $SMC(\overline{E})$ and φ' for $SMC(\overline{E'})$ are

$$\varphi = \left(0, \frac{1}{2}, \frac{1}{2}\right), \ \varphi' = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Consider $\mathcal{H} = \{s_3, s_3', s_4'\}$. The steady-state probabilities for $\mathcal{H}$ coincide:
$\sum_{s \in \mathcal{H} \cap DR(\overline{E})} \varphi(s) = \varphi(s_3) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \varphi'(s_3') + \varphi'(s_4') = \sum_{s' \in \mathcal{H} \cap DR(\overline{E'})} \varphi'(s')$.

Let $\Sigma = \{\{c\}\}$. The steady-state probabilities to enter into the equivalence class $\mathcal{H}$ and start the derived step trace Σ from it coincide:
$\varphi(s_3)(PT(\{(\{c\}, \frac{1}{3})_1\}, s_3) + PT(\{(\{c\}, \frac{1}{3})_2\}, s_3)) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4} = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \varphi'(s_3')PT(\{(\{c\}, \frac{1}{2})_1\}, s_3') + \varphi'(s_4')PT(\{(\{c\}, \frac{1}{2})_2\}, s_4')$.

The sojourn time averages in the equivalence class $\mathcal{H}$ coincide:
$SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\mathcal{H}\cap DR(G)) =$
$SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\{s_3\}) = \frac{1}{1-PM(\{s_3\},\{s_3\})} =$
$\frac{1}{1-PM(s_3,s_3)} = \frac{1}{1-\frac{1}{2}} = 2 = \frac{1}{1-\frac{1}{2}} = \frac{1}{1-PM(s_3',s_3')} = \frac{1}{1-PM(s_4',s_4')} =$
$\frac{1}{1-PM(\{s_3',s_4'\},\{s_3',s_4'\})} = SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\{s_3', s_4'\}) =$
$SJ_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\mathcal{H}\cap DR(G'))$.

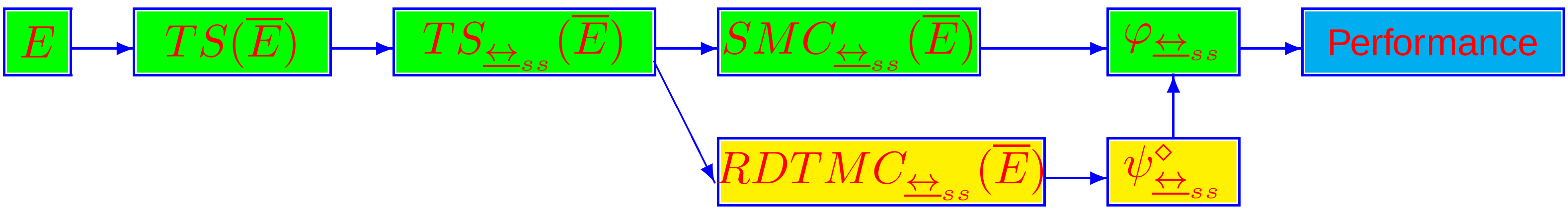
The sojourn time variances in the equivalence class $\mathcal{H}$ coincide:
$VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\mathcal{H}\cap DR(G)) =$
$VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E}))^2}(\{s_3\}) = \frac{PM(\{s_3\},\{s_3\})}{(1-PM(\{s_3\},\{s_3\}))^2} =$
$\frac{PM(s_3,s_3)}{(1-PM(s_3,s_3))^2} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2 = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{PM(s_3',s_3')}{(1-PM(s_3',s_3'))^2} =$
$\frac{PM(s_4',s_4')}{(1-PM(s_4',s_4'))^2} = \frac{PM(\{s_3',s_4'\},\{s_3',s_4'\})}{(1-PM(\{s_3',s_4'\},\{s_3',s_4'\}))^2} =$
$VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\{s_3', s_4'\}) =$
$VAR_{\mathcal{R}_{ss}(\overline{E},\overline{E'})\cap(DR(\overline{E'}))^2}(\mathcal{H}\cap DR(G'))$.

Simplification of performance analysis

The method of performance analysis simplification.

1. The investigated system is specified by a static expression of $dtsiPBC$.
2. The transition system of the expression is constructed.
3. After treating the transition system for self-similarity,
   a step stochastic autobisimulation equivalence for the expression is determined.
4. The quotient underlying SMC is constructed from the quotient transition system.
5. Stationary probabilities and performance indices are calculated using the SMC.

Simplification of the steps 4 and 5:
constructing the reduced quotient DTMC from the quotient transition system,
calculating the stationary probabilities of the quotient underlying SMC
using this DTMC and obtaining the performance indices.



EQPEVA: Equivalence-based simplification of performance evaluation

The limitation of the method: the expressions with underlying SMCs containing one closed communication class of states, which is ergodic, to ensure uniqueness of the stationary distribution.

If an SMC contains several closed communication classes of states that are all ergodic: several stationary distributions may exist, depending on the initial PMF.

The general steady-state probabilities are then calculated as
the sum of the stationary probabilities of all the ergodic classes of states,
weighted by the probabilities to enter these classes,
starting from the initial state and passing through transient states.

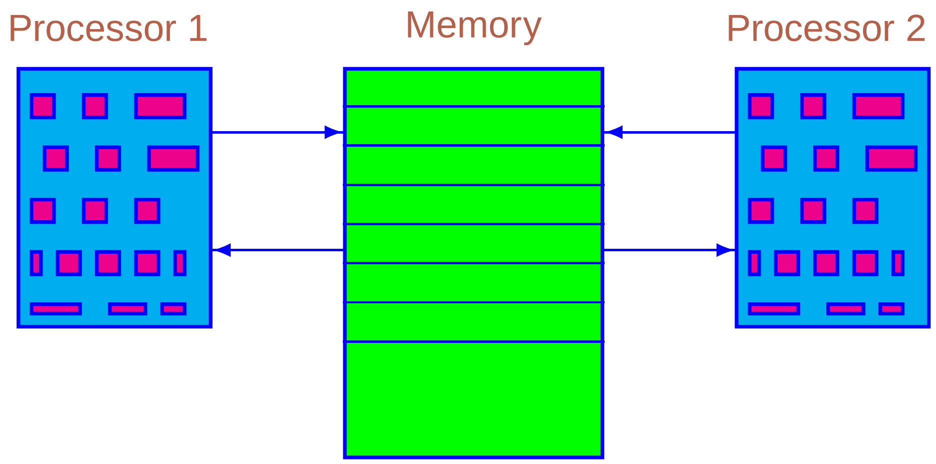
The underlying SMC of each process expression has one initial PMF
(that at the time moment 0): the stationary distribution is unique.

It is worth applying the method to the systems with similar subprocesses.

Shared memory system

A model of two processors accessing a common shared memory [MBCDF95]

The standard system



SHMDIA: The diagram of the shared memory system

After activation of the system (turning the computer on), two processors are active, and the common memory is available. Each processor can request an access to the memory after which the instantaneous decision is made.

When the decision is made in favour of a processor, it starts an acquisition of the memory, and another processor waits until the former one ends its operations, and the system returns to the state with both active processors and the available memory.

a corresponds to the system activation.

r_i $(1 \leq i \leq 2)$ represent the common memory request of processor i.

d_i correspond to the instantaneous decision on the memory allocation in favour of the processor i.

m_i represent the common memory access of processor i.

The other actions are used for communication purpose only.

The static expression of the first processor is

$E_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r_1\}, \frac{1}{2}); (\{d_1, y_1\}, 1); (\{m_1, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is

$E_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r_2\}, \frac{1}{2}); (\{d_2, y_2\}, 1); (\{m_2, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is

$E_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, 1); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, 1); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the shared memory system with two processors is

$E = (E_1 \| E_2 \| E_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

Effect of synchronization

The synchronization of $(\{d_i, y_i\}, 1)$ and $(\{\widehat{y_i}\}, 1)$ produces $(\{d_i\}, 2)$ $(1 \leq i \leq 2)$.

The synchronization of $(\{m_i, z_i\}, \frac{1}{2})$ and $(\{\widehat{z_i}\}, \frac{1}{2})$ produces $(\{m_i\}, \frac{1}{4})$ $(1 \leq i \leq 2)$.

The synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a, \widehat{x_2}\}, \frac{1}{4})$,

Synchronization of $(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2})$ and $(\{x_2\}, \frac{1}{2})$ produces $(\{a, \widehat{x_1}\}, \frac{1}{4})$.

Synchronization of $(\{a, \widehat{x_2}\}, \frac{1}{4})$ and $(\{x_2\}, \frac{1}{2})$, as well as $(\{a, \widehat{x_1}\}, \frac{1}{4})$ and $(\{x_1\}, \frac{1}{2})$ produces $(\{a\}, \frac{1}{8})$.

$DR(\overline{E})$ consists of

$$s_1 = [([\overline{(\{x_1\},\frac{1}{2})} * ((\{r_1\},\frac{1}{2});(\{d_1,y_1\},1);(\{m_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[\overline{(\{x_2\},\frac{1}{2})} * ((\{r_2\},\frac{1}{2});(\{d_2,y_2\},1);(\{m_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[\overline{(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2})} * (((\{\widehat{y_1}\},1);(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},1);(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_2 = [([(\{x_1\},\frac{1}{2}) * (\overline{(\{r_1\},\frac{1}{2})};(\{d_1,y_1\},1);(\{m_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\frac{1}{2}) * (\overline{(\{r_2\},\frac{1}{2})};(\{d_2,y_2\},1);(\{m_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},1);(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},1);(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_3 = [([(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});\overline{(\{d_1,y_1\},1)};(\{m_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\frac{1}{2}) * (\overline{(\{r_2\},\frac{1}{2})};(\{d_2,y_2\},1);(\{m_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * ((\overline{(\{\widehat{y_1}\},1)};(\{\widehat{z_1}\},\frac{1}{2}))[]((\{\widehat{y_2}\},1);(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_4 = [([(\{x_1\},\frac{1}{2}) * (\overline{(\{r_1\},\frac{1}{2})};(\{d_1,y_1\},1);(\{m_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\frac{1}{2}) * ((\{r_2\},\frac{1}{2});\overline{(\{d_2,y_2\},1)};(\{m_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},1);(\{\widehat{z_1}\},\frac{1}{2}))[](\overline{(\{\widehat{y_2}\},1)};(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_5 = [([(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});(\{d_1,y_1\},1);\overline{(\{m_1,z_1\},\frac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\frac{1}{2}) * (\overline{(\{r_2\},\frac{1}{2})};(\{d_2,y_2\},1);(\{m_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * (((\{\widehat{y_1}\},1);\overline{(\{\widehat{z_1}\},\frac{1}{2})})[]((\{\widehat{y_2}\},1);(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_6 = [([(\{x_1\},\frac{1}{2}) * ((\{r_1\},\frac{1}{2});\overline{(\{d_1,y_1\},1)};(\{m_1,z_1\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\frac{1}{2}) * ((\{r_2\},\frac{1}{2});\overline{(\{d_2,y_2\},1)};(\{m_2,z_2\},\frac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\frac{1}{2}) * ((\overline{(\{\widehat{y_1}\},1)};(\{\widehat{z_1}\},\frac{1}{2}))[](\overline{(\{\widehat{y_2}\},1)};(\{\widehat{z_2}\},\frac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_\approx,$$

$$s_7 = [([(\{x_1\},\tfrac{1}{2}) * ((\overline{(\{r_1\},\tfrac{1}{2})};(\{d_1,y_1\},1);(\{m_1,z_1\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * ((\{r_2\},\tfrac{1}{2});(\{d_2,y_2\},1);\overline{(\{m_2,z_2\},\tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * ((((\{\widehat{y_1}\},1);(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},1);\overline{(\{\widehat{z_2}\},\tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_8 = [([(\{x_1\},\tfrac{1}{2}) * ((\{r_1\},\tfrac{1}{2});(\{d_1,y_1\},1);\overline{(\{m_1,z_1\},\tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * ((\{r_2\},\tfrac{1}{2});(\{d_2,y_2\},1);(\{m_2,z_2\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * ((((\{\widehat{y_1}\},1);\overline{(\{\widehat{z_1}\},\tfrac{1}{2})})[]((\{\widehat{y_2}\},1);(\{\widehat{z_2}\},\tfrac{1}{2}))) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx},$$

$$s_9 = [([(\{x_1\},\tfrac{1}{2}) * ((\{r_1\},\tfrac{1}{2});(\{d_1,y_1\},1);(\{m_1,z_1\},\tfrac{1}{2})) * \mathsf{Stop}]$$
$$\|[(\{x_2\},\tfrac{1}{2}) * ((\{r_2\},\tfrac{1}{2});(\{d_2,y_2\},1);\overline{(\{m_2,z_2\},\tfrac{1}{2})}) * \mathsf{Stop}]$$
$$\|[(\{a,\widehat{x_1},\widehat{x_2}\},\tfrac{1}{2}) * ((((\{\widehat{y_1}\},1);(\{\widehat{z_1}\},\tfrac{1}{2}))[]((\{\widehat{y_2}\},1);\overline{(\{\widehat{z_2}\},\tfrac{1}{2})})) * \mathsf{Stop}])$$
$$\mathsf{sy}\ x_1\ \mathsf{sy}\ x_2\ \mathsf{sy}\ y_1\ \mathsf{sy}\ y_2\ \mathsf{sy}\ z_1\ \mathsf{sy}\ z_2\ \mathsf{rs}\ x_1\ \mathsf{rs}\ x_2\ \mathsf{rs}\ y_1\ \mathsf{rs}\ y_2\ \mathsf{rs}\ z_1\ \mathsf{rs}\ z_2]_{\approx}.$$

Interpretation of the states

$DR_T(\overline{E}) = \{s_1, s_2, s_5, s_7, s_8, s_9\}$ and $DR_V(\overline{E}) = \{s_3, s_4, s_6\}$.

s_1: the initial state,

s_2: the system is activated and the memory is not requested,

s_3: the memory is requested by the first processor,

s_4: the memory is requested by the second processor,

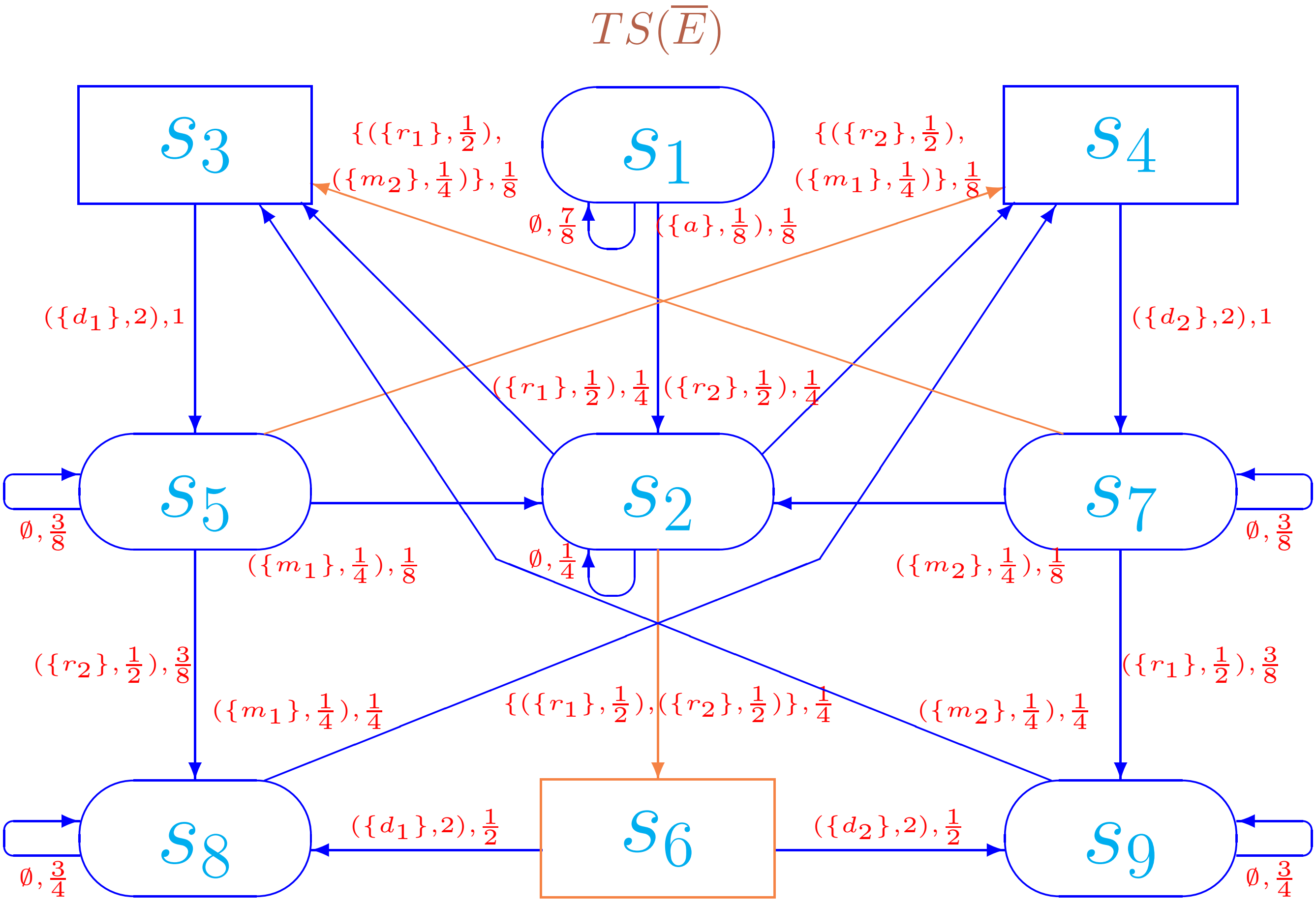
s_5: the memory is allocated to the first processor,

s_6: the memory is requested by two processors,

s_7: the memory is allocated to the second processor,

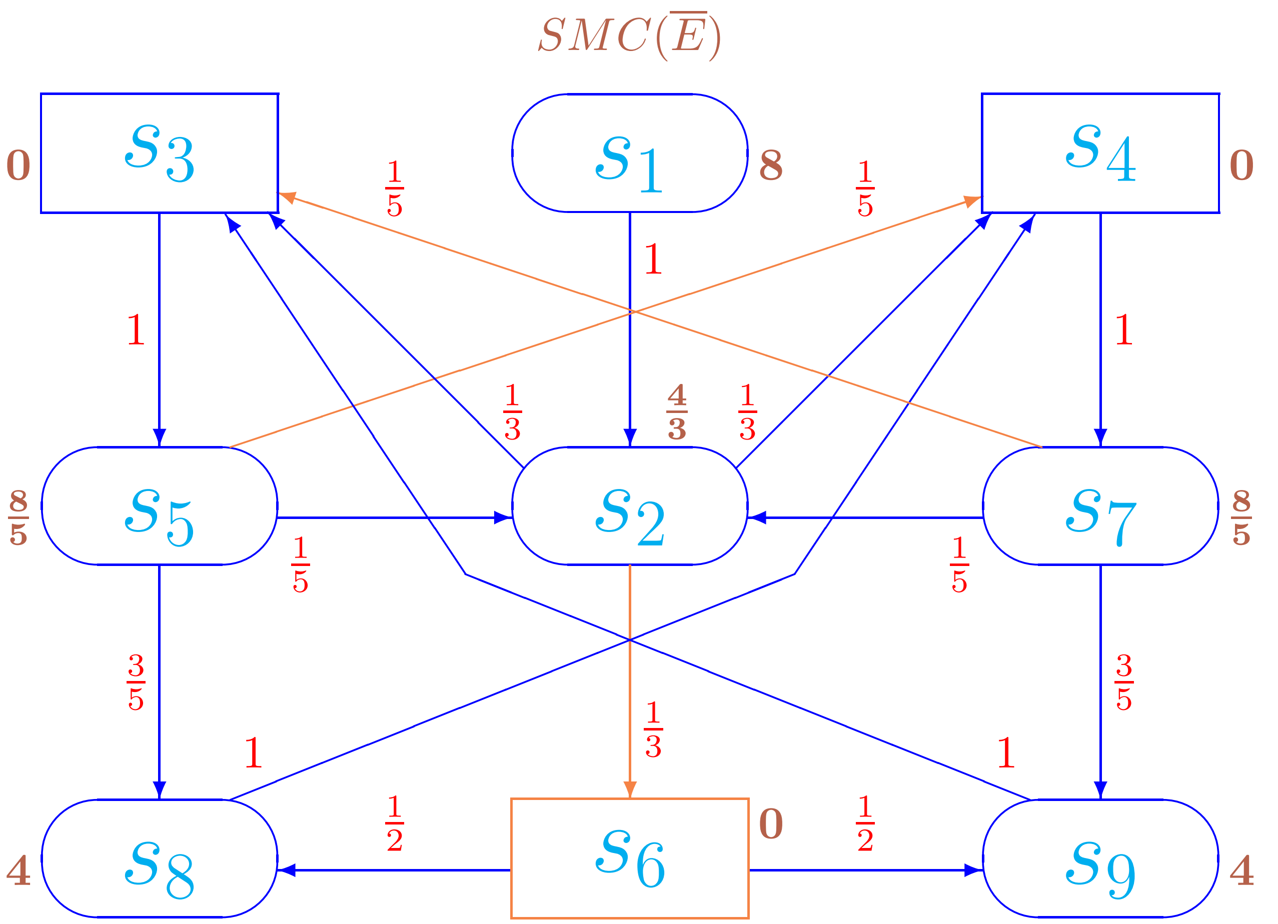
s_8: the memory is allocated to the first processor and the memory is requested by the second processor,

s_9: the memory is allocated to the second processor and the memory is requested by the first processor.



