A Class of Stochastic Petri Nets with Step Semantics and Related Equivalence Notions

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Abstract: A class of Stochastic Petri Nets with concurrent transition firings is proposed. It is assumed that transitions occur in steps and that for every step each enabled transition decides probabilistically whether it wants to participate in the step or not. Among the transitions which want to participate in a step, a maximal number is chosen to perform the firing step. The observable behavior of a net is described by labels associated with transitions. For this class of nets the dynamic behavior is defined and equivalence relations are introduced. The equivalences extend the well-known trace and bisimulation ones for systems with step semantics to Stochastic Petri Nets with concurrent transition firing. It is shown that the equivalence notions form a lattice of interrelations. In addition, we demonstrate how the equivalences can be used to compare stationary behavior of nets.

Keywords: Stochastic Petri Nets, Step Semantics, Equivalence Relations, Bisimulation, Stationary Behavior.

Previous work

1. Continuous time: the stochastic process is a *Continuous Time Markov Chain (CTMC)*.

   - *Continuous Time Stochastic Petri Nets (CTSPNs)* [FlNa85,Moll82]:

     transitions with exponential firing delay,

     interleaving semantics.

   - *Generalized Continuous Time Stochastic Petri Nets (GCTSPNs)* [AjBC84,CABC93]:

     transitions with exponential and zero firing delay,

     interleaving semantics.

2. Discrete time: the stochastic process is a *Discrete Time Markov Chain (DTMC)*.

   - *Discrete Time Stochastic Petri Nets (DTSPNs)* [Moll85,ZiCH97,ZiGe94]:

     transitions with exponential firing delay,

     step semantics.

Transition labeling in SPNs and GSPNs [Buch95,Buch98].

Bisimulation equivalences for SPNs and GSPNs [Buch94, Buch95,HeRe94,Hill94].

Formal model

Definition 1 *A DTSPN is a seven tuple $N = (P, T, W, \Lambda, \Omega, L, M_{in})$ where:*

- P *and* T *are finite sets of places and transitions respectively such that* $P \cup T \neq \emptyset$ *and* $P \cap T = \emptyset$*;*

- $W : (P \times T) \cup (T \times P) \rightarrow \mathbb{N}$ *is function describing the weights of arcs between places and transitions and vice versa;*

- $\Lambda : T \rightarrow \mathbb{R}$ *is the transition weight function;*

- $\Omega : T \rightarrow (0, 1]$ *is the transition probability function;*

- $L : T \rightarrow Act_\tau$ *is the transition labeling function assigning labels from a finite set of visible actions* Act *or an invisible action* τ *to transitions (i.e.,* $Act_\tau = Act \cup \{\tau\}$*);*

- $M_{in} : P \rightarrow \mathbb{N}$ *is the initial marking.*

Behavior of the model

$Ena(M)$ denotes the set of *all transitions that are enabled in marking* M. The transitions from $U \subseteq Ena(M)$ try to fire in the next step with probability:

$$PF[U] = \prod_{t \in U} \Omega(t) \cdot \prod_{t \in Ena(M) \setminus U} (1 - \Omega(t)).$$

All transitions from U can fire if the following condition (*) holds:

$$\forall p \in P : M(p) \geq \sum_{t \in U} W(p,t).$$

A set $V \subseteq Ena(M)$ is a *maximal fireable subset in marking* M if (*) holds for V and no more transitions from $Ena(M) \setminus V$ can be added when the condition has to hold.

$MaxFire(M)$ denotes the set of *all maximal fireable subsets in marking* M.

A set $V \subseteq U$ is a *maximal fireable subset of* U *in marking* M if the condition holds for V and no more transitions from $U \setminus V$ can be added.

$MaxFire(U, M)$ denotes the set of *all maximal fireable subsets of* U *in marking* M.