SHMTS: The transition system of the shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)



SHMSMC: The underlying SMC of the shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The average sojourn time vector of $\overline{E}$:

$$SJ = \left(8, \frac{4}{3}, 0, 0, \frac{8}{5}, 0, \frac{8}{5}, 4, 4\right).$$

The sojourn time variance vector of $\overline{E}$:

$$VAR = \left(56, \frac{4}{9}, 0, 0, \frac{24}{25}, 0, \frac{24}{25}, 12, 12\right).$$

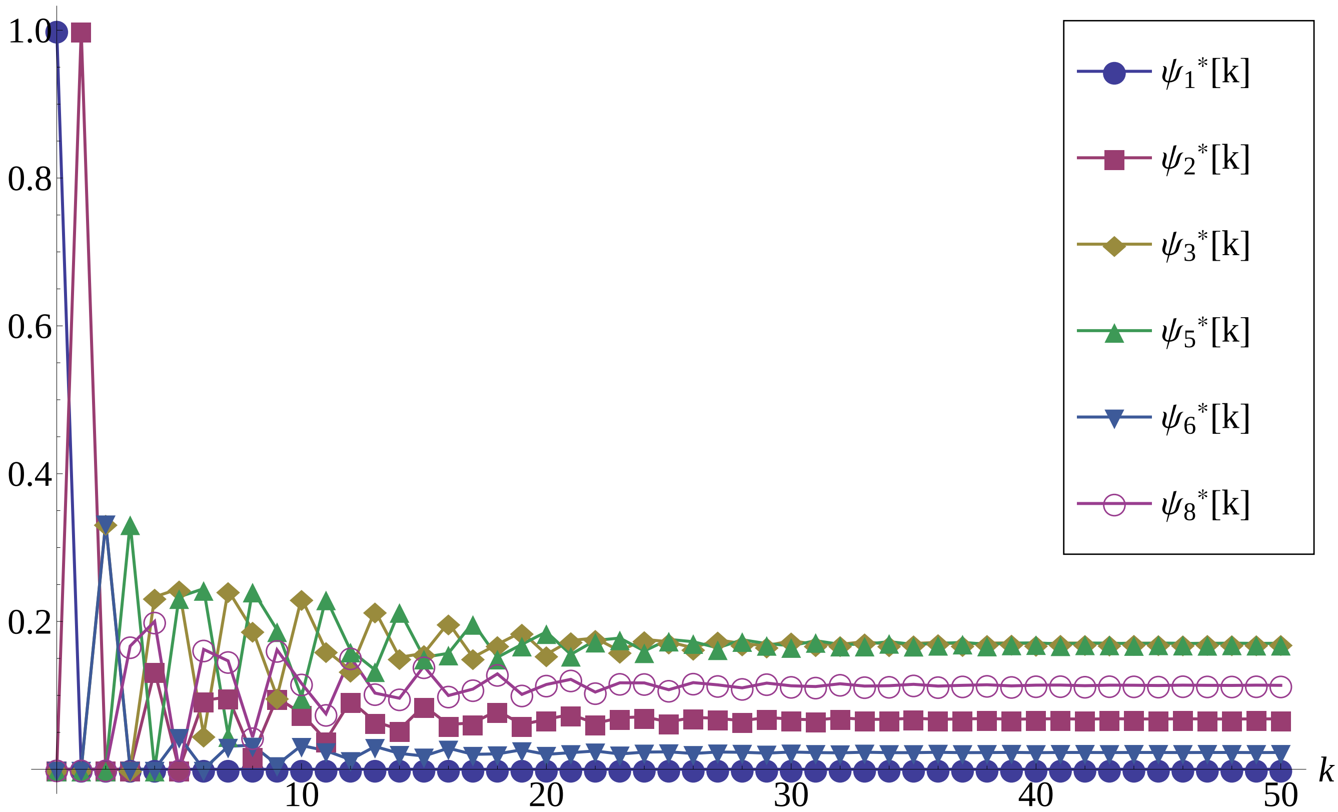
The TPM for $EDTMC(\overline{E})$:

$$\mathbf{P}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

SHMTP: Transient and steady-state probabilities for the EDTMC of the shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	
$\psi_1^*[k]$	1	0	0	0	0	0	0	0	0	0	0	
$\psi_2^*[k]$	0	0	0.0754	0.0859	0.0677	0.0641	0.0680	0.0691	0.0683	0.0680	0.0681	0.0
$\psi_3^*[k]$	0	0.2444	0.2316	0.1570	0.1554	0.1726	0.1741	0.1702	0.1696	0.1705	0.1707	0.1
$\psi_5^*[k]$	0	0.2333	0.0982	0.1516	0.1859	0.1758	0.1672	0.1690	0.1711	0.1708	0.1703	0.1
$\psi_6^*[k]$	0	0.0444	0.0323	0.0179	0.0202	0.0237	0.0234	0.0226	0.0226	0.0228	0.0228	0.0
$\psi_8^*[k]$	0	0	0.1163	0.1395	0.1147	0.1077	0.1130	0.1150	0.1139	0.1133	0.1136	0.1

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.



SHMTP: Transient probabilities alteration diagram for the EDTMC of the shared memory system

The steady-state PMF for $EDTMC(\overline{E})$:

$$\psi^* = \left(0, \frac{3}{44}, \frac{15}{88}, \frac{15}{88}, \frac{15}{88}, \frac{1}{44}, \frac{15}{88}, \frac{5}{44}, \frac{5}{44}\right).$$

The steady-state PMF ψ^* weighted by SJ:

$$\left(0, \frac{1}{11}, 0, 0, \frac{3}{11}, 0, \frac{3}{11}, \frac{5}{11}, \frac{5}{11}\right).$$

We normalize the steady-state weighted PMF dividing it by the sum of its components $\psi^* SJ^T = \frac{17}{11}$.

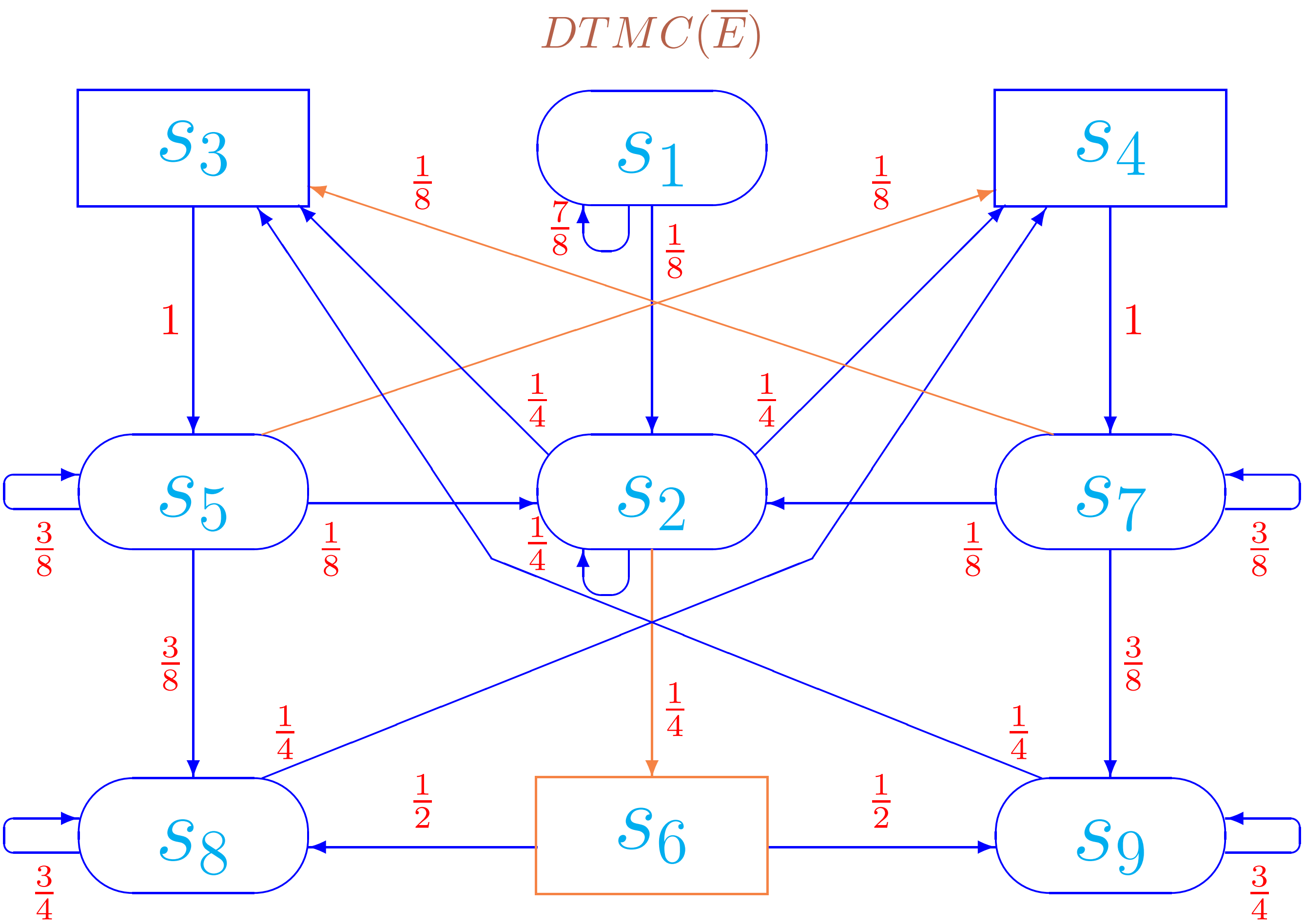
The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \left(0, \frac{1}{17}, 0, 0, \frac{3}{17}, 0, \frac{3}{17}, \frac{5}{17}, \frac{5}{17}\right).$$

Otherwise, from $TS(\overline{E})$, we can construct $DTMC(\overline{E})$ and calculate φ using it.

The TPM for $DTMC(\overline{E})$:

$$\mathbf{P} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{3}{8} & 0 & 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$



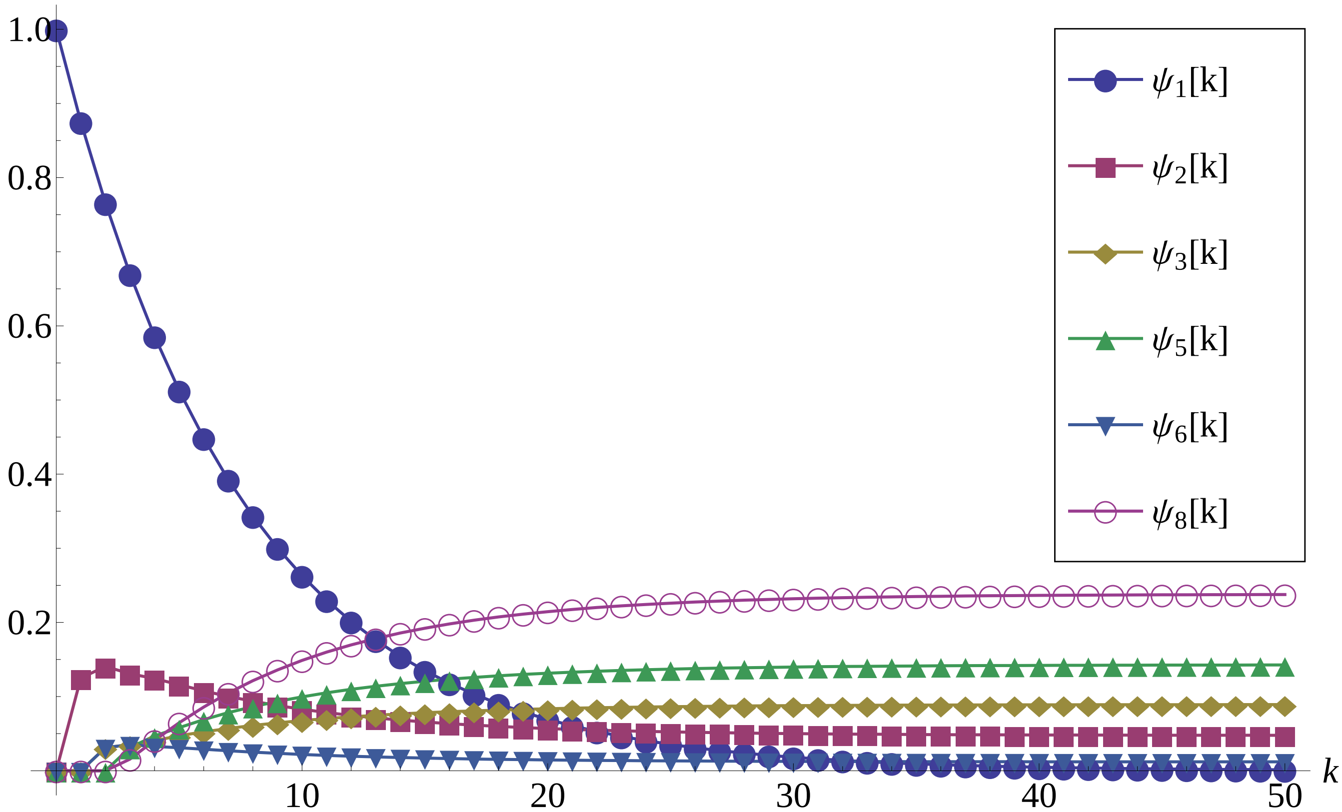
SHMDTMC: The DTMC of the shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

SHMTPDTMC: Transient and steady-state probabilities for the DTMC of the shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	
$\psi_1[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	
$\psi_2[k]$	0	0.1161	0.0829	0.0657	0.0569	0.0524	0.0501	0.0489	0.0483	0.0479	0.0478	0.0
$\psi_3[k]$	0	0.0472	0.0677	0.0782	0.0836	0.0864	0.0878	0.0885	0.0889	0.0891	0.0892	0.0
$\psi_5[k]$	0	0.0581	0.0996	0.1207	0.1315	0.1370	0.1399	0.1413	0.1421	0.1425	0.1427	0.1
$\psi_6[k]$	0	0.0311	0.0220	0.0171	0.0146	0.0133	0.0126	0.0123	0.0121	0.0120	0.0120	0.0
$\psi_8[k]$	0	0.0647	0.1487	0.1923	0.2146	0.2260	0.2319	0.2349	0.2365	0.2373	0.2377	0.2

We depict the probabilities for the states $s_1, s_2, s_3, s_5, s_6, s_8$ only, since the corresponding values coincide for s_3, s_4 as well as for s_5, s_7 as well as for s_8, s_9.



SHMTPDTMC: Transient probabilities alteration diagram for the DTMC of the shared memory system

The steady-state PMF for $DTMC(\overline{E})$:

$$\psi = \left(0, \frac{1}{21}, \frac{5}{56}, \frac{5}{56}, \frac{1}{7}, \frac{1}{84}, \frac{1}{7}, \frac{5}{21}, \frac{5}{21}\right).$$

Remember that $DR_T(\overline{E}) = \{s_1, s_2, s_5, s_7, s_8, s_9\}$ and $DR_V(\overline{E}) = \{s_3, s_4, s_6\}$. Hence,

$$\sum_{s \in DR_T(\overline{E})} \psi(s) = \psi(s_1) + \psi(s_2) + \psi(s_5) + \psi(s_7) + \psi(s_8) + \psi(s_9) = \frac{17}{21}.$$

By Proposition PMFSMC $DTMC(G)$:

$$\varphi(s_1) = 0 \cdot \tfrac{21}{17} = 0,$$

$$\varphi(s_2) = \tfrac{1}{21} \cdot \tfrac{21}{17} = \tfrac{1}{17},$$

$$\varphi(s_3) = 0,$$

$$\varphi(s_4) = 0,$$

$$\varphi(s_5) = \tfrac{1}{7} \cdot \tfrac{21}{17} = \tfrac{3}{17},$$

$$\varphi(s_6) = 0,$$

$$\varphi(s_7) = \tfrac{1}{7} \cdot \tfrac{21}{17} = \tfrac{3}{17},$$

$$\varphi(s_8) = \tfrac{5}{21} \cdot \tfrac{21}{17} = \tfrac{5}{17},$$

$$\varphi(s_9) = \tfrac{5}{21} \cdot \tfrac{21}{17} = \tfrac{5}{17}.$$

Alternatively, from $TS(\overline{E})$, we can construct the reduced DTMC of $\overline{E}$, $RDTMC(\overline{E})$, and calculate φ using it.

$DR_T(\overline{E}) = \{s_1, s_2, s_5, s_7, s_8, s_9\}$ and $DR_V(\overline{E}) = \{s_3, s_4, s_6\}$.

We reorder the elements of $DR(\overline{E})$ by

moving the equivalence classes of vanishing states to the first positions:

$s_3, s_4, s_6, s_1, s_2, s_5, s_7, s_8, s_9$.

The reordered TPM for $DTMC(\overline{E})$:

$$\mathbf{P}_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

The result of the decomposing $\mathbf{P}_r$:

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

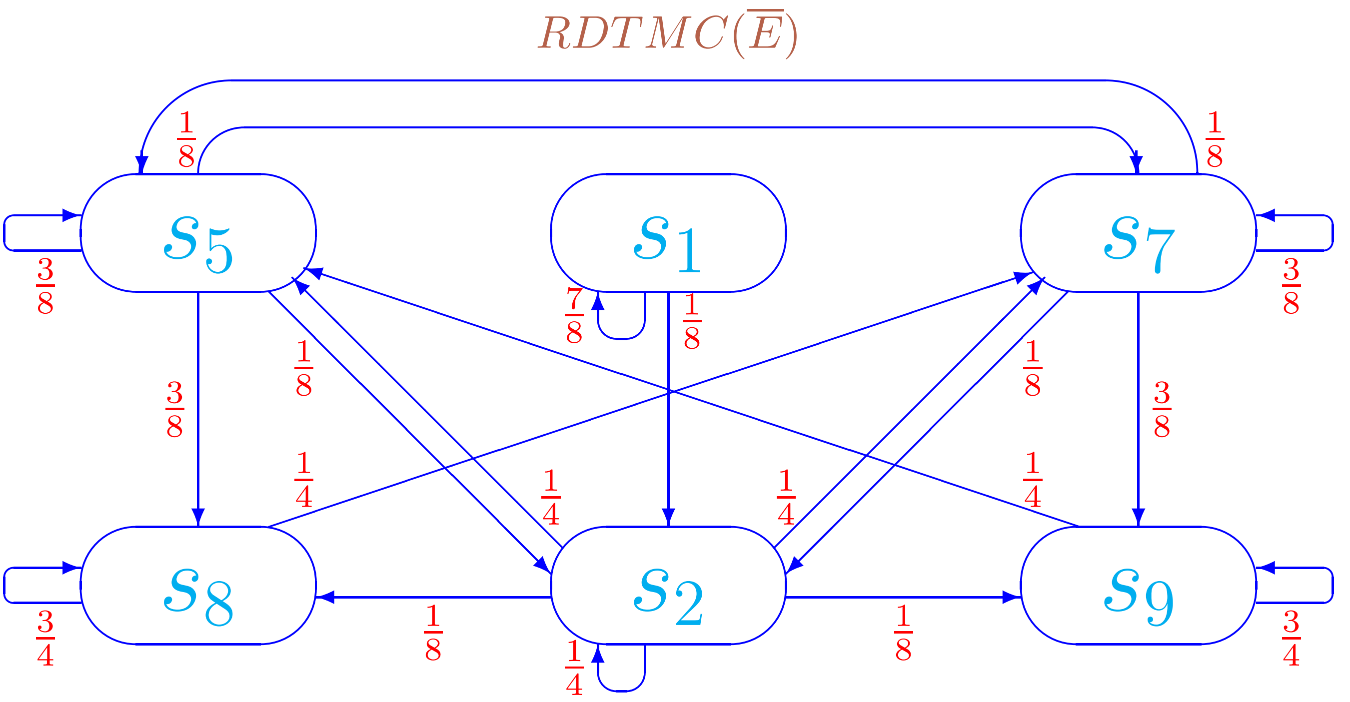
$$\mathbf{E} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} & 0 \\ 0 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

Since $\mathbf{C}^1 = \mathbf{0}$, we have $\forall k > 0,\ \mathbf{C}^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\mathbf{G} = \sum_{k=0}^{l} \mathbf{C}^k = \mathbf{C}^0 = \mathbf{I}.$$

The TPM for $RDTMC(\overline{E})$:

$$\mathbf{P}^\diamond = \mathbf{F} + \mathbf{EGD} = \mathbf{F} + \mathbf{EID} = \mathbf{F} + \mathbf{ED} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

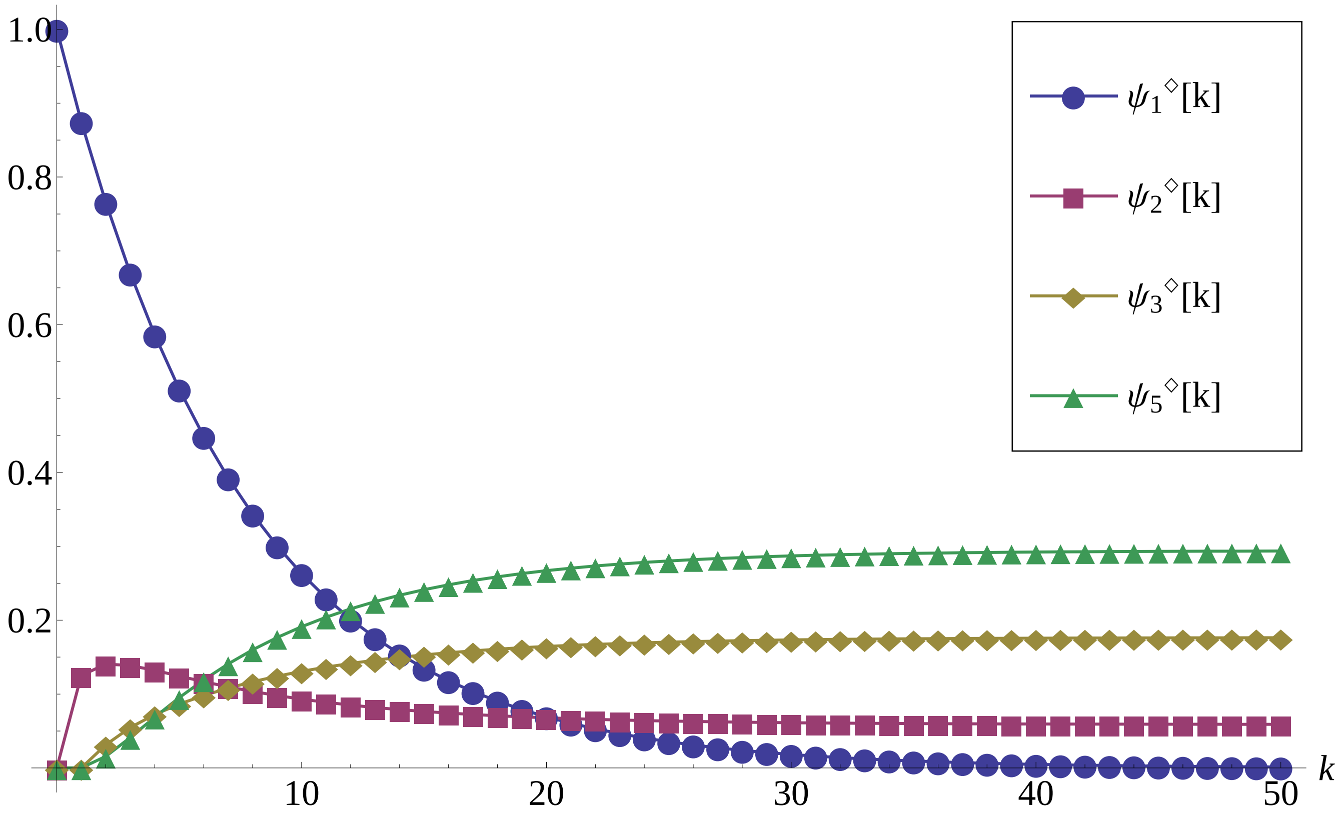


SHMRDTMC: The reduced DTMC of the shared memory system

SHMTRPR: Transient and steady-state probabilities for the RDTMC of the shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	[illegible]
$\psi_1^\diamond[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	
$\psi_2^\diamond[k]$	0	0.1244	0.0931	0.0764	0.0679	0.0635	0.0612	0.0600	0.0594	0.0591	0.0590	0.0[illegible]
$\psi_3^\diamond[k]$	0	0.0863	0.1307	0.1530	0.1644	0.1703	0.1733	0.1748	0.1756	0.1760	0.1763	0.1[illegible]
$\psi_5^\diamond[k]$	0	0.0951	0.1912	0.2413	0.2670	0.2802	0.2870	0.2905	0.2922	0.2932	0.2936	0.2[illegible]

We depict the probabilities for states s_1, s_2, s_5, s_8 only, since the corresponding values coincide for s_5, s_7, as well as for s_8, s_9.



SHMTRPR: Transient probabilities alteration diagram for the RDTMC of the shared memory system

The steady-state PMF for $RDTMC(\overline{E})$:

$$\psi^{\diamond} = \left(0, \frac{1}{17}, \frac{3}{17}, \frac{3}{17}, \frac{5}{17}, \frac{5}{17}\right).$$

Note that $\psi^{\diamond} = (\psi^{\diamond}(s_1), \psi^{\diamond}(s_2), \psi^{\diamond}(s_5), \psi^{\diamond}(s_7), \psi^{\diamond}(s_8), \psi^{\diamond}(s_9))$.

By Proposition PMFSMCT:

$$\varphi(s_1) = 0, \quad \varphi(s_2) = \tfrac{1}{17}, \quad \varphi(s_5) = \tfrac{3}{17},$$

$$\varphi(s_7) = \tfrac{3}{17}, \quad \varphi(s_8) = \tfrac{5}{17}, \quad \varphi(s_9) = \tfrac{5}{17}.$$

The steady-state PMF for $SMC(\overline{E})$:

$$\varphi = \left(0, \frac{1}{17}, 0, 0, \frac{3}{17}, 0, \frac{3}{17}, \frac{5}{17}, \frac{5}{17}\right).$$

This coincides with the result obtained with the use of ψ^* and SJ.

Performance indices

- The average recurrence time in the state s_2, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\varphi_2} = 17$.

- The common memory is available only in the states s_2, s_3, s_4, s_6.

  The steady-state probability that the memory is available is $\varphi_2 + \varphi_3 + \varphi_4 + \varphi_6 = \frac{1}{17} + 0 + 0 + 0 = \frac{1}{17}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{1}{17} = \frac{16}{17}$.

- After activation of the system, we leave the state s_1 for ever, and the common memory is either requested or allocated in every remaining state, with exception of s_2.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving s_2, calculated as $\frac{\varphi_2}{SJ_2} = \frac{1}{17} \cdot \frac{3}{4} = \frac{3}{68}$.

- The parallel common memory request of two processors $\{(\{r_1\}, \frac{1}{2}), (\{r_2\}, \frac{1}{2})\}$ is only possible from the state s_2.

  The request probability in this state is the sum of the execution probabilities for all multisets of activities containing both $(\{r_1\}, \frac{1}{2})$ and $(\{r_2\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from two processors* is

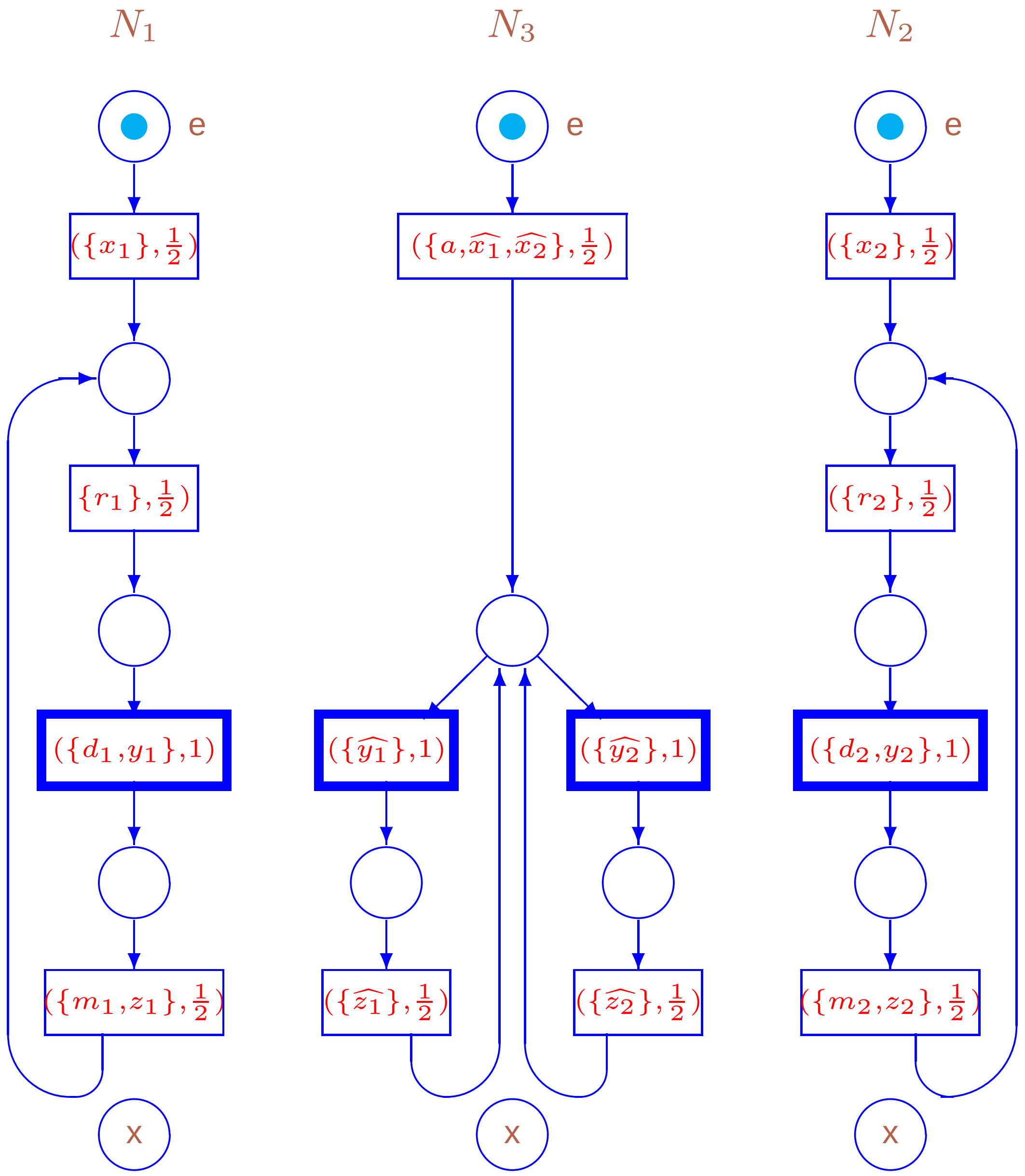
  $\varphi_2 \sum_{\{\Upsilon | (\{(\{r_1\}, \frac{1}{2}), (\{r_2\}, \frac{1}{2})\} \subseteq \Upsilon\}} PT(\Upsilon, s_2) = \frac{1}{17} \cdot \frac{1}{4} = \frac{1}{68}$.

- The common memory request of the first processor $(\{r_1\}, \frac{1}{2})$ is only possible from the states s_2, s_7.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r_1\}, \frac{1}{2})$.

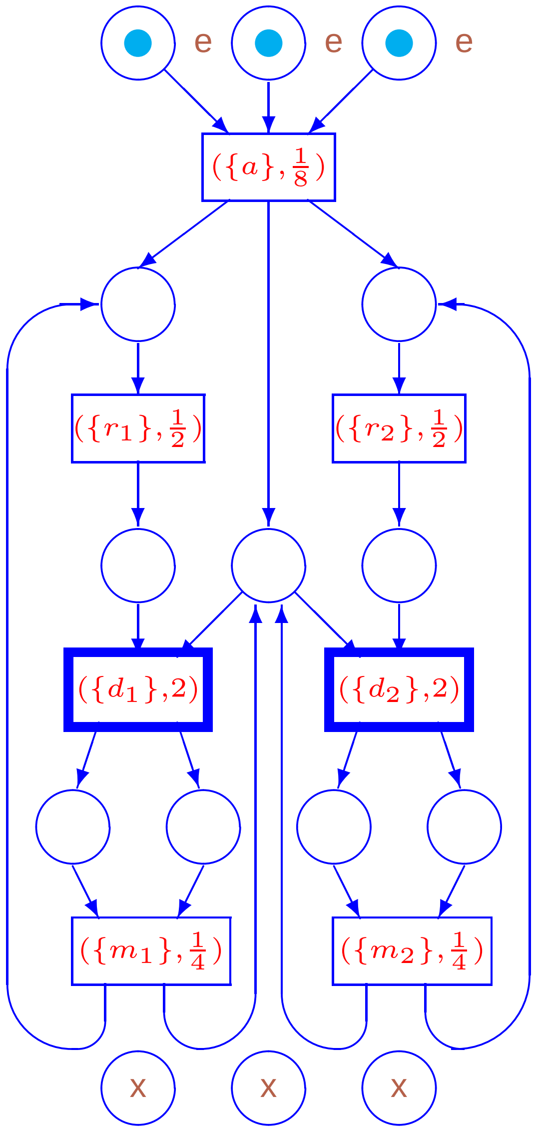
  The *steady-state probability of the shared memory request from the first processor* is

  $\varphi_2 \sum_{\{\Gamma | (\{r_1\}, \frac{1}{2}) \in \Gamma\}} PT(\Gamma, s_2) + \varphi_7 \sum_{\{\Gamma | (\{r_1\}, \frac{1}{2}) \in \Gamma\}} PT(\Gamma, s_7) = \frac{1}{17}\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{3}{17}\left(\frac{3}{8} + \frac{1}{8}\right) = \frac{2}{17}$.



SHMPMBOX: The marked dtsi-boxes of two processors and shared memory

N



SHMBOX: The marked dtsi-box of the shared memory system

The abstract system and its reduction

The static expression of the first processor is
$F_1 = [(\{x_1\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{d, y_1\}, 1); (\{m, z_1\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the second processor is
$F_2 = [(\{x_2\}, \frac{1}{2}) * ((\{r\}, \frac{1}{2}); (\{d, y_2\}, 1); (\{m, z_2\}, \frac{1}{2})) * \mathsf{Stop}]$.

The static expression of the shared memory is $F_3 =$
$[(\{a, \widehat{x_1}, \widehat{x_2}\}, \frac{1}{2}) * (((\{\widehat{y_1}\}, 1); (\{\widehat{z_1}\}, \frac{1}{2}))[]((\{\widehat{y_2}\}, 1); (\{\widehat{z_2}\}, \frac{1}{2}))) * \mathsf{Stop}]$.

The static expression of the abstract shared memory system with two processors:
$F = (F_1 \| F_2 \| F_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{F})$ resembles $DR(\overline{E})$, and $TS(\overline{F})$ is similar to $TS(\overline{E})$.

$SMC(\overline{F}) \simeq SMC(\overline{E})$, thus, the average sojourn time vectors of $\overline{F}$ and $\overline{E}$, the TPMs and the steady-state PMFs for $EDTMC(\overline{F})$ and $EDTMC(\overline{E})$ coincide.

Performance indices

The first, second and third performance indices are the same for the standard and abstract systems.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \frac{1}{2})$ is only possible from the states s_2, s_5, s_7.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r\}, \frac{1}{2})$.

  The *steady-state probability of the shared memory request from a processor* is $\varphi_2 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_2) + \varphi_5 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_5) + \varphi_7 \sum_{\{\Gamma|(\{r\},\frac{1}{2})\in\Gamma\}} PT(\Gamma, s_7) =$
  $\frac{1}{17}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\frac{3}{17}\left(\frac{3}{8}+\frac{1}{8}\right)+\frac{3}{17}\left(\frac{3}{8}+\frac{1}{8}\right)=\frac{15}{68}$.

The quotient of the abstract system

$DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6\}$, where

$\mathcal{K}_1 = \{s_1\}$ (the initial state),

$\mathcal{K}_2 = \{s_2\}$ (the system is activated and the memory is not requested),

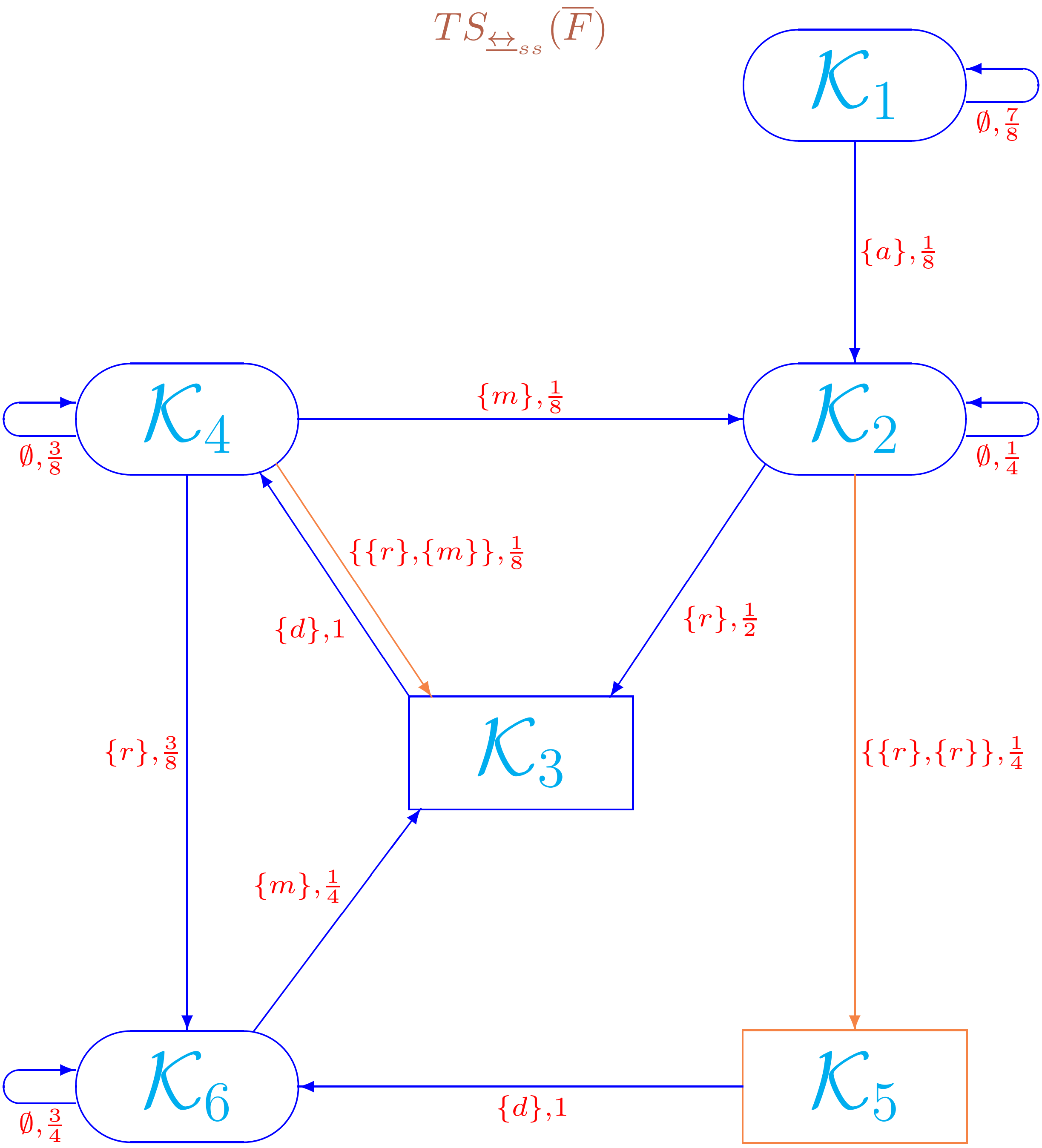
$\mathcal{K}_3 = \{s_3, s_4\}$ (the memory is requested by one processor),

$\mathcal{K}_4 = \{s_5, s_7\}$ (the memory is allocated to a processor),

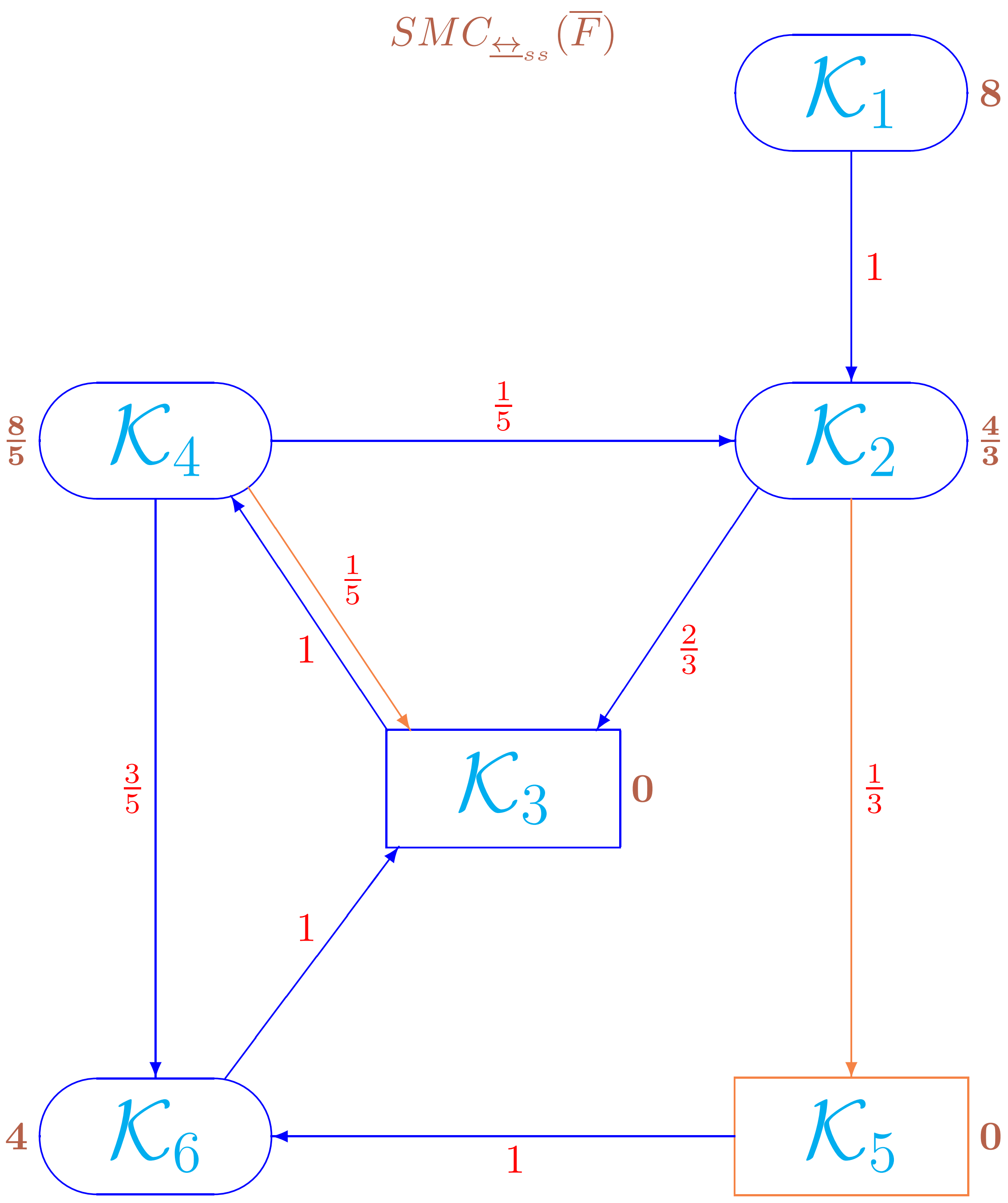
$\mathcal{K}_5 = \{s_6\}$ (the memory is requested by two processors),

$\mathcal{K}_6 = \{s_8, s_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6\}$ and
$DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3, \mathcal{K}_5\}$.



SHMQTS: The quotient transition system of the abstract shared memory system (parallel executions of activities and the exclusively reachable states are marked with orange)



SHMQSMC: The quotient underlying SMC of the abstract shared memory system (parallel executions of activities and the exclusively reachable states are marked with orange)

The quotient average sojourn time vector of $\overline{F}$:

$$SJ' = \left(8, \frac{4}{3}, 0, \frac{8}{5}, 0, 4\right).$$

The quotient sojourn time variance vector of $\overline{F}$:

$$VAR' = \left(56, \frac{4}{9}, 0, \frac{24}{25}, 0, 12\right).$$

The TPM for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}'^{*} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi'^{*} = \left(0, \frac{3}{44}, \frac{15}{44}, \frac{15}{44}, \frac{1}{44}, \frac{5}{22}\right).$$

The steady-state PMF ψ'^* weighted by SJ':

$$\left(0, \frac{1}{11}, 0, \frac{6}{11}, 0, \frac{10}{11}\right).$$

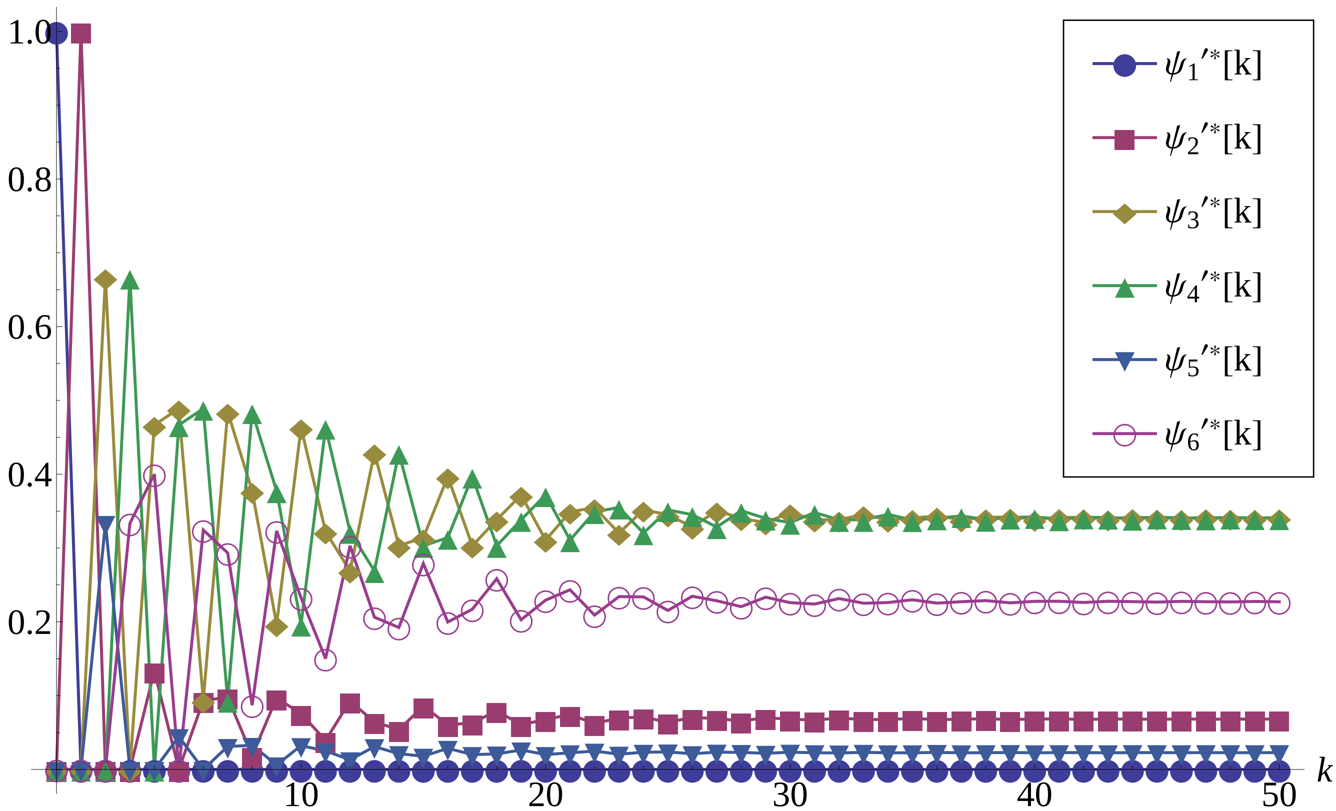
We normalize the steady-state weighted PMF dividing it by the sum of its components $\psi'^* SJ'^T = \frac{17}{11}$.