The weight function is extended to sets of transitions $V \subseteq T$:

$$\Lambda(V) = \sum_{t \in V} \Lambda(t).$$

If transitions from the set U try to fire, but cannot fire concurrently since (*) does not hold, then a maximal fireable subset of transitions, i.e., $V \in MaxFire(U, M)$ is chosen with probability:

$$PC[V, U] = \Lambda(V) \Bigg/ \left(\sum_{W \in MaxFire(U,M)} \Lambda(W) \right).$$

For $V \in MaxFire(M)$ let $SubEna(V, M)$ be the set of *all subsets of* $Ena(M)$ *that include* V. The probability of firing $V \in MaxFire(M)$ is:

$$PT[V, M] = \sum_{U \in SubEna(V,M)} PF[U] \cdot PC[V, U].$$

If no transition wants to fire at the next step, then $U = \emptyset = V$ and:

$$PT[\emptyset, M] = PF[\emptyset] = \prod_{t \in Ena(M)} (1 - \Omega(t)).$$

We define the *visible labeling* function $VisL$ on sets of transitions $V \subseteq T$:

$$VisL(V) = \sum_{(t \in V) \wedge (L(t) \neq \tau)} L(t).$$

Denote a set of *all multisets over a set* X by $\mathcal{M}(X)$. Let $A \in \mathcal{M}(Act)$. The set of *all subsets of transitions which are labeled by* A is:

$$Trans(A) = \{V \subseteq T \mid VisL(V) = A\}.$$

The probability of observing A in marking M is:

$$PL[A, M] = \sum_{V \in Trans(A) \cap MaxFire(M)} PT[V, M].$$

If V fires in M, then the successor marking $\widetilde{M}$ is defined as:

$$\widetilde{M}(p) = M(p) - \sum_{t \in V} W(p, t) + \sum_{t \in V} W(t, p).$$

Let V be a set of transitions which can fire concurrently in marking M resulting to $\widetilde{M}$ and $\mathcal{P} = PT[V, M]$, notation $M \xrightarrow{V}_{\mathcal{P}} \widetilde{M}$.

We write $M \xrightarrow{V} \widetilde{M}$ if $M \xrightarrow{V}_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For one-element set of transitions $V = \{t\}$ we write $M \xrightarrow{t}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{t} \widetilde{M}$.

$M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ describes a step starting in marking M, performing transitions labeled by A and ending in $\widetilde{M}$. The probability $\mathcal{P} = PS[A, M, \widetilde{M}]$ is:

$$PS[A, M, \widetilde{M}] = \sum_{\{V \in Trans(A) \mid M \xrightarrow{V}_{\mathcal{Q}} \widetilde{M}\}} \mathcal{Q}.$$

We write $M \xrightarrow{A} \widetilde{M}$ if $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For one-element multiset of actions $A = \{a\}$ we write $M \xrightarrow{a}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{a} \widetilde{M}$.

Definition 2 *For a DTSPN N we define:*

- *The* reachability set $RS(N)$ *as the minimal set of markings M for which the following conditions hold:*
  - $M_{in} \in RS(N)$;
  - *if $M \in RS(N)$ and $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ for $\mathcal{P} > 0$, then $\widetilde{M} \in RS(N)$.*

- *The* reachability graph $RG(N)$ *as a directed labeled graph with a set of nodes $RS(N)$ and an arc labeled by A, $\mathcal{P}$ between nodes M and $\widetilde{M}$ whenever $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ holds.*

- *The underlying* Discrete Time Markov Chain (DTMC) *$DT(N)$ with state space $RS(N)$ and a transition $M \longrightarrow_{\mathcal{P}} \widetilde{M}$ whenever at least one arc between M and $\widetilde{M}$ exists in $RG(N)$. In this case, the probability $\mathcal{P} = PS[M, \widetilde{M}]$ is computed as:*

$$PS[M, \widetilde{M}] = \sum_{A \in \mathcal{M}(Act)} PS[A, M, \widetilde{M}].$$

An *internal step* $M \xrightarrow{\emptyset}_{\mathcal{P}} \widetilde{M}$ with $\mathcal{P} > 0$ takes place when $\widetilde{M}$ is reachable from M by firing a set of internal transitions or if no transition fires.