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

SHMQTP: Transient and steady-state probabilities for the quotient EDTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	
$\psi_1'^*[k]$	1	0	0	0	0	0	0	0	0	0	0	
$\psi_2'^*[k]$	0	0	0.0754	0.0859	0.0677	0.0641	0.0680	0.0691	0.0683	0.0680	0.0681	0
$\psi_3'^*[k]$	0	0.4889	0.4633	0.3140	0.3108	0.3452	0.3482	0.3404	0.3392	0.3409	0.3413	0
$\psi_4'^*[k]$	0	0.4667	0.1964	0.3031	0.3719	0.3517	0.3344	0.3380	0.3422	0.3417	0.3407	0
$\psi_5'^*[k]$	0	0.0444	0.0323	0.0179	0.0202	0.0237	0.0234	0.0226	0.0226	0.0228	0.0228	0
$\psi_6'^*[k]$	0	0	0.2325	0.2791	0.2294	0.2154	0.2260	0.2299	0.2277	0.2267	0.2271	0



SHMQTP: Transient probabilities alteration diagram for the quotient EDTMC of the abstract shared memory system

The steady-state PMF for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi'^{*} = \left(0, \frac{3}{44}, \frac{15}{44}, \frac{15}{44}, \frac{1}{44}, \frac{5}{22}\right).$$

The steady-state PMF ψ'^{*} weighted by SJ':

$$\left(0, \frac{1}{11}, 0, \frac{6}{11}, 0, \frac{10}{11}\right).$$

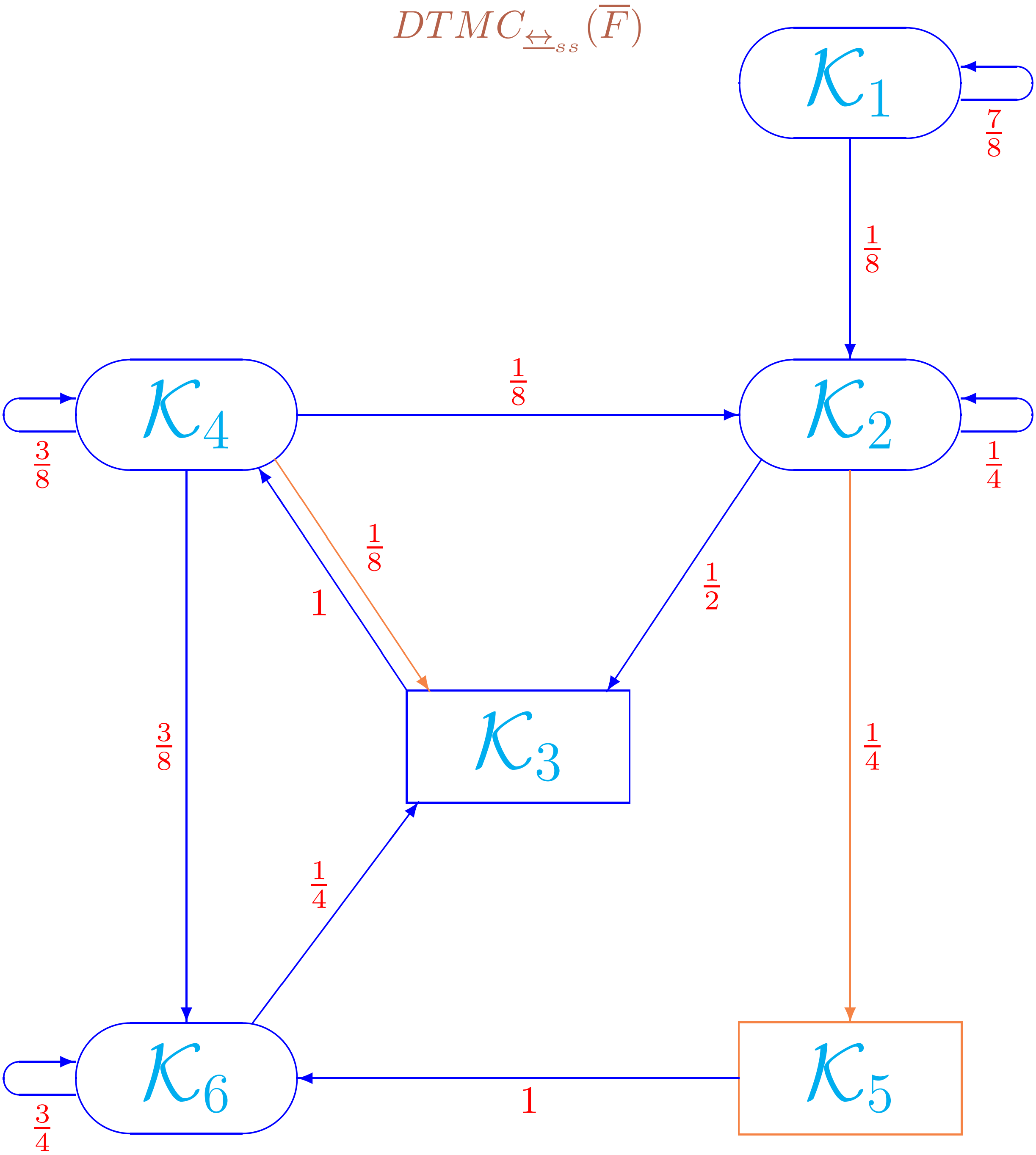
We normalize the steady-state weighted PMF dividing it by the sum of its components

$$\psi'^{*} SJ'^{T} = \frac{17}{11}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

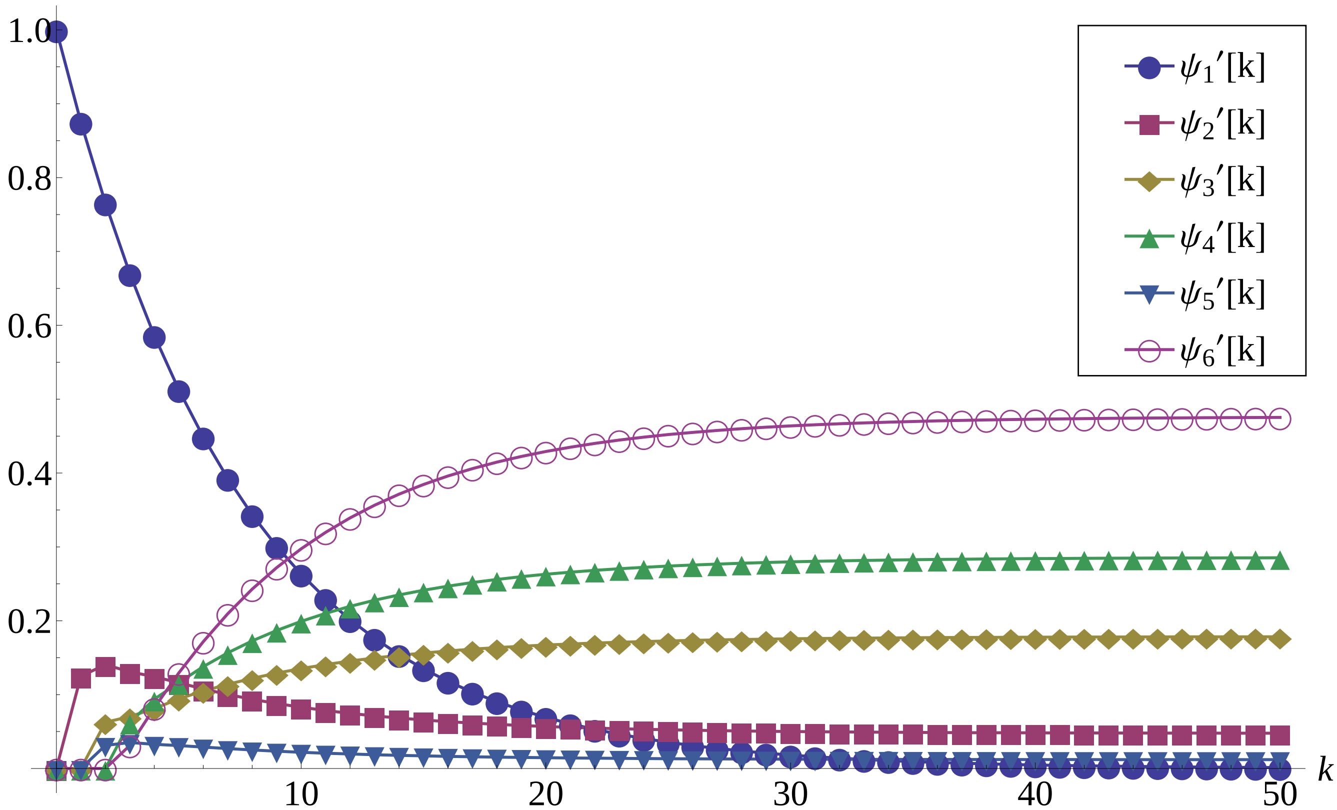
Otherwise, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{F})$, we can construct the quotient DTMC of $\overline{F}$, $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$, and calculate φ' using it.



SHMQDTMC: The quotient DTMC of the abstract shared memory system (parallel executions of activities and the exclusively reachable states are marked with orange)

SHMTPQDTMC: Transient and steady-state probabilities for the quotient DTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	[illegible]
$\psi'_1[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	[illegible]
$\psi'_2[k]$	0	0.1161	0.0829	0.0657	0.0569	0.0524	0.0501	0.0489	0.0483	0.0479	0.0478	0.0
$\psi'_3[k]$	0	0.0944	0.1353	0.1564	0.1672	0.1727	0.1756	0.1770	0.1778	0.1782	0.1784	0.1
$\psi'_4[k]$	0	0.1162	0.1992	0.2414	0.2630	0.2740	0.2797	0.2826	0.2841	0.2849	0.2853	0.2
$\psi'_5[k]$	0	0.0311	0.0220	0.0171	0.0146	0.0133	0.0126	0.0123	0.0121	0.0120	0.0120	0.0
$\psi'_6[k]$	0	0.1294	0.2974	0.3845	0.4292	0.4521	0.4638	0.4698	0.4729	0.4745	0.4753	0.4



SHMTPQDTMC: Transient probabilities alteration diagram for the quotient DTMC of the abstract shared memory system

The TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}' = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

The steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi' = \left(0, \frac{1}{21}, \frac{5}{28}, \frac{2}{7}, \frac{1}{84}, \frac{10}{21}\right).$$

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6\}$ and $DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3, \mathcal{K}_5\}$. Hence,

$$\sum_{\mathcal{K} \in DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})}} \psi'(\mathcal{K}) = \psi'(\mathcal{K}_1) + \psi'(\mathcal{K}_2) + \psi'(\mathcal{K}_4) + \psi'(\mathcal{K}_6) = \frac{17}{21}.$$

By the “quotient” analogue of Proposition PMFSMC:

$$\varphi'(\mathcal{K}_1) = 0 \cdot \frac{21}{17} = 0,$$
$$\varphi'(\mathcal{K}_2) = \frac{1}{21} \cdot \frac{21}{17} = \frac{1}{17},$$
$$\varphi'(\mathcal{K}_3) = 0,$$
$$\varphi'(\mathcal{K}_4) = \frac{2}{7} \cdot \frac{21}{17} = \frac{6}{17},$$
$$\varphi'(\mathcal{K}_5) = 0,$$
$$\varphi'(\mathcal{K}_6) = \frac{10}{21} \cdot \frac{21}{17} = \frac{10}{17}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

This coincides with the result obtained with the use of ψ'^* and SJ'.

Alternatively, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{F})$, we can construct $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$ and calculate φ' using it.

$DR_T(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6\}$ and
$DR_V(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})} = \{\mathcal{K}_3, \mathcal{K}_5\}$.

We reorder the elements of $DR(\overline{F})/_{\mathcal{R}_{ss}(\overline{F})}$ by moving the equivalence classes of vanishing states to the first positions: $\mathcal{K}_3, \mathcal{K}_5, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_4, \mathcal{K}_6$.

The reordered TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\mathbf{P}'_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

The result of the decomposing $\mathbf{P}'_r$:

$$\mathbf{C}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \mathbf{D}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathbf{E}' = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & 0 \\ \frac{1}{4} & 0 \end{pmatrix},$$

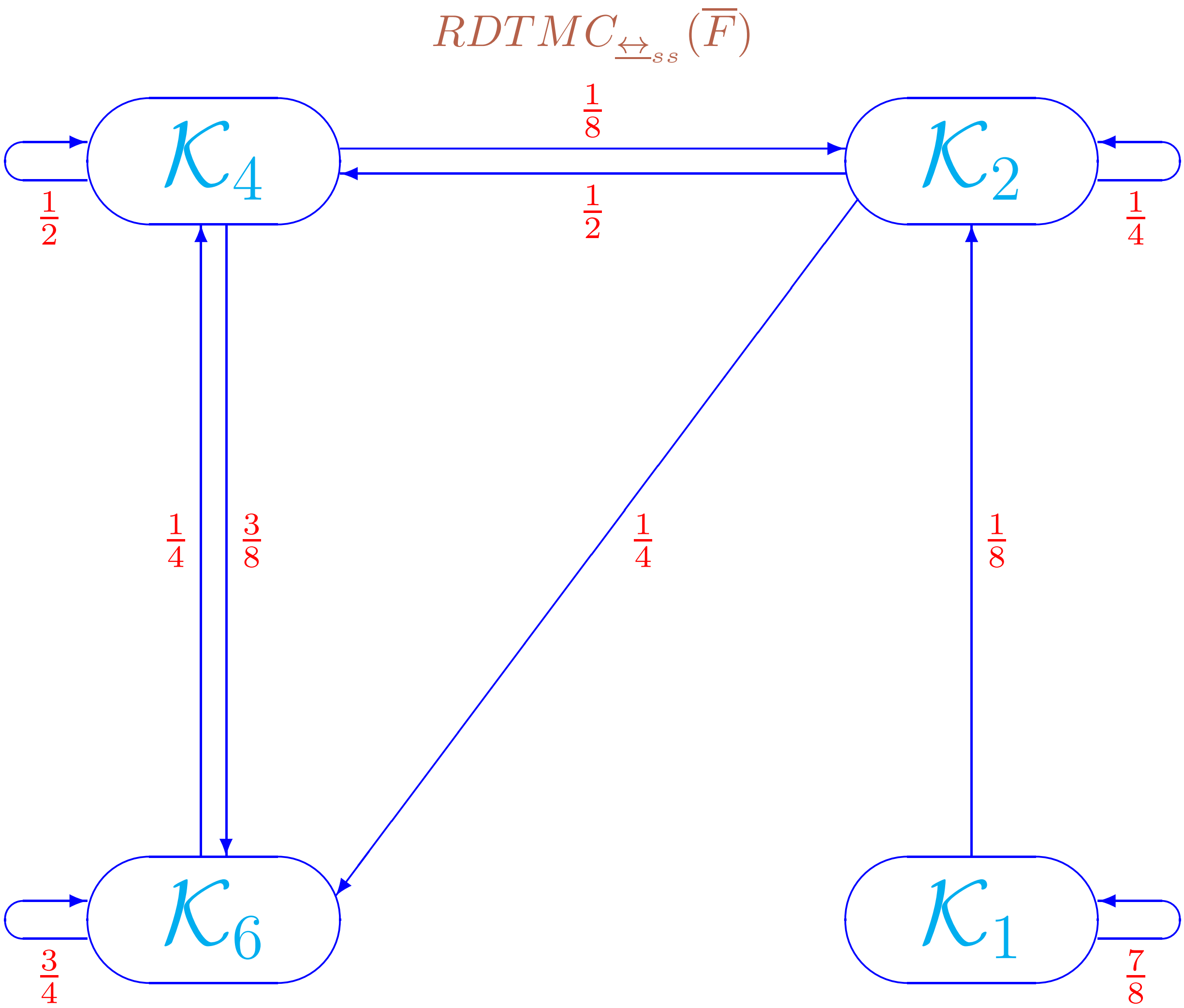
$$\mathbf{F}' = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

Since $\mathbf{C}'^1 = \mathbf{0}$, we have $\forall k > 0,\ \mathbf{C}'^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\mathbf{G}' = \sum_{k=0}^{l} \mathbf{C}'^l = \mathbf{C}'^0 = \mathbf{I}.$$

The TPM for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

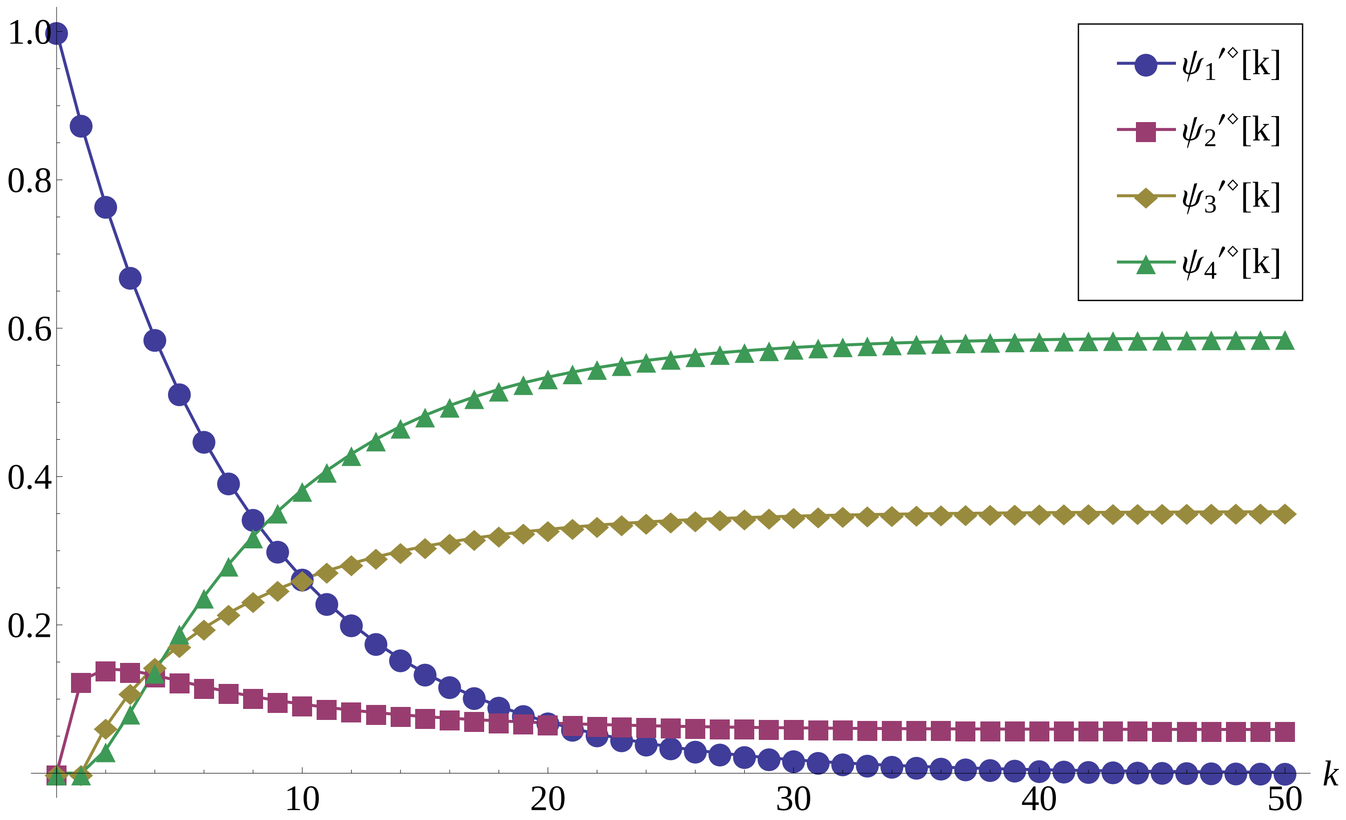
$$\mathbf{P}'^{\diamond} = \mathbf{F}' + \mathbf{E}'\mathbf{G}'\mathbf{D}' = \mathbf{F}' + \mathbf{E}'\mathbf{I}\mathbf{D}' = \mathbf{F}' + \mathbf{E}'\mathbf{D}' = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$



SHMQRDTMC: The reduced quotient DTMC of the abstract shared memory system

SHMQRTP: Transient and steady-state probabilities for the reduced quotient DTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	
$\psi_1'^{\diamond}[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	
$\psi_2'^{\diamond}[k]$	0	0.1244	0.0931	0.0764	0.0679	0.0635	0.0612	0.0600	0.0594	0.0591	0.0590	0.
$\psi_3'^{\diamond}[k]$	0	0.1726	0.2614	0.3060	0.3289	0.3406	0.3466	0.3497	0.3513	0.3521	0.3525	0.
$\psi_4'^{\diamond}[k]$	0	0.1901	0.3824	0.4826	0.5341	0.5605	0.5740	0.5810	0.5845	0.5863	0.5872	0.



SHMQRTP: Transient probabilities alteration diagram for the reduced quotient DTMC of the abstract shared memory system

The steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi'^{\diamond} = \left(0, \frac{1}{17}, \frac{6}{17}, \frac{10}{17}\right).$$

Note that $\psi'^{\diamond} = (\psi'^{\diamond}(\mathcal{K}_1), \psi'^{\diamond}(\mathcal{K}_2), \psi'^{\diamond}(\mathcal{K}_4), \psi'^{\diamond}(\mathcal{K}_6))$.

By the "quotient" analogue of Proposition PMFSMCT:

$$\varphi'(\mathcal{K}_1) = 0,$$
$$\varphi'(\mathcal{K}_2) = \tfrac{1}{17},$$
$$\varphi'(\mathcal{K}_3) = 0,$$
$$\varphi'(\mathcal{K}_4) = \tfrac{6}{17},$$
$$\varphi'(\mathcal{K}_5) = 0,$$
$$\varphi'(\mathcal{K}_6) = \tfrac{10}{17}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\varphi' = \left(0, \frac{1}{17}, 0, \frac{6}{17}, 0, \frac{10}{17}\right).$$

This coincides with the result obtained with the use of ψ'^{*} and SJ'.

Performance indices

- The average recurrence time in the state $\mathcal{K}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\varphi_2'} = \frac{17}{1} = 17$.
- The common memory is available only in the states $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5$.

  The steady-state probability that the memory is available is
  $\varphi_2' + \varphi_3' + \varphi_5' = \frac{1}{17} + 0 + 0 = \frac{1}{17}$.

  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{1}{17} = \frac{16}{17}$.
- After activation of the system, we leave the state $\mathcal{K}_1$ for all, and the common memory is either requested or allocated in every remaining state, with exception of $\mathcal{K}_2$.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving $\mathcal{K}_2$, calculated as $\frac{\varphi_2'}{SJ_2'} = \frac{1}{17} \cdot \frac{3}{4} = \frac{3}{68}$.
- The parallel common memory request of two processors $\{\{r\}, \{r\}\}$ is only possible from the state $\mathcal{K}_2$.

  The request probability in this state is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$ twice.

  The *steady-state probability of the shared memory request from two processors* is
  $\varphi_2' \sum_{\{A,\mathcal{K} \mid \{\{r\},\{r\}\} \subseteq A,\ \mathcal{K}_2 \xrightarrow{A} \mathcal{K}\}} PM_A(\mathcal{K}_2, \mathcal{K}) = \frac{1}{17} \cdot \frac{1}{4} = \frac{1}{68}$.
- The common memory request of a processor $\{r\}$ is only possible from the states $\mathcal{K}_2, \mathcal{K}_4$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is $\varphi_2' \sum_{\{A,\mathcal{K} \mid \{r\} \in A,\ \mathcal{K}_2 \xrightarrow{A} \mathcal{K}\}} PM_A(\mathcal{K}_2, \mathcal{K}) +$
  $\varphi_4' \sum_{\{A,\mathcal{K} \mid \{r\} \in A,\ \mathcal{K}_4 \xrightarrow{A} \mathcal{K}\}} PM_A(\mathcal{K}_4, \mathcal{K}) =$
  $\frac{1}{17}\left(\frac{1}{2} + \frac{1}{4}\right) + \frac{6}{17}\left(\frac{3}{8} + \frac{1}{8}\right) = \frac{15}{68}$.

The performance indices are the same for the complete and quotient abstract shared memory systems.

The coincidence of the first and second performance indices illustrates Proposition STPROB.

The coincidence of the third performance index illustrates Proposition STPROB and Proposition SJAVVA.

The coincidence of the fourth performance index is by Theorem STTRAC:

one should apply its result to the derived step trace $\{\{r\},\{r\}\}$ of $\overline{F}$ and itself.

The coincidence of the fifth performance index is by Theorem STTRAC:

one should apply its result to the derived step traces $\{\{r\}\}$, $\{\{r\},\{r\}\}$, $\{\{r\},\{m\}\}$ of $\overline{F}$ and itself,

and sum the left and right parts of the three resulting equalities.

The generalized system

The static expression of the first processor is

$K_1 = [(\{x_1\}, \rho) * ((\{r_1\}, \rho); (\{d_1, y_1\}, l); (\{m_1, z_1\}, \rho)) * \mathsf{Stop}]$.

The static expression of the second processor is

$K_2 = [(\{x_2\}, \rho) * ((\{r_2\}, \rho); (\{d_2, y_2\}, l); (\{m_2, z_2\}, \rho)) * \mathsf{Stop}]$.

The static expression of the shared memory is

$K_3 = [(\{a, \widehat{x_1}, \widehat{x_2}\}, \rho) * (((\{\widehat{y_1}\}, l); (\{\widehat{z_1}\}, \rho))[](((\{\widehat{y_2}\}, l); (\{\widehat{z_2}\}, \rho)))* \mathsf{Stop}]$.

The static expression of the generalized shared memory system with two processors is

$K = (K_1 \| K_2 \| K_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

Interpretation of the states

$DR_T(\overline{K}) = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9\}$ and $DR_V(\overline{K}) = \{\tilde{s}_3, \tilde{s}_4, \tilde{s}_6\}$.

$\tilde{s}_1$: the initial state,

$\tilde{s}_2$: the system is activated and the memory is not requested,

$\tilde{s}_3$: the memory is requested by the first processor,

$\tilde{s}_4$: the memory is requested by the second processor,

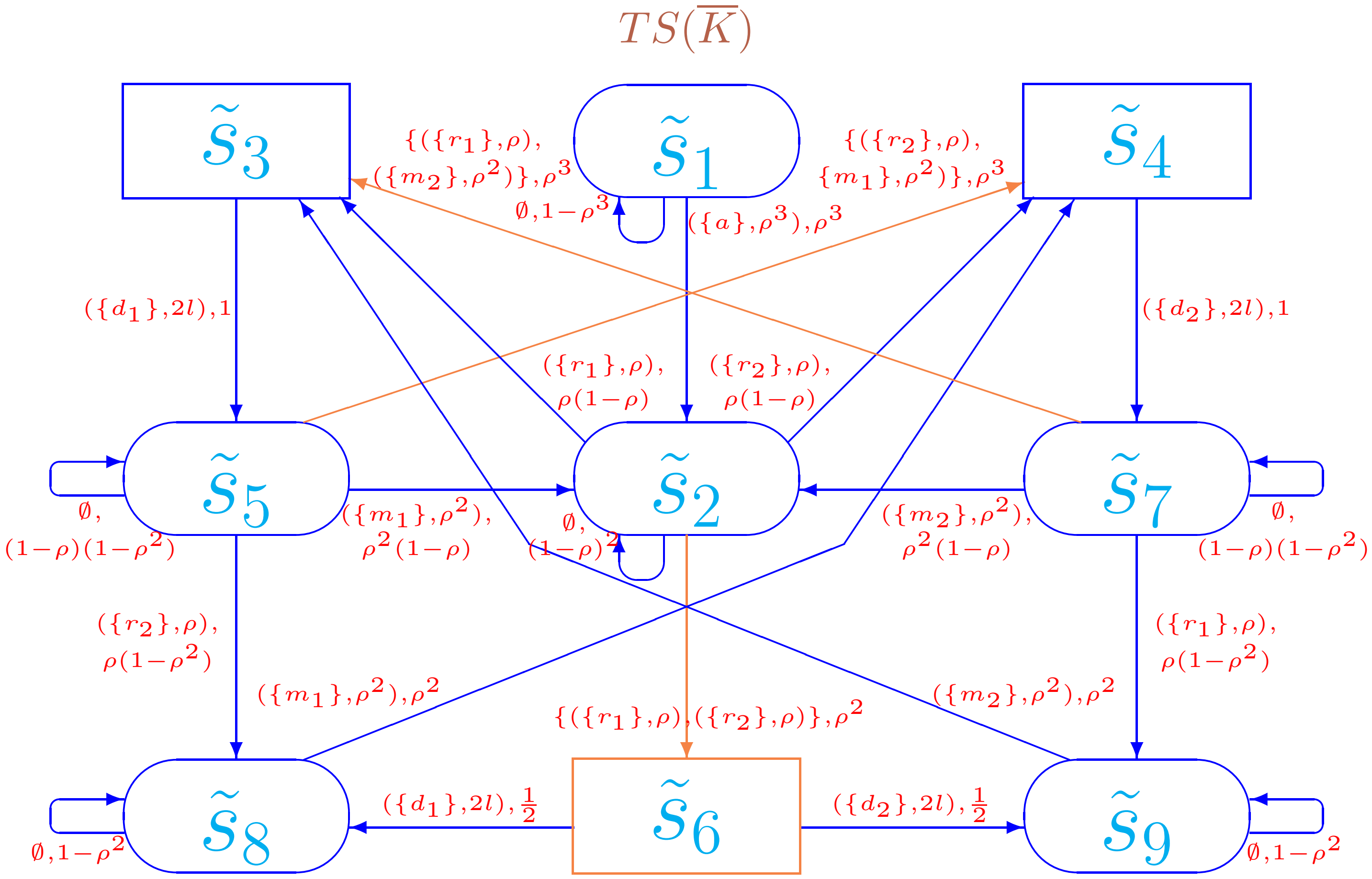
$\tilde{s}_5$: the memory is allocated to the first processor,

$\tilde{s}_6$: the memory is requested by two processors,

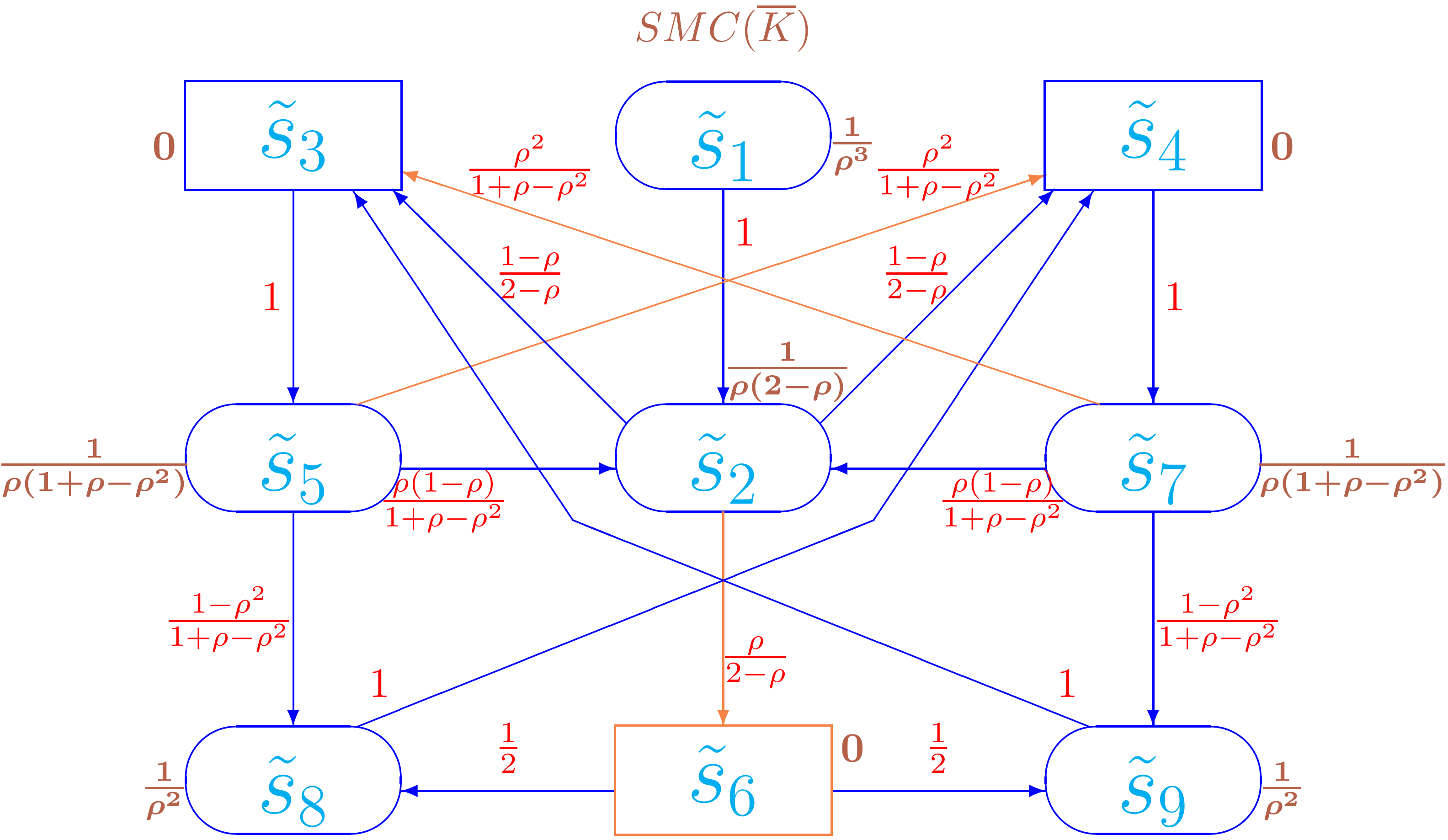
$\tilde{s}_7$: the memory is allocated to the second processor,

$\tilde{s}_8$: the memory is allocated to the first processor and the memory is requested by the second processor,

$\tilde{s}_9$: the memory is allocated to the second processor and the memory is requested by the first processor.



SHMGTS: The transition system of the generalized shared memory system (parallel executions of activities and the exclusively reachable states are marked with orange)



SHMGSMC: The underlying SMC of the generalized shared memory system (parallel executions of activities and the exclusively reachable states are marked with orange)

The average sojourn time vector of $\overline{K}$:

$$\widetilde{SJ} = \left(\frac{1}{\rho^3}, \frac{1}{\rho(2-\rho)}, 0, 0, \frac{1}{\rho(1+\rho-\rho^2)}, 0, \frac{1}{\rho(1+\rho-\rho^2)}, \frac{1}{\rho^2}, \frac{1}{\rho^2}\right).$$

The sojourn time variance vector of $\overline{K}$:

$$\widetilde{VAR} = \left(\frac{1-\rho^3}{\rho^6}, \frac{(1-\rho)^2}{\rho^2(2-\rho)^2}, 0, 0, \frac{(1-\rho)^2(1+\rho)}{\rho^2(1+\rho-\rho^2)^2}, 0, \frac{(1-\rho)^2(1+\rho)}{\rho^2(1+\rho-\rho^2)^2}, \frac{1-\rho^2}{\rho^4}, \frac{1-\rho^2}{\rho^4}\right).$$

The TPM for $EDTMC(\overline{K})$:

$$\widetilde{\mathbf{P}}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\rho}{2-\rho} & \frac{1-\rho}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & 0 & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & 0 & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC(\overline{K})$:

$$\tilde{\psi}^* = \frac{1}{2(6+3\rho-9\rho^2+2\rho^3)}(0, 2\rho(2-3\rho-\rho^2), 2+\rho-3\rho^2+\rho^3, 2+\rho-3\rho^2+\rho^3, 2+\rho-3\rho^2+\rho^3, 2\rho^2(1-\rho), 2+\rho-3\rho^2+\rho^3, 2-\rho-\rho^2, 2-\rho-\rho^2).$$

The steady-state PMF $\tilde{\psi}^*$ weighted by $\widetilde{SJ}$:

$$\frac{1}{2\rho^2(6+3\rho-9\rho^2+2\rho^3)}(0, 2\rho^2(1-\rho), 0, 0, \rho(2-\rho), 0, \rho(2-\rho), 2-\rho-\rho^2, 2-\rho-\rho^2).$$

We normalize the steady-state weighted PMF dividing it by the sum of its components

$$\tilde{\psi}^*\widetilde{SJ}^T = \frac{2+\rho-\rho^2-\rho^3}{\rho^2(6+3\rho-9\rho^2+2\rho^3)}.$$

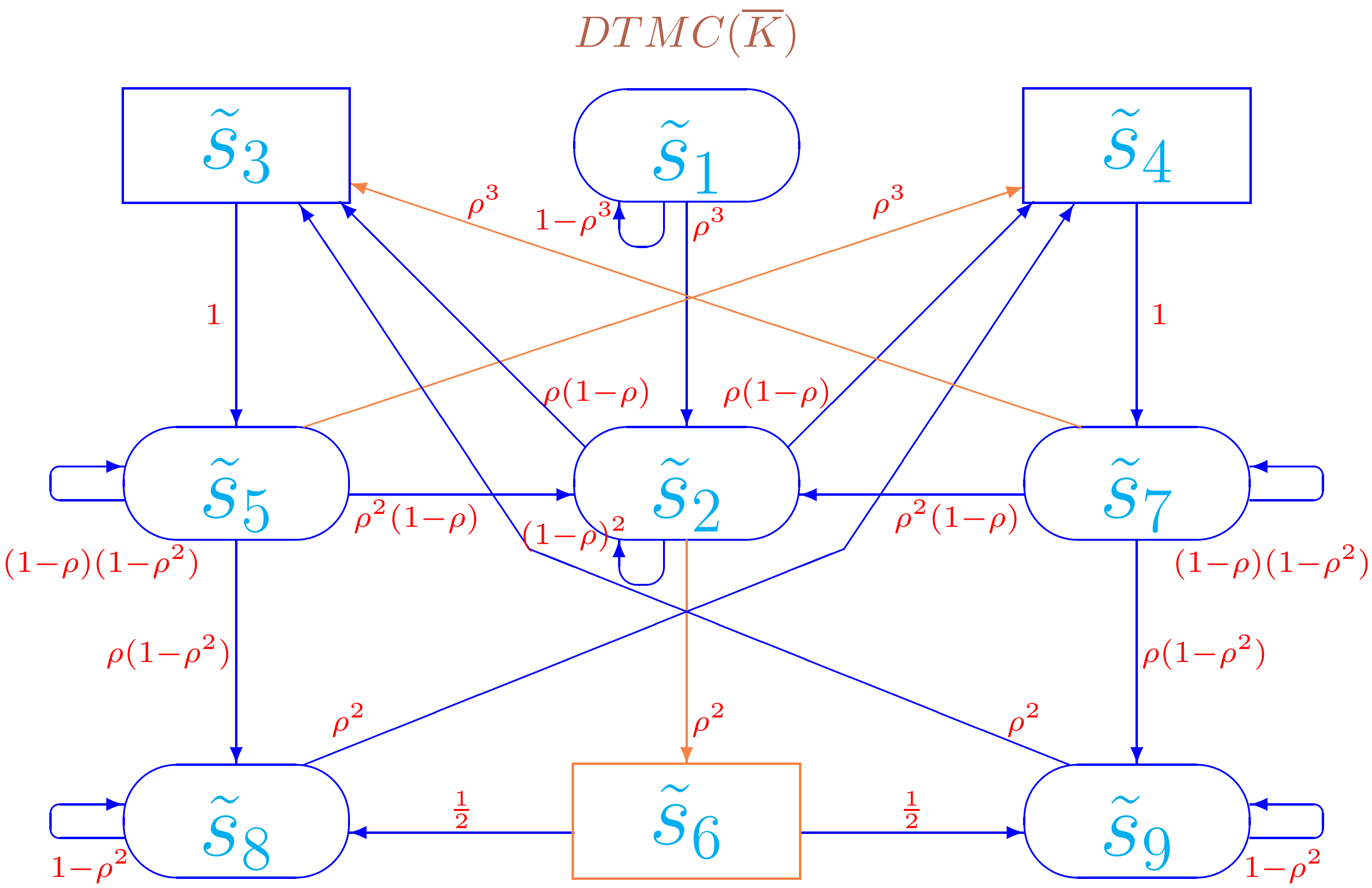
The steady-state PMF for $SMC(\overline{K})$:

$$\tilde{\varphi} = \frac{1}{2(2+\rho-\rho^2-\rho^3)}(0, 2\rho^2(1-\rho), 0, 0, \rho(2-\rho), 0, \rho(2-\rho), 2-\rho-\rho^2, 2-\rho-\rho^2).$$

Otherwise, from $TS(\overline{K})$, we can construct $DTMC(\overline{K})$ and calculate $\tilde{\varphi}$ using it.

The TPM for $DTMC(\overline{K})$: $\widetilde{\mathbf{P}} =$

$$
\begin{array}{ccccccccc}
1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1-\rho)^2 & \rho(1-\rho) & \rho(1-\rho) & 0 & \rho^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & \rho^2(1-\rho) & 0 & \rho^3 & (1-\rho)(1-\rho^2) & 0 & 0 & \rho(1-\rho^2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & \rho^2(1-\rho) & \rho^3 & 0 & 0 & 0 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\
0 & 0 & 0 & \rho^2 & 0 & 0 & 0 & 1-\rho^2 & 0 \\
0 & 0 & \rho^2 & 0 & 0 & 0 & 0 & 0 & 1-\rho^2
\end{array}
$$



SHMGDTMC: The DTMC of the generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The steady-state PMF for $DTMC(\overline{K})$:

$$\tilde{\psi} = \frac{1}{2(2+\rho+\rho^2-2\rho^4)}(0, 2\rho^2(1-\rho), \rho^2(2+\rho-3\rho^2+\rho^3),$$
$$\rho^2(2+\rho-3\rho^2+\rho^3), \rho(2-\rho), 2\rho^4(1-\rho), \rho(2-\rho),$$
$$2-\rho-\rho^2, 2-\rho-\rho^2).$$

Remember that $DR_T(\overline{K}) = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9\}$ and $DR_V(\overline{K}) = \{\tilde{s}_3, \tilde{s}_4, \tilde{s}_6\}$. Hence,

$$\sum_{\tilde{s}\in DR_T(\overline{K})} \tilde{\psi}(\tilde{s}) =$$
$$\tilde{\psi}(\tilde{s}_1) + \tilde{\psi}(\tilde{s}_2) + \tilde{\psi}(\tilde{s}_5) + \tilde{\psi}(\tilde{s}_7) + \tilde{\psi}(\tilde{s}_8) + \tilde{\psi}(\tilde{s}_9) = \frac{2+\rho-\rho^2-\rho^3}{2+\rho+\rho^2-2\rho^4}.$$

By Proposition PMFSMC:

$$\tilde{\varphi}(\tilde{s}_1) = 0 \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = 0,$$
$$\tilde{\varphi}(\tilde{s}_2) = \frac{\rho^2(1-\rho)}{2+\rho+\rho^2-2\rho^4} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3},$$
$$\tilde{\varphi}(\tilde{s}_3) = 0,$$
$$\tilde{\varphi}(\tilde{s}_4) = 0,$$
$$\tilde{\varphi}(\tilde{s}_5) = \frac{\rho(2-\rho)}{2(2+\rho+\rho^2-2\rho^4)} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)},$$
$$\tilde{\varphi}(\tilde{s}_6) = 0,$$
$$\tilde{\varphi}(\tilde{s}_7) = \frac{\rho(2-\rho)}{2(2+\rho+\rho^2-2\rho^4)} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)},$$
$$\tilde{\varphi}(\tilde{s}_8) = \frac{2-\rho-\rho^2}{2(2+\rho+\rho^2-2\rho^4)} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{2-\rho-\rho^2}{2(2+\rho-\rho^2-\rho^3)},$$
$$\tilde{\varphi}(\tilde{s}_9) = \frac{2-\rho-\rho^2}{2(2+\rho+\rho^2-2\rho^4)} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{2-\rho-\rho^2}{2(2+\rho-\rho^2-\rho^3)}.$$

The steady-state PMF for $SMC(\overline{K})$:

$$\tilde{\varphi} = \frac{1}{2(2+\rho-\rho^2-\rho^3)}(0, 2\rho^2(1-\rho), 0, 0, \rho(2-\rho), 0, \rho(2-\rho), 2-\rho-\rho^2, 2-\rho-\rho^2).$$

This coincides with the result obtained with the use of $\tilde{\psi}^*$ and $\widetilde{SJ}$.

Alternatively, from $TS(\overline{K})$, we can construct the reduced DTMC of $\overline{K}$, $RDTMC(\overline{K})$, and calculate $\tilde{\varphi}$ using it.

$DR_T(\overline{K}) = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_7, \tilde{s}_8, \tilde{s}_9\}$ and $DR_V(\overline{K}) = \{\tilde{s}_3, \tilde{s}_4, \tilde{s}_6\}$.

We reorder the elements of $DR(\overline{K})$ by moving the equivalence classes of vanishing states to the first positions:

$\tilde{s}_3, \tilde{s}_4, \tilde{s}_6, \tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_7, \tilde{s}_8, \tilde{s}_9$.