The *probability of reaching $\widetilde{M}$ from M by k internal steps* is:

$$PS^k[\emptyset, M, \widetilde{M}] = \begin{cases} \sum_{\overline{M} \in RS(N)} PS^{k-1}[\emptyset, M, \overline{M}] \cdot \\ PS[\emptyset, \overline{M}, \widetilde{M}] & \text{if } k \geq 1; \\ 1 & \text{if } k = 0 \text{ and} \\ & M = \widetilde{M}; \\ 0 & \text{otherwise.} \end{cases}$$

The *probability of reaching $\widetilde{M}$ from M by internal steps* is:

$$PS^*[\emptyset, M, \widetilde{M}] = \sum_{k=0}^{\infty} PS^k[\emptyset, M, \widetilde{M}].$$

The *probability of reaching $\widetilde{M}$ from M by internal steps, followed by an observable step A* is:

$$PS^*[A, M, \widetilde{M}] = \sum_{\overline{M} \in RS(N)} PS^*[\emptyset, M, \overline{M}] \cdot PS[A, \overline{M}, \widetilde{M}].$$

We define a new transition relation $M \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$ where $\mathcal{P} = PS^*[A, M, \widetilde{M}]$ and $A \neq \emptyset$.

We write $M \xrightarrow{A}\!\!\!\!\rightarrow \widetilde{M}$ if $M \xrightarrow{A}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For one-element multiset of actions $A = \{a\}$ we write $M \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{a}\!\!\!\!\rightarrow \widetilde{M}$.

$RS^*(N)$ and $RG^*(N)$ denote the *observable reachability set* and *graph* respectively.

We define the *embedded DTMC* $DT^*(N)$ with state space $RS^*(N)$ and transition probabilities:

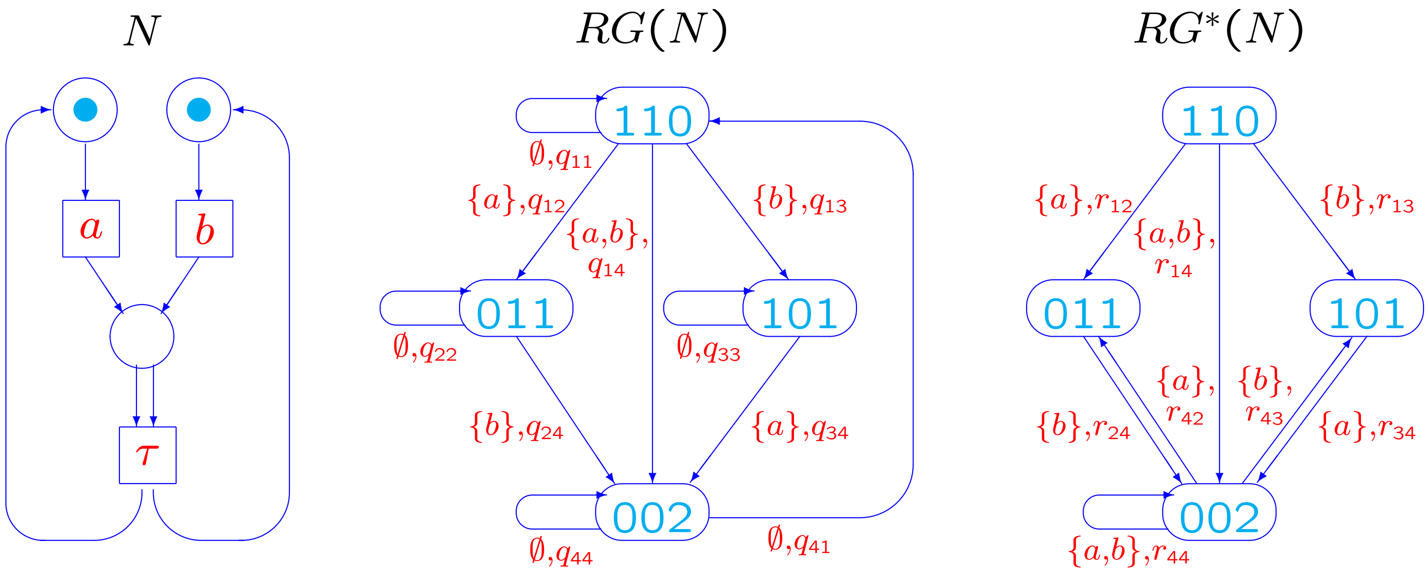
$$PS^*[M, \widetilde{M}] = \sum_{A \in \mathcal{M}(Act) \setminus \emptyset} PS^*[A, M, \widetilde{M}].$$

A *trap* is a loop of internal transitions starting and ending in some marking M which occurs with probability 1. $PS^*[\emptyset, M, \widetilde{M}]$ is finite as long as no traps exist.