The reordered TPM for $DTMC(\overline{K})$ $\widetilde{\mathbf{P}}_r =$

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\
\rho(1-\rho) & \rho(1-\rho) & \rho^2 & 0 & (1-\rho)^2 & 0 & 0 & 0 & 0 \\
0 & \rho^3 & 0 & 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) & 0 \\
\rho^3 & 0 & 0 & 0 & \rho^2(1-\rho) & 0 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\
0 & \rho^2 & 0 & 0 & 0 & 0 & 0 & 1-\rho^2 & 0 \\
\rho^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\rho^2
\end{matrix}$$

The result of the decomposing $\widetilde{\mathbf{P}}_r$:

$$\widetilde{\mathbf{C}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \widetilde{\mathbf{D}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \widetilde{\mathbf{E}} = \begin{pmatrix} 0 & 0 & 0 \\ \rho(1-\rho) & \rho(1-\rho) & \rho^2 \\ 0 & \rho^3 & 0 \\ \rho^3 & 0 & 0 \\ 0 & \rho^2 & 0 \\ \rho^2 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathbf{F}} = \begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\ 0 & (1-\rho)^2 & 0 & 0 & 0 & 0 \\ 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) & 0 \\ 0 & \rho^2(1-\rho) & 0 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & 0 & 0 & 0 & 1-\rho^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

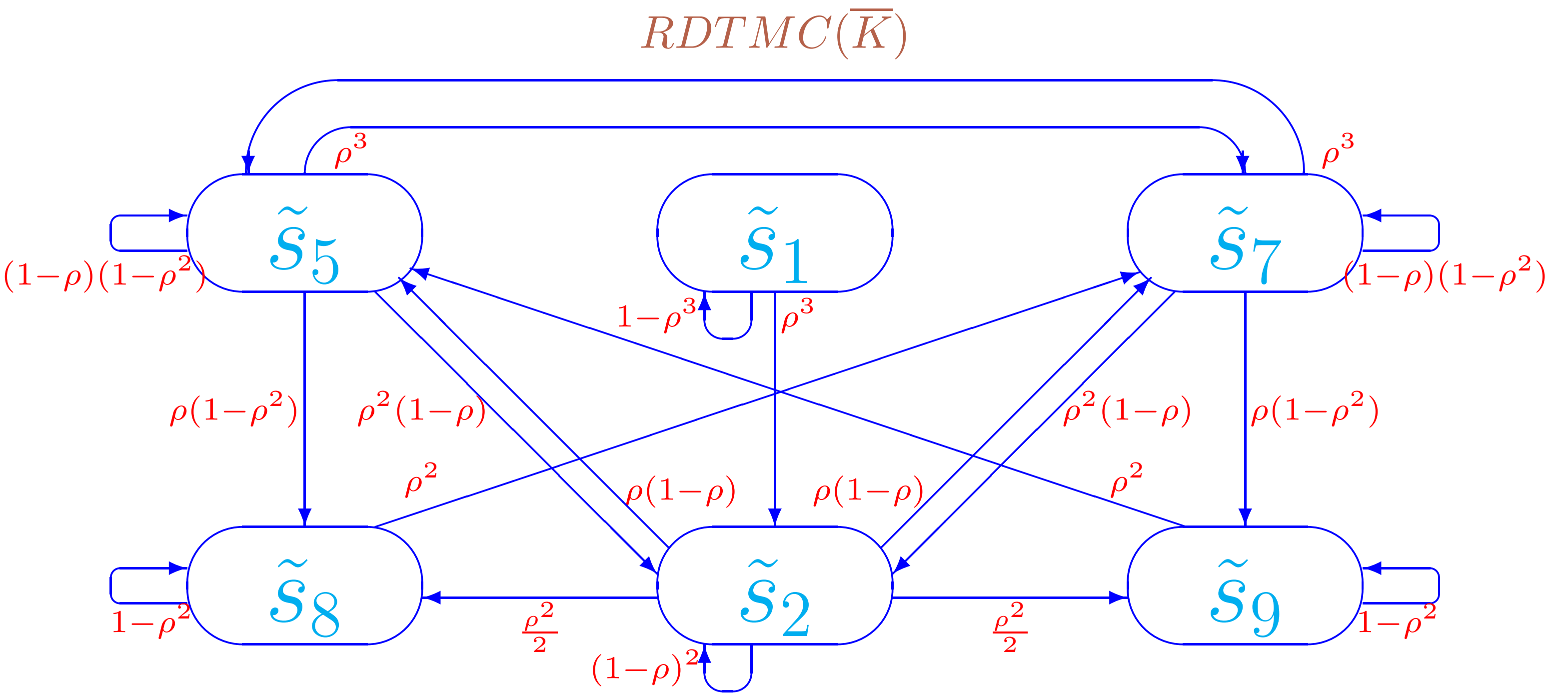
Since $\widetilde{\mathbf{C}}^1 = \mathbf{0}$, we have $\forall k > 0,\ \widetilde{\mathbf{C}}^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\widetilde{\mathbf{G}} = \sum_{k=0}^{l} \widetilde{\mathbf{C}}^k = \widetilde{\mathbf{C}}^0 = \mathbf{I}.$$

The TPM for $RDTMC(\overline{K})$:

$$\widetilde{\mathbf{P}}^{\diamond} = \widetilde{\mathbf{F}} + \widetilde{\mathbf{E}}\widetilde{\mathbf{G}}\widetilde{\mathbf{D}} = \widetilde{\mathbf{F}} + \widetilde{\mathbf{E}}\mathbf{I}\widetilde{\mathbf{D}} = \widetilde{\mathbf{F}} + \widetilde{\mathbf{E}}\widetilde{\mathbf{D}} =$$

$$\begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\ 0 & (1-\rho)^2 & \rho(1-\rho) & \rho(1-\rho) & \frac{\rho^2}{2} & \frac{\rho^2}{2} \\ 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & \rho^3 & \rho(1-\rho^2) & 0 \\ 0 & \rho^2(1-\rho) & \rho^3 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & 0 & 0 & \rho^2 & 1-\rho^2 & 0 \\ 0 & 0 & \rho^2 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$



SHMGRDTMC: The reduced DTMC of the generalized shared memory system

The steady-state PMF for $RDTMC(\overline{K})$:

$$\tilde{\psi}^\diamond = \frac{1}{2(2+\rho-\rho^2-\rho^3)}(0, 2\rho^2(1-\rho), \rho(2-\rho), \rho(2-\rho),\\ 2-\rho-\rho^2, 2-\rho-\rho^2).$$

Note that $\tilde{\psi}^\diamond = (\tilde{\psi}^\diamond(\tilde{s}_1), \tilde{\psi}^\diamond(\tilde{s}_2), \tilde{\psi}^\diamond(\tilde{s}_5), \tilde{\psi}^\diamond(\tilde{s}_7), \tilde{\psi}^\diamond(\tilde{s}_8), \tilde{\psi}^\diamond(\tilde{s}_9))$.

By Proposition PMFSMCT:

$$\tilde{\varphi}(\tilde{s}_1) = 0, \quad \tilde{\varphi}(\tilde{s}_2) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}, \quad \tilde{\varphi}(\tilde{s}_5) = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)},$$

$$\tilde{\varphi}(\tilde{s}_7) = \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}, \quad \tilde{\varphi}(\tilde{s}_8) = \frac{2-\rho-\rho^2}{2(2+\rho-\rho^2-\rho^3)}, \quad \tilde{\varphi}(\tilde{s}_9) = \frac{2-\rho-\rho^2}{2(2+\rho-\rho^2-\rho^3)}.$$

The steady-state PMF for $SMC(\overline{K})$:

$$\tilde{\varphi} = \frac{1}{2(2+\rho-\rho^2-\rho^3)}(0, 2\rho^2(1-\rho), 0, 0, \rho(2-\rho), 0, \rho(2-\rho),\\ 2-\rho-\rho^2, 2-\rho-\rho^2).$$

This coincides with the result obtained with the use of $\tilde{\psi}^*$ and $\widetilde{SJ}$.

Performance indices

- The average recurrence time in the state $\tilde{s}_2$, where no processor requests the memory, the *average system run-through*, is $\frac{1}{\tilde{\varphi}_2} = \frac{2+\rho-\rho^2-\rho^3}{\rho^2(1-\rho)}$.
- The common memory is available only in the states $\tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_6$.

  The steady-state probability that the memory is available is
  $\tilde{\varphi}_2 + \tilde{\varphi}_3 + \tilde{\varphi}_4 + \tilde{\varphi}_6 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} + 0 + 0 + 0 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}$.
  The steady-state probability that the memory is used (i.e. not available), the *shared memory utilization*, is $1 - \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} = \frac{2+\rho-2\rho^2}{2+\rho-\rho^2-\rho^3}$.
- After activation of the system, we leave the state $\tilde{s}_1$ for all, and the common memory is either requested or allocated in every remaining state, with exception of $\tilde{s}_2$.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving $\tilde{s}_2$, calculated as
  $\frac{\tilde{\varphi}_2}{\widetilde{SJ}_2} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} \cdot \frac{\rho(2-\rho)}{1} = \frac{\rho^3(1-\rho)(2-\rho)}{2+\rho-\rho^2-\rho^3}$.
- The parallel common memory request of two processors $\{(\{r_1\},\rho),(\{r_2\},\rho)\}$ is only possible from the state $\tilde{s}_2$.

  The request probability in this state is the sum of the execution probabilities for all multisets of activities containing both $(\{r_1\},\rho)$ and $(\{r_2\},\rho)$.

  The *steady-state probability of the shared memory request from two processors* is $\tilde{\varphi}_2 \sum_{\{\Upsilon \mid \{(\{r_1\},\rho),(\{r_2\},\rho)\} \subseteq \Upsilon\}} PT(\Upsilon, \tilde{s}_2) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}\rho^2 = \frac{\rho^4(1-\rho)}{2+\rho-\rho^2-\rho^3}$.
- The common memory request of the first processor $(\{r_1\},\rho)$ is only possible from the states $\tilde{s}_2, \tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r_1\},\rho)$.

  The *steady-state probability of the shared memory request from the first processor* is $\tilde{\varphi}_2 \sum_{\{\Gamma \mid (\{r_1\},\rho) \in \Gamma\}} PT(\Gamma, \tilde{s}_2) +$
  $\tilde{\varphi}_7 \sum_{\{\Gamma \mid (\{r_1\},\rho) \in \Gamma\}} PT(\Gamma, \tilde{s}_7) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}(\rho(1-\rho)+\rho^2) +$
  $\frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}(\rho(1-\rho^2)+\rho^3) = \frac{\rho^2(2+\rho-2\rho^2)}{2(2+\rho-\rho^2-\rho^3)}$.

The abstract generalized system and its reduction

The static expression of the first processor is

$L_1 = [(\{x_1\},\rho) * ((\{r\},\rho);(\{d,y_1\},l);(\{m,z_1\},\rho)) * \mathsf{Stop}]$.

The static expression of the second processor is

$L_2 = [(\{x_2\},\rho) * ((\{r\},\rho);(\{d,y_2\},l);(\{m,z_2\},\rho)) * \mathsf{Stop}]$.

The static expression of the shared memory is

$L_3 =$
$[(\{a,\widehat{x_1},\widehat{x_2}\},\rho) * (((\{\widehat{y_1}\},l);(\{\widehat{z_1}\},\rho))[]((\{\widehat{y_2}\},l);(\{\widehat{z_2}\},\rho))) * \mathsf{Stop}]$.

The static expression of the abstract generalized shared memory system with two processors is

$L = (L_1 \| L_2 \| L_3)$ sy x_1 sy x_2 sy y_1 sy y_2 sy z_1 sy z_2 rs x_1 rs x_2 rs y_1 rs y_2 rs z_1 rs z_2.

$DR(\overline{L})$ resembles $DR(\overline{K})$, and $TS(\overline{L})$ is similar to $TS(\overline{K})$.

$SMC(\overline{L}) \simeq SMC(\overline{K})$, thus, the average sojourn time vectors of $\overline{L}$ and $\overline{K}$, the TPMs and the steady-state PMFs for $EDTMC(\overline{L})$ and $EDTMC(\overline{K})$ coincide.

Performance indices

The first, second and third performance indices are the same for the generalized system and its abstract modification.

The following performance index: non-identified viewpoint to the processors.

- The common memory request of a processor $(\{r\}, \rho)$ is only possible from the states $\tilde{s}_2, \tilde{s}_5, \tilde{s}_7$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of activities containing $(\{r\}, \rho)$.

  The *steady-state probability of the shared memory request from a processor* is $\tilde{\varphi}_2 \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT(\Gamma, \tilde{s}_2) + \tilde{\varphi}_5 \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT(\Gamma, \tilde{s}_5) + \tilde{\varphi}_7 \sum_{\{\Gamma|(\{r\},\rho)\in\Gamma\}} PT(\Gamma, \tilde{s}_7) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}(\rho(1-\rho) + \rho(1-\rho) + \rho^2) + \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}(\rho(1-\rho^2) + \rho^3) + \frac{\rho(2-\rho)}{2(2+\rho-\rho^2-\rho^3)}(\rho(1-\rho^2) + \rho^3) = \frac{\rho^2(2-\rho)(1+\rho-\rho^2)}{2+\rho-\rho^2-\rho^3}$.

The quotient of the abstract generalized system

$DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_5, \widetilde{\mathcal{K}}_6\}$, where

$\widetilde{\mathcal{K}}_1 = \{\tilde{s}_1\}$ (the initial state),

$\widetilde{\mathcal{K}}_2 = \{\tilde{s}_2\}$ (the system is activated and the memory is not requested),

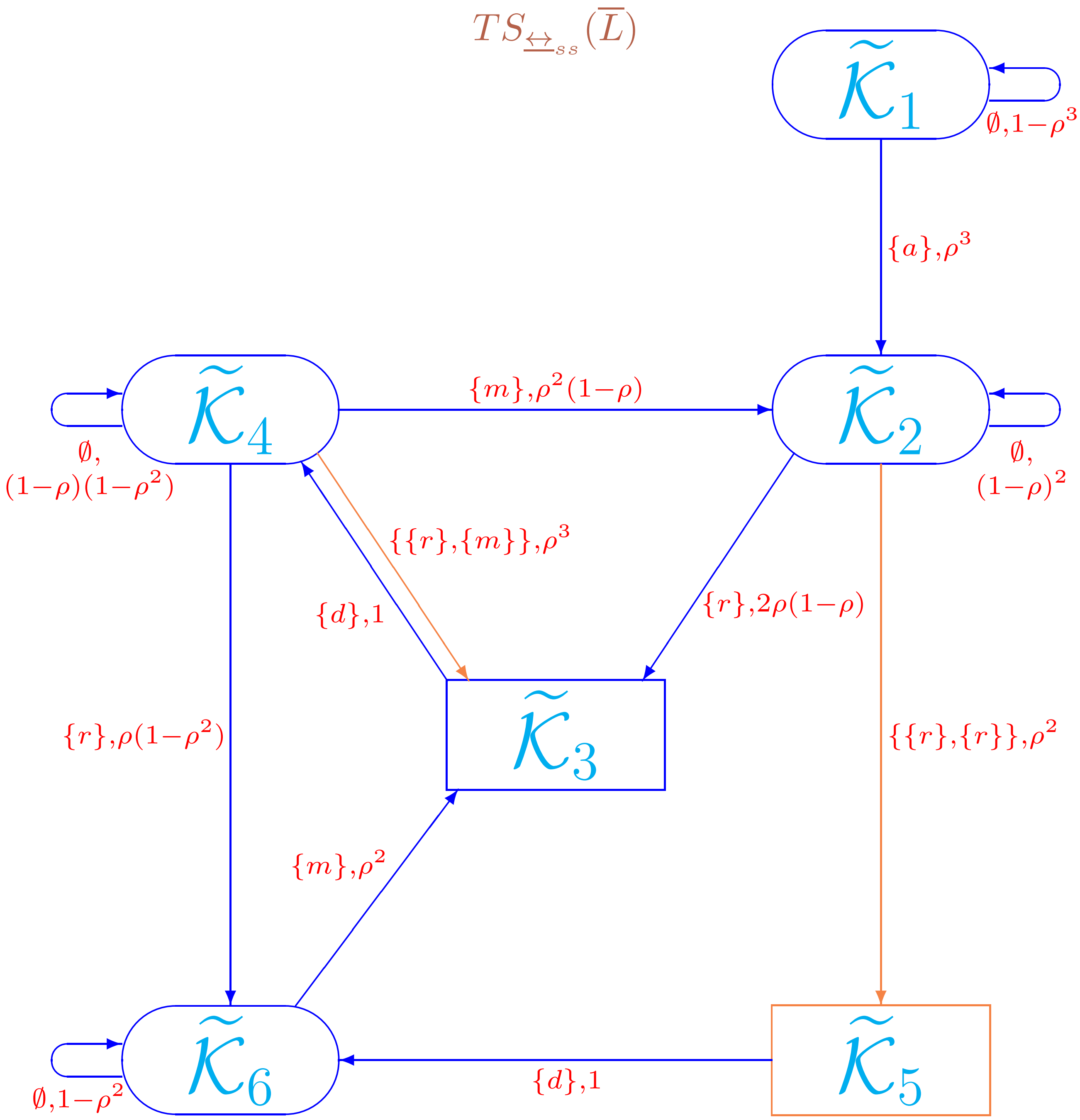
$\widetilde{\mathcal{K}}_3 = \{\tilde{s}_3, \tilde{s}_4\}$ (the memory is requested by one processor),

$\widetilde{\mathcal{K}}_4 = \{\tilde{s}_5, \tilde{s}_7\}$ (the memory is allocated to a processor),

$\widetilde{\mathcal{K}}_5 = \{\tilde{s}_6\}$ (the memory is requested by two processors),

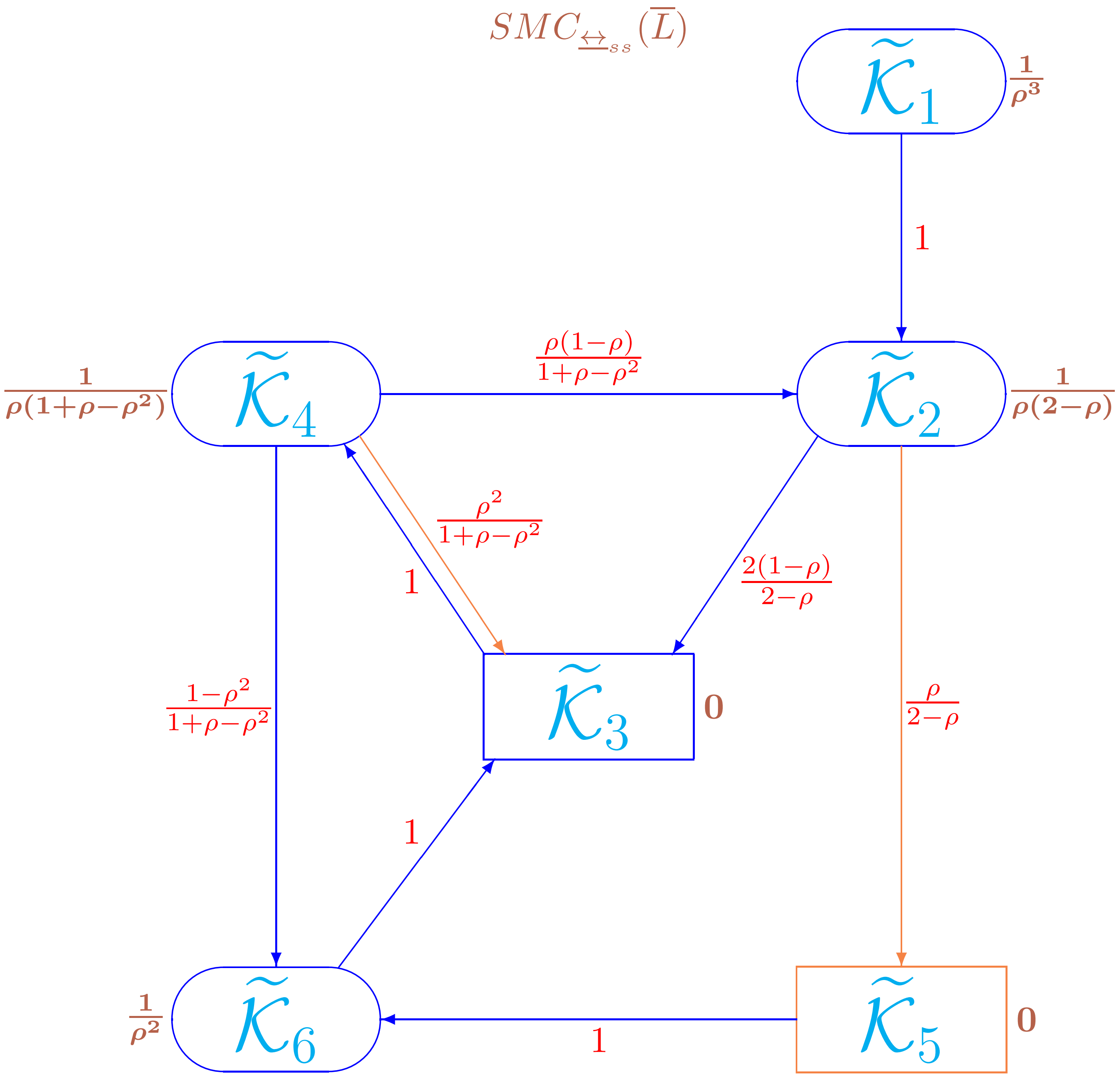
$\widetilde{\mathcal{K}}_6 = \{\tilde{s}_8, \tilde{s}_9\}$ (the memory is allocated to a processor and the memory is requested by another processor).

$DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6\}$ and $DR_V(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5\}$.



SHMGQTS: The quotient transition system of the abstract generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)



SHMGQSMC: The quotient underlying SMC of the abstract generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The quotient average sojourn time vector of $\overline{F}$:

$$\widetilde{SJ}' = \left(\frac{1}{\rho^3}, \frac{1}{\rho(2-\rho)}, 0, \frac{1}{\rho(1+\rho-\rho^2)}, 0, \frac{1}{\rho^2}\right).$$

The quotient sojourn time variance vector of $\overline{F}$:

$$\widetilde{VAR}' = \left(\frac{1-\rho^3}{\rho^6}, \frac{(1-\rho)^2}{\rho^2(2-\rho)^2}, 0, \frac{(1-\rho)^2(1+\rho)}{\rho^2(1+\rho-\rho^2)^2}, 0, \frac{1-\rho^2}{\rho^4}\right).$$

The TPM for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}'^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2(1-\rho)}{2-\rho} & 0 & \frac{\rho}{2-\rho} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\rho(1-\rho)}{1+\rho-\rho^2} & \frac{\rho^2}{1+\rho-\rho^2} & 0 & 0 & \frac{1-\rho^2}{1+\rho-\rho^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The steady-state PMF for $EDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\psi}'^* = \frac{1}{6+3\rho-9\rho^2+2\rho^3}(0, \rho(2-3\rho+\rho^2), 2+\rho-3\rho^2+\rho^3,$$
$$2+\rho-3\rho^2+\rho^3, \rho^2(1-\rho), 2-\rho-\rho^2).$$

The steady-state PMF $\tilde{\psi}'^*$ weighted by $\widetilde{SJ}'$:

$$\frac{1}{\rho^2(6+3\rho-9\rho^2+2\rho^3)}(0, \rho^2(1-\rho), 0, \rho(2-\rho), 0, 2-\rho-\rho^2).$$

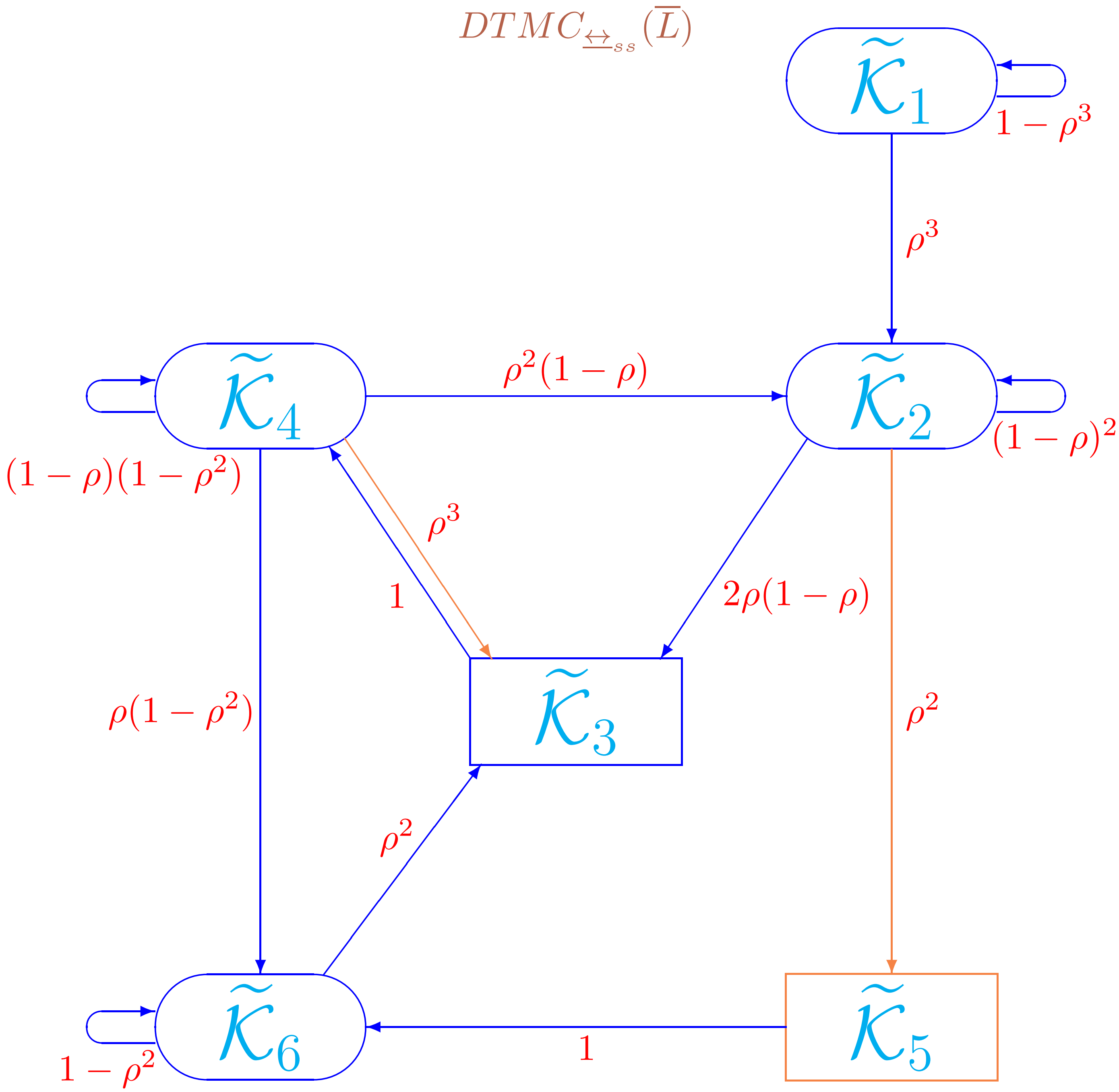
We normalize the steady-state weighted PMF dividing it by the sum of its components

$$\tilde{\psi}'^* \widetilde{SJ}'^T = \frac{2+\rho-\rho^2-\rho^3}{\rho^2(6+3\rho-9\rho^2+2\rho^3)}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\varphi}' = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), 0, \rho(2-\rho), 0, 2-\rho-\rho^2).$$

Otherwise, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{L})$, we can construct the quotient DTMC of $\overline{L}$, $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$, and calculate $\tilde{\varphi}'$ using it.



SHMGQDTMC: The quotient DTMC of the abstract generalized shared memory system

(parallel executions of activities and the exclusively reachable states are marked with orange)

The TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}' = \begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 & 0 & 0 \\ 0 & (1-\rho)^2 & 2\rho(1-\rho) & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \rho^2(1-\rho) & \rho^3 & (1-\rho)(1-\rho^2) & 0 & \rho(1-\rho^2) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \rho^2 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

The steady-state PMF for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\psi}' = \frac{1}{2+\rho+\rho^2-2\rho^4}(0, \rho^2(1-\rho), \rho^2(2+\rho-3\rho^2+\rho^3), \rho(2-\rho), \rho^4(1-\rho), 2-\rho-\rho^2).$$

$DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6\}$ and $DR_V(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5\}$. Hence,

$$\sum_{\widetilde{\mathcal{K}} \in DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})}} \tilde{\psi}'(\widetilde{\mathcal{K}}) = \tilde{\psi}'(\widetilde{\mathcal{K}}_1) + \tilde{\psi}'(\widetilde{\mathcal{K}}_2) + \tilde{\psi}'(\widetilde{\mathcal{K}}_4) + \tilde{\psi}'(\widetilde{\mathcal{K}}_6) = \frac{2+\rho-\rho^2-\rho^3}{2+\rho+\rho^2-2\rho^4}.$$

By the “quotient” analogue of Proposition PMFSMC:

$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_1) = 0 \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = 0,$$
$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_2) = \frac{\rho^2(1-\rho)}{2+\rho+\rho^2-2\rho^4} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3},$$
$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_3) = 0,$$
$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_4) = \frac{\rho(2-\rho)}{2+\rho+\rho^2-2\rho^4} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3},$$
$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_5) = 0,$$
$$\tilde{\varphi}'(\widetilde{\mathcal{K}}_6) = \frac{2-\rho-\rho^2}{2+\rho+\rho^2-2\rho^4} \cdot \frac{2+\rho+\rho^2-2\rho^4}{2+\rho-\rho^2-\rho^3} = \frac{2-\rho-\rho^2}{2+\rho-\rho^2-\rho^3}.$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\varphi}' = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), 0, \rho(2-\rho), 0, 2-\rho-\rho^2).$$

This coincides with the result obtained with the use of $\tilde{\psi}'^*$ and $\widetilde{SJ}'$.

Alternatively, from $TS_{\underline{\leftrightarrow}_{ss}}(\overline{L})$, we can construct $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$ and calculate $\tilde{\varphi}'$ using it.

$DR_T(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6\}$ and
$DR_V(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})} = \{\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5\}$.

We reorder the elements of $DR(\overline{L})/_{\mathcal{R}_{ss}(\overline{L})}$ by moving the equivalence classes of vanishing states to the first positions: $\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5, \widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_4, \widetilde{\mathcal{K}}_6$.

The reordered TPM for $DTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}'_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-\rho^3 & \rho^3 & 0 & 0 \\ 2\rho(1-\rho) & \rho^2 & 0 & (1-\rho)^2 & 0 & 0 \\ \rho^3 & 0 & 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & \rho(1-\rho^2) \\ \rho^2 & 0 & 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

The result of the decomposing $\widetilde{\mathbf{P}}'_r$:

$$\widetilde{\mathbf{C}}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \widetilde{\mathbf{D}}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \widetilde{\mathbf{E}}' = \begin{pmatrix} 0 & 0 \\ 2\rho(1-\rho) & \rho^2 \\ \rho^3 & 0 \\ \rho^2 & 0 \end{pmatrix},$$

$$\widetilde{\mathbf{F}}' = \begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 \\ 0 & (1-\rho)^2 & 0 & 0 \\ 0 & \rho^2(1-\rho) & (1-\rho)(1-\rho^2) & \rho(1-\rho^2) \\ 0 & 0 & 0 & 1-\rho^2 \end{pmatrix}.$$

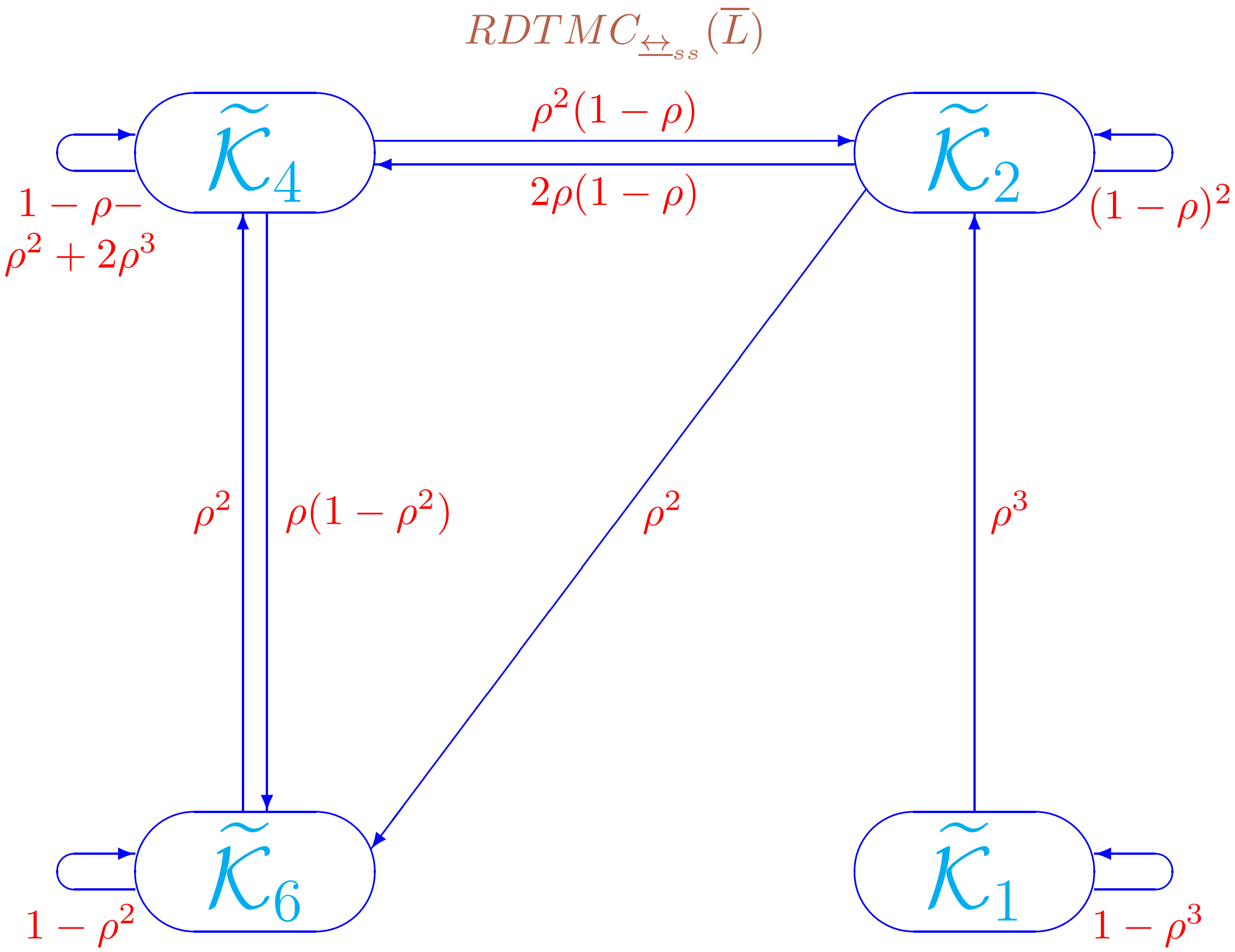
Since $\widetilde{\mathbf{C}}'^1 = \mathbf{0}$, we have $\forall k > 0,\ \widetilde{\mathbf{C}}'^k = \mathbf{0}$, hence, $l = 0$ and there are no loops among vanishing states. Then

$$\widetilde{\mathbf{G}}' = \sum_{k=0}^{l} \widetilde{\mathbf{C}}'^l = \widetilde{\mathbf{C}}'^0 = \mathbf{I}.$$

The TPM for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\widetilde{\mathbf{P}}'^{\diamond} = \widetilde{\mathbf{F}}' + \widetilde{\mathbf{E}}'\widetilde{\mathbf{G}}'\widetilde{\mathbf{D}}' = \widetilde{\mathbf{F}}' + \widetilde{\mathbf{E}}'\mathbf{I}\widetilde{\mathbf{D}}' = \widetilde{\mathbf{F}}' + \widetilde{\mathbf{E}}'\widetilde{\mathbf{D}}' =$$

$$\begin{pmatrix} 1-\rho^3 & \rho^3 & 0 & 0 \\ 0 & (1-\rho)^2 & 2\rho(1-\rho) & \rho^2 \\ 0 & \rho^2(1-\rho) & 1-\rho-\rho^2+2\rho^3 & \rho(1-\rho^2) \\ 0 & 0 & \rho^2 & 1-\rho^2 \end{pmatrix}.$$



SHMGQRDTMC: The reduced quotient DTMC of the abstract generalized shared memory system

The steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\psi}'^{\diamond} = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), \rho(2-\rho), 2-\rho-\rho^2).$$

Note that $\tilde{\psi}'^{\diamond} = (\tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_1), \tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_2), \tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_4), \tilde{\psi}'^{\diamond}(\widetilde{\mathcal{K}}_6))$.

By the “quotient” analogue of Proposition PMFSMCT:

$$\begin{aligned}
\tilde{\varphi}'(\widetilde{\mathcal{K}}_1) &= 0, \\
\tilde{\varphi}'(\widetilde{\mathcal{K}}_2) &= \tfrac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}, \\
\tilde{\varphi}'(\widetilde{\mathcal{K}}_3) &= 0, \\
\tilde{\varphi}'(\widetilde{\mathcal{K}}_4) &= \tfrac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3}, \\
\tilde{\varphi}'(\widetilde{\mathcal{K}}_5) &= 0, \\
\tilde{\varphi}'(\widetilde{\mathcal{K}}_6) &= \tfrac{2-\rho-\rho^2}{2+\rho-\rho^2-\rho^3}.
\end{aligned}$$

The steady-state PMF for $SMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\varphi}' = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), 0, \rho(2-\rho), 0, 2-\rho-\rho^2).$$

This coincides with the result obtained with the use of $\tilde{\psi}'^{*}$ and $\widetilde{SJ}'$.

Performance indices

- The average recurrence time in the state $\widetilde{\mathcal{K}}_2$, where no processor requests the memory,
  the *average system run-through*, is $\frac{1}{\tilde{\varphi}'_2} = \frac{2+\rho-\rho^2-\rho^3}{\rho^2(1-\rho)}$.

- The common memory is available only in the states $\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_5$.

  The steady-state probability that the memory is available is
  $\tilde{\varphi}'_2 + \tilde{\varphi}'_3 + \tilde{\varphi}'_5 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} + 0 + 0 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}$.
  The steady-state probability that the memory is used (i.e. not available),
  the *shared memory utilization*, is $1 - \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} = \frac{2+\rho-2\rho^2}{2+\rho-\rho^2-\rho^3}$.

- After activation of the system, we leave the state $\widetilde{\mathcal{K}}_1$ for all, and the common memory is either requested or allocated in every remaining state, with exception of $\widetilde{\mathcal{K}}_2$.

  The *rate with which the necessity of shared memory emerges* coincides with the rate of leaving $\widetilde{\mathcal{K}}_2$, calculated as
  $\frac{\tilde{\varphi}'_2}{\widetilde{SJ}'_2} = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3} \cdot \frac{\rho(2-\rho)}{1} = \frac{\rho^3(1-\rho)(2-\rho)}{2+\rho-\rho^2-\rho^3}$.

- The parallel common memory request of two processors $\{\{r\},\{r\}\}$ is only possible from the state $\widetilde{\mathcal{K}}_2$.

  The request probability in this state is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$ twice.

  The *steady-state probability of the shared memory request from two processors* is $\tilde{\varphi}'_2 \sum_{\{A,\widetilde{\mathcal{K}}\mid\{\{r\},\{r\}\}\subseteq A,\ \widetilde{\mathcal{K}}_2\overset{A}{\to}\widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{K}}) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}\rho^2 = \frac{\rho^4(1-\rho)}{2+\rho-\rho^2-\rho^3}$.

- The common memory request of a processor $\{r\}$ is only possible from the states $\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{K}}_4$.

  The request probability in each of the states is the sum of the execution probabilities for all multisets of multiactions containing $\{r\}$.

  The *steady-state probability of the shared memory request from a processor* is $\tilde{\varphi}'_2 \sum_{\{A,\widetilde{\mathcal{K}}\mid\{r\}\in A,\ \widetilde{\mathcal{K}}_2\overset{A}{\to}\widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{K}}) +$
  $\tilde{\varphi}'_4 \sum_{\{A,\widetilde{\mathcal{K}}\mid\{r\}\in A,\ \widetilde{\mathcal{K}}_4\overset{A}{\to}\widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_4,\widetilde{\mathcal{K}}) = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}(2\rho(1-\rho)+\rho^2) + \frac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3}(\rho(1-\rho^2)+\rho^3) = \frac{\rho^2(2-\rho)(1+\rho-\rho^2)}{2+\rho-\rho^2-\rho^3}$.

The performance indices are the same for the complete and quotient abstract generalized shared memory systems.

The coincidence of the first and second performance indices illustrates Proposition STPROB.

The coincidence of the third performance index illustrates Proposition STPROB and Proposition SJAVVA.

The coincidence of the fourth performance index is by Theorem STTRAC:

one should apply its result to the derived step trace $\{\{r\},\{r\}\}$ of $\overline{L}$ and itself.

The coincidence of the fifth performance index is by Theorem STTRAC:

one should apply its result to the derived step traces $\{\{r\}\},\ \{\{r\},\{r\}\},\ \{\{r\},\{m\}\}$ of $\overline{L}$ and itself,

and sum the left and right parts of the three resulting equalities.

Effect of quantitative changes of ρ to performance of the quotient abstract generalized shared memory system in its steady state

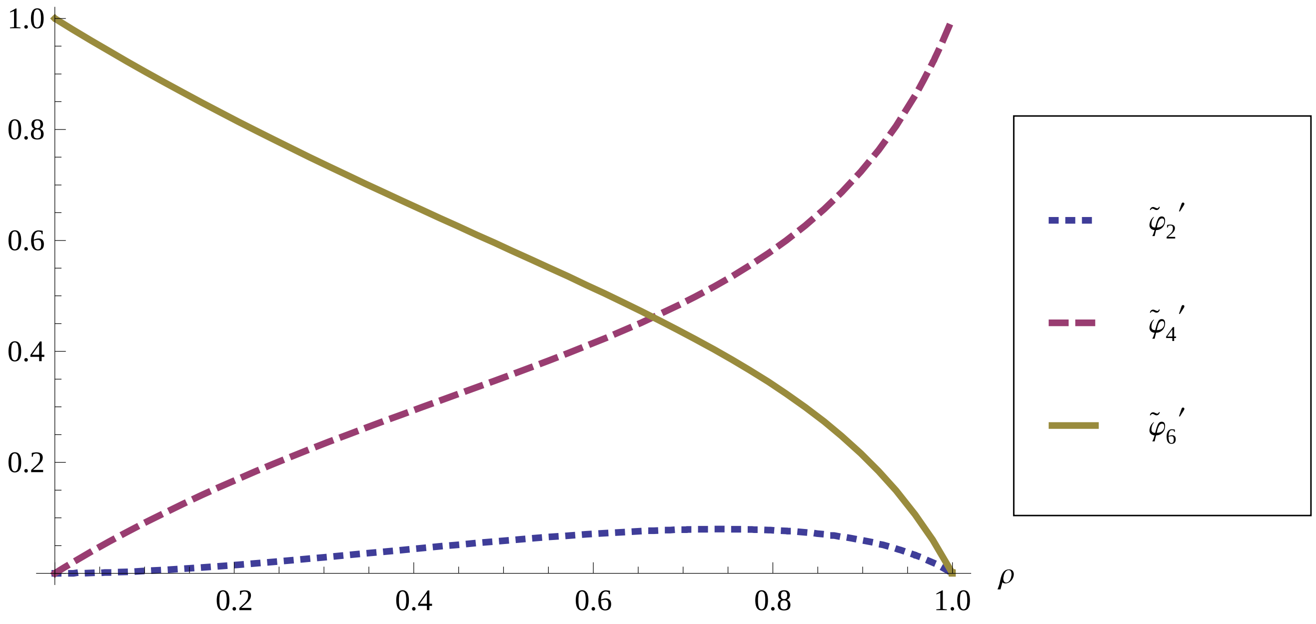
$\rho \in (0; 1)$ is the probability of every multiaction of the system.

The closer is ρ to 0, the less is the probability to execute some activities at every discrete time step: the system will most probably *stand idle*.

The closer is ρ to 1, the greater is the probability to execute some activities at every discrete time step: the system will most probably *operate*.

$\tilde{\varphi}'_1 = \tilde{\varphi}'_3 = \tilde{\varphi}'_5 = 0$ are constants, and they do not depend on ρ.

$\tilde{\varphi}'_2 = \frac{\rho^2(1-\rho)}{2+\rho-\rho^2-\rho^3}$, $\tilde{\varphi}'_4 = \frac{\rho(2-\rho)}{2+\rho-\rho^2-\rho^3}$, $\tilde{\varphi}'_6 = \frac{2-\rho-\rho^2}{2+\rho-\rho^2-\rho^3}$ depend on ρ.



SHMGQSSP: Steady-state probabilities $\tilde{\varphi}_2', \tilde{\varphi}_4', \tilde{\varphi}_6'$ as functions of the parameter ρ

$\tilde{\varphi}_2', \tilde{\varphi}_4'$ tend to 0 and $\tilde{\varphi}_6'$ tends to 1 when ρ approaches 0.

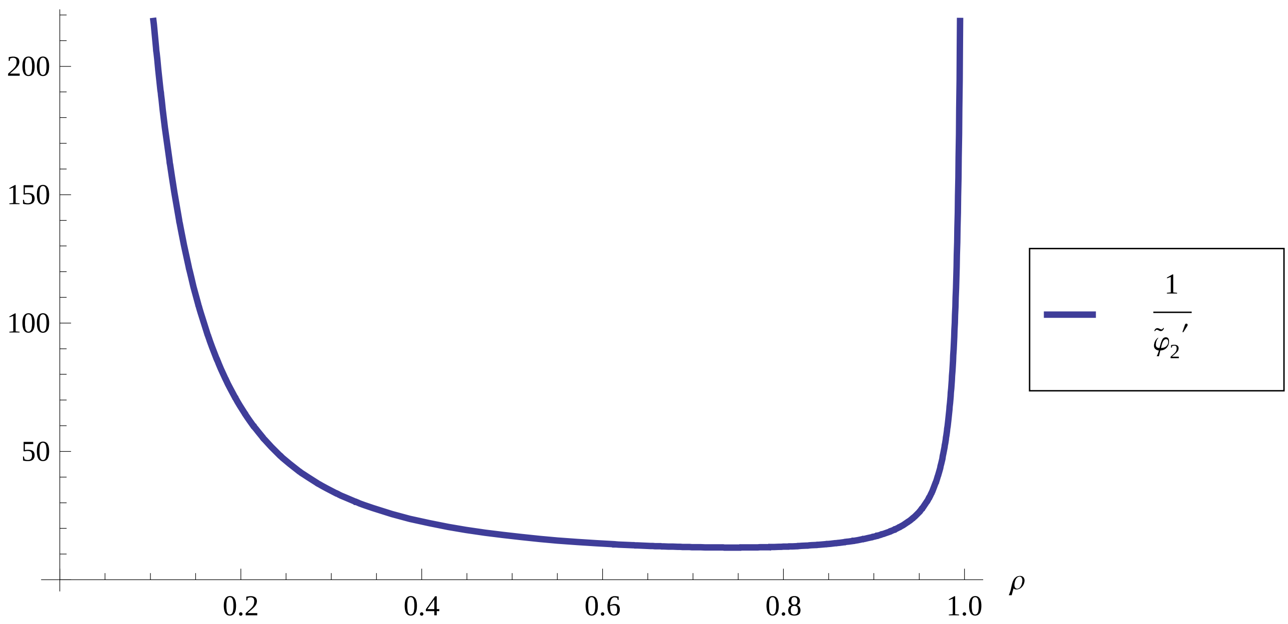
When ρ is closer to 0, the probability that the memory is allocated to a processor and the memory is requested by another processor increases: *more unsatisfied memory requests*.

$\tilde{\varphi}_2', \tilde{\varphi}_6'$ tend to 0 and $\tilde{\varphi}_4'$ tends to 1 when ρ approaches 1.

When ρ is closer to 1, the probability that the memory is allocated to a processor (and not requested by another one) increases: *less unsatisfied memory requests*.

The maximal value 0.0797 of $\tilde{\varphi}_2'$ is reached when $\rho \approx 0.7433$.

In this case, the probability that the system is activated and the memory is not requested is maximal: *maximal shared memory availability* is about 8%.



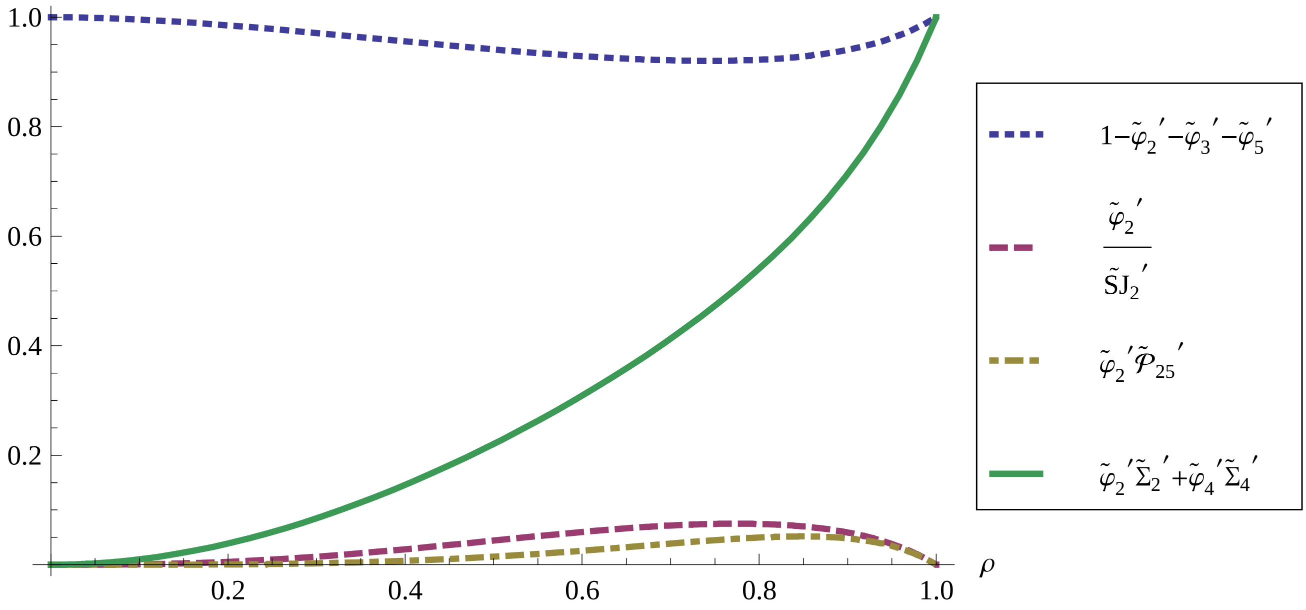
SHMGQART: Average system run-through $\frac{1}{\tilde{\varphi}'_2}$ as a function of the parameter ρ

The average system run-through is $\frac{1}{\tilde{\varphi}'_2}$.

It tends to ∞ when ρ approaches 0 or 1.

The minimal value 12.5516 of $\frac{1}{\tilde{\varphi}'_2}$ is reached when $\rho \approx 0.7433$.

To speed up the system's operation: take the parameter ρ closer to 0.7433.



SHMGQIND: Some performance indices as functions of the parameter ρ

The shared memory utilization is $1 - \tilde{\varphi}_2' - \tilde{\varphi}_3' - \tilde{\varphi}_5'$.

It tends to 1 when ρ approaches 0 and when ρ approaches 1.

The minimal value 0.9203 of the utilization is reached when $\rho \approx 0.7433$.

The *minimal shared memory utilization* is about 92%.

To increase the utilization: take the parameter ρ closer to 0 or 1.

The rate with which the necessity of shared memory emerges is $\frac{\tilde{\varphi}_2'}{\widetilde{SJ}_2'}$.

It tends to 0 when ρ approaches 0 and when ρ approaches 1.

The maximal value 0.0751 of the rate is reached when $\rho \approx 0.7743$.

The *maximal rate with which the necessity of shared memory emerges* is about $\frac{1}{13}$.

To decrease the rate: take the parameter ρ closer to 0 or 1.

The steady-state probability of the shared memory request from two processors is $\tilde{\varphi}'_2\widetilde{\mathcal{P}}'_{25}$, where
$\widetilde{\mathcal{P}}'_{25} = \sum_{\{A,\widetilde{\mathcal{K}}\mid\{\{r\},\{r\}\}\subseteq A,\ \widetilde{\mathcal{K}}_2\stackrel{A}{\to}\widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{K}}) = PM(\widetilde{\mathcal{K}}_2,\widetilde{\mathcal{K}}_5)$.

It tends to 0 when ρ approaches 0 and when ρ approaches 1.

The maximal value 0.0517 of the rate is reached when $\rho \approx 0.8484$.

To decrease the probability: take the parameter ρ closer to 0 or 1.

The steady-state probability of the shared memory request from a processor is $\tilde{\varphi}'_2\widetilde{\Sigma}'_2 + \tilde{\varphi}'_4\widetilde{\Sigma}'_4$, where
$\widetilde{\Sigma}'_i = \sum_{\{A,\widetilde{\mathcal{K}}\mid\{r\}\in A,\ \widetilde{\mathcal{K}}_i\stackrel{A}{\to}\widetilde{\mathcal{K}}\}} PM_A(\widetilde{\mathcal{K}}_i,\widetilde{\mathcal{K}}),\ i\in\{2,4\}$.

It tends to 0 when ρ approaches 0 and it tends to 1 when ρ approaches 1.

To increase the probability: take the parameter ρ closer to 1.

Overview and open questions

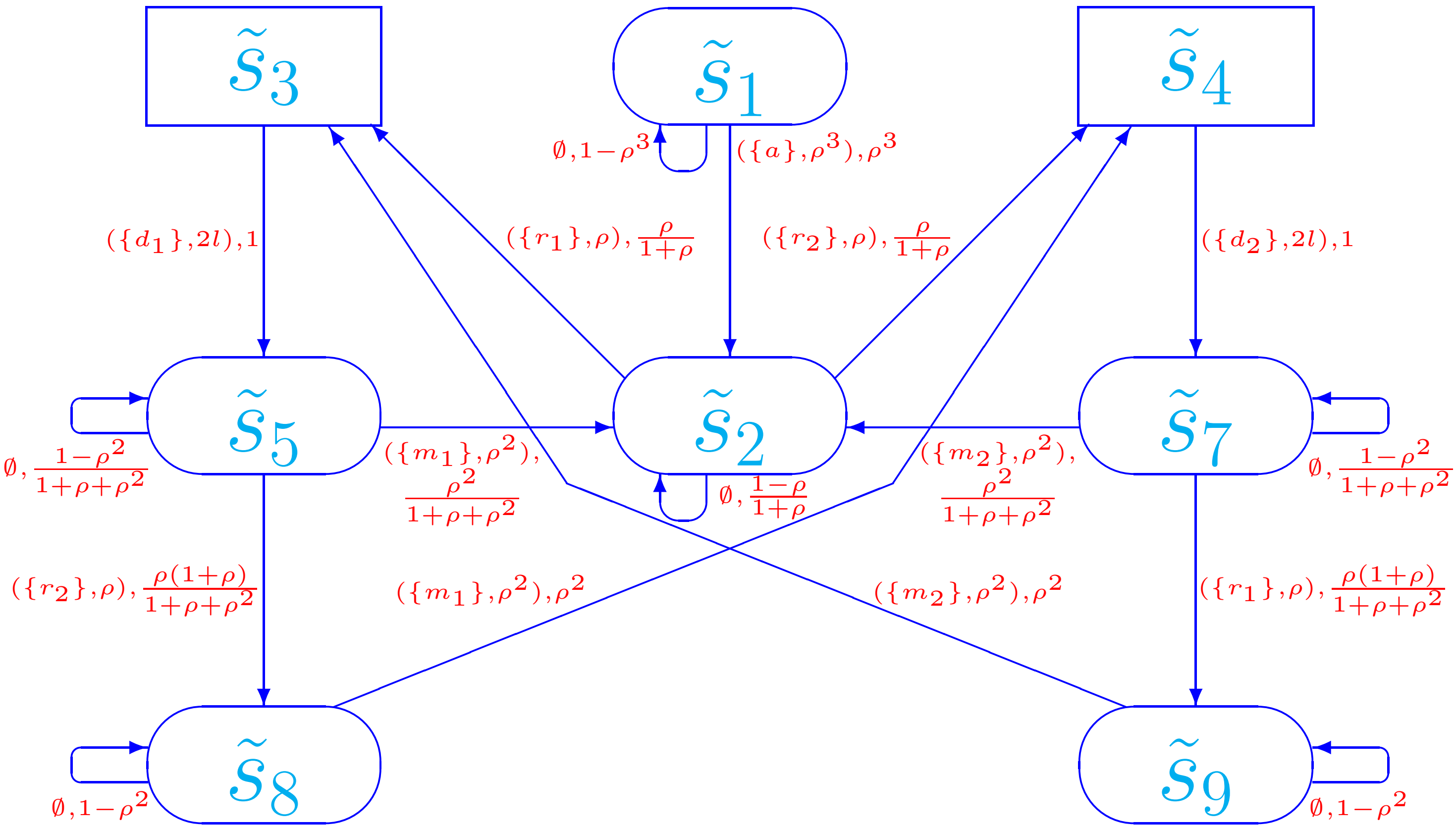
Concurrency interpretation

Interleaving transition relation

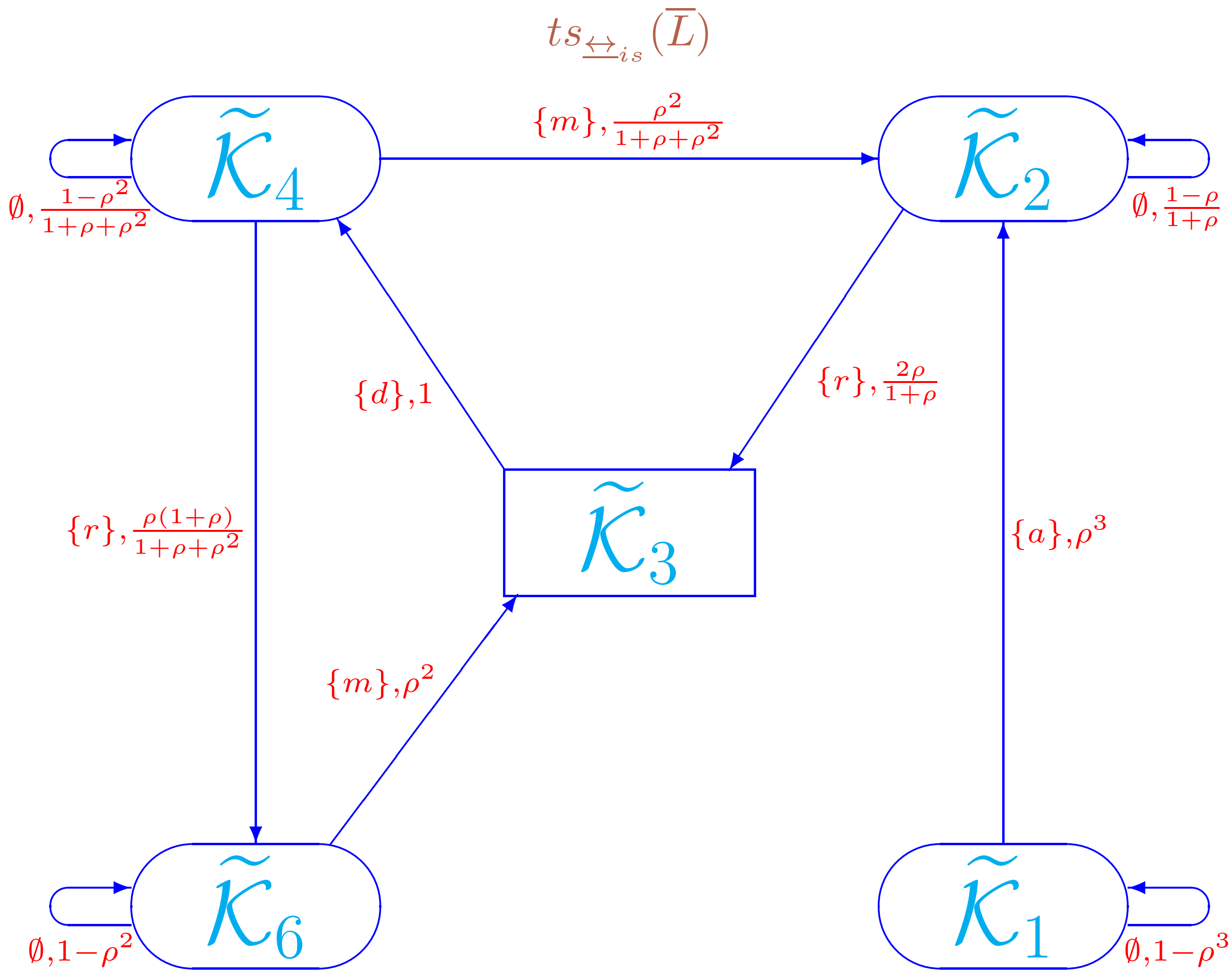
Let G be a dynamic expression, $s \in DR(G)$, $\Upsilon \in Exec(s)$ and $|\Upsilon| \leq 1$.
The *probability to execute the multiset of activities Υ in s, when only zero-element steps (i.e. empty loops) or one-element steps are allowed*:

$$pt(\Upsilon, s) = \frac{PT(\Upsilon, s)}{\sum_{\{\Xi \mid |\Xi| \leq 1\}} PT(\Xi, s)}.$$

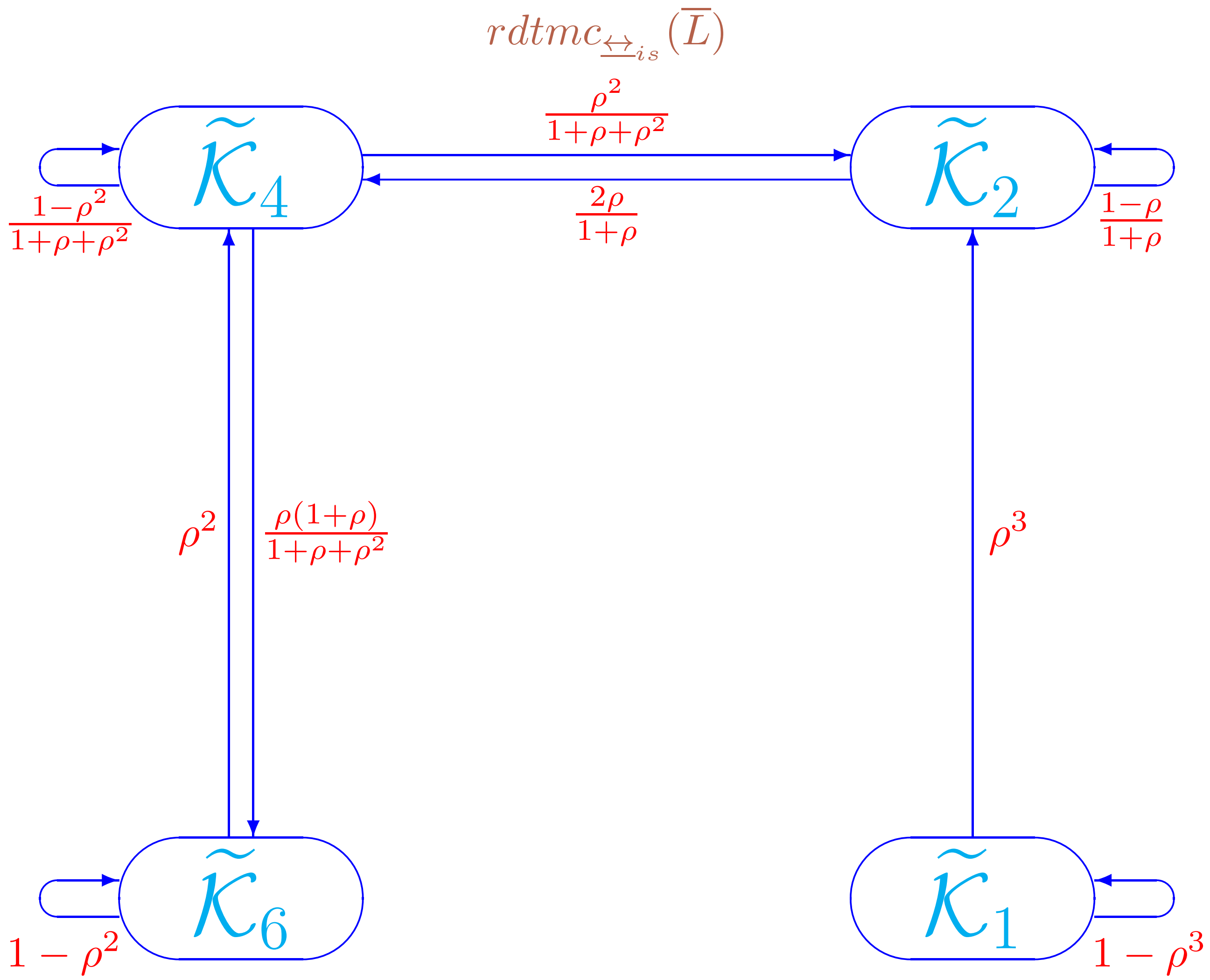
$ts(\overline{K})$



SHMGTSI: The interleaving transition system of the generalized shared memory system



SHMGQTSI: The interleaving quotient transition system of the abstract generalized shared memory system



SHMGQRDTMCI: The interleaving reduced quotient DTMC of the abstract generalized shared memory system

The steady-state PMF for $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{L})$:

$$\tilde{\phi}'^{\diamond} = \frac{1}{2+4\rho+3\rho^2+3\rho^3}(0, \rho^2(1+\rho), 2\rho(1+\rho+\rho^2), 2(1+\rho)),$$

whereas the steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{L})$:

$$\tilde{\psi}'^{\diamond} = \frac{1}{2+\rho-\rho^2-\rho^3}(0, \rho^2(1-\rho), \rho(2-\rho), 2-\rho-\rho^2).$$

SHMQRTPI: Transient and steady-state probabilities for the interleaving reduced quotient DTMC of the abstract shared memory system

k	0	5	10	15	20	25	30	35	40	45	50	
$\phi_1'^{\diamond}[k]$	1	0.5129	0.2631	0.1349	0.0692	0.0355	0.0182	0.0093	0.0048	0.0025	0.0013	
$\phi_2'^{\diamond}[k]$	0	0.1499	0.1155	0.0950	0.0844	0.0789	0.0761	0.0747	0.0739	0.0736	0.0734	0.
$\phi_3'^{\diamond}[k]$	0	0.1992	0.2722	0.3061	0.3233	0.3322	0.3367	0.3390	0.3402	0.3408	0.3411	0.
$\phi_4'^{\diamond}[k]$	0	0.1379	0.3493	0.4640	0.5231	0.5534	0.5690	0.5770	0.5811	0.5832	0.5842	0.

Let $\rho = \frac{1}{2}$ and $l = 1$ in the above interleaving transition systems and DTMC.

The result: the interleaving transition system $ts(\overline{E})$,
quotient transition system $ts_{\underline{\leftrightarrow}_{is}}(\overline{F})$,
reduced quotient DTMC $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{F})$
of the concrete and abstract *standard* shared memory system.

The steady-state PMF for $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{F})$:

$$\phi'^{\diamond} = \left(0, \frac{3}{41}, \frac{14}{41}, \frac{24}{41}\right),$$

whereas the steady-state PMF for $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$:

$$\psi'^{\diamond} = \left(0, \frac{1}{17}, \frac{6}{17}, \frac{10}{17}\right).$$

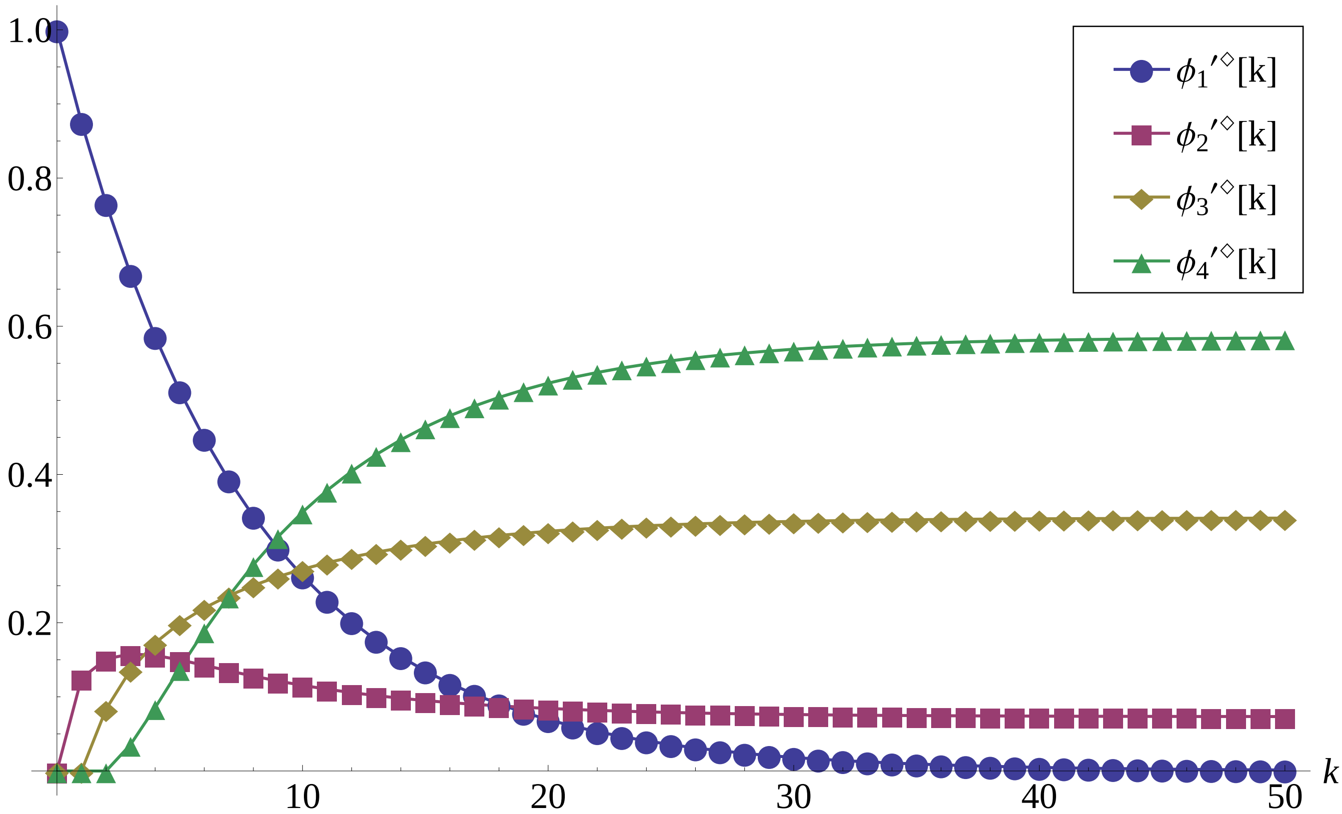
With k growing, $\phi_4'^{\diamond}[k] = \phi'^{\diamond}[k](\mathcal{K}_6)$ stabilizes slower than $\psi_4'^{\diamond}[k] = \psi'^{\diamond}[k](\mathcal{K}_6)$ from Table SHMQRTP and Figure SHMQRTP.

One reason: $rdtmc_{\underline{\leftrightarrow}_{is}}(\overline{F})$ has no transition from $\mathcal{K}_2$ to $\mathcal{K}_6$, unlike $RDTMC_{\underline{\leftrightarrow}_{ss}}(\overline{F})$.

The absolute relative differences for $k = 5$:

$\left|\frac{\phi_4'^{\diamond} - \phi_4'^{\diamond}[5]}{\phi_4'^{\diamond}}\right| = \left|\frac{0.5854 - 0.1379}{0.5854}\right| = \frac{0.4475}{0.5854} \approx 0.7644$ (76%),

$\left|\frac{\psi_4'^{\diamond} - \psi_4'^{\diamond}[5]}{\psi_4'^{\diamond}}\right| = \left|\frac{0.5882 - 0.1901}{0.5882}\right| = \frac{0.3981}{0.5882} \approx 0.6768$ (68%, i.e. 8% less).



SHMQRTPI: Transient probabilities alteration diagram for the interleaving reduced quotient DTMC of the abstract shared memory system

The results obtained

- A discrete time stochastic and immediate extension $dtsiPBC$ of finite PBC enriched with iteration.
- The step operational semantics based on labeled probabilistic transition systems.
- The denotational semantics in terms of a subclass of LDTSIPNs.
- The method of performance analysis based on underlying SMCs.
- Step stochastic bisimulation equivalence of the expressions and dtsi-boxes.
- The transition systems and SMCs reduction modulo the equivalence.
- An application of the equivalence to comparison of stationary behaviour.
- The case study: the shared memory system.

Further research

- Constructing a congruence relation: the equivalence that withstands application of the algebraic operations.
- Introducing the deterministically timed multiactions with fixed time delays (including the zero delay).
- Extending the syntax with recursion operator.

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