If $PS^*[\emptyset, M, \widetilde{M}]$ is finite, then $PS^*[A, M, \widetilde{M}]$ defines a probability distribution, i.e.:

$$\sum_{A \in \mathcal{M}(Act) \setminus \emptyset} \sum_{\widetilde{M} \in RS^*(N)} PS^*[A, M, \widetilde{M}] = 1.$$

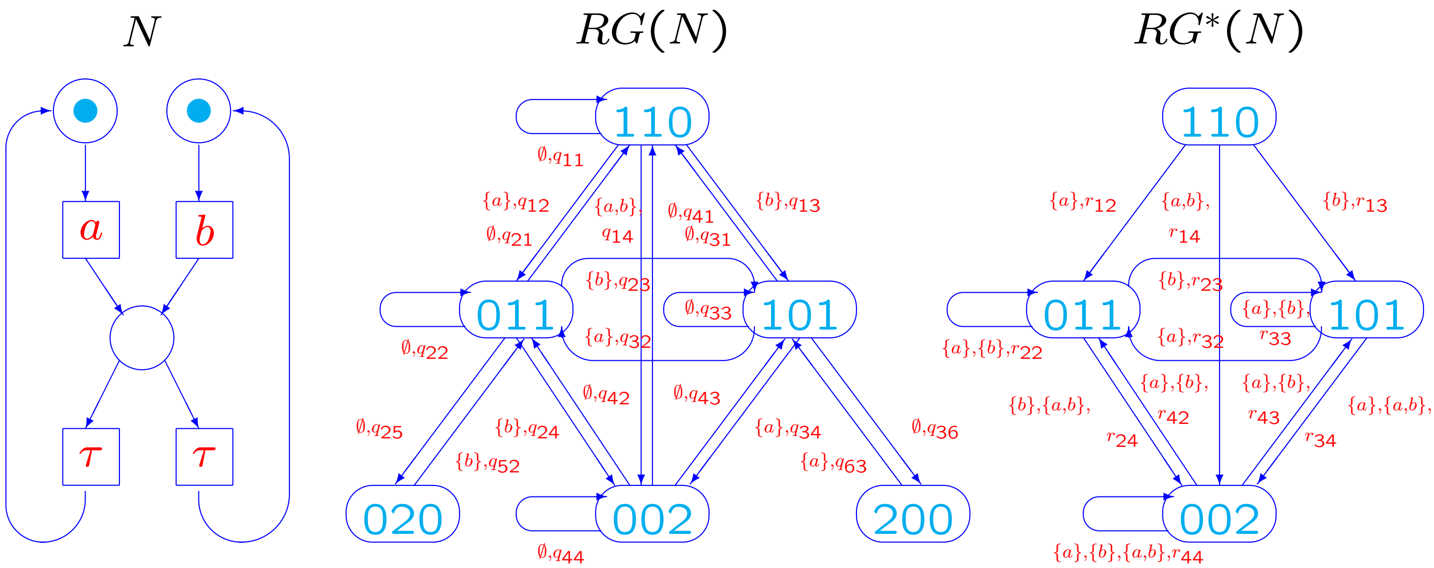
Examples of DTSPNs



First example net and the corresponding reachability graphs

$$q_{11} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_2) \quad q_{12} = \Omega(t_1) \cdot \overline{\Omega}(t_2) \quad q_{13} = \overline{\Omega}(t_1) \cdot \Omega(t_2)$$
$$q_{14} = \Omega(t_1) \cdot \Omega(t_2) \quad q_{22} = \overline{\Omega}(t_2) \quad q_{24} = \Omega(t_2)$$
$$q_{33} = \overline{\Omega}(t_1) \quad q_{34} = \Omega(t_1) \quad q_{41} = \Omega(t_3)$$
$$q_{44} = \overline{\Omega}(t_3)$$

$$r_{12} = r_{42} = \frac{q_{12}}{1-q_{11}} \quad r_{13} = r_{43} = \frac{q_{13}}{1-q_{11}} \quad r_{14} = r_{44} = \frac{q_{14}}{1-(1-q_{11})} = \frac{q_{14}}{q_{11}}$$
$$r_{24} = 1 \quad r_{34} = 1$$



Second example net and the corresponding reachability graphs

$$q_{11} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_2)$$
$$q_{12} = \Omega(t_1) \cdot \overline{\Omega}(t_2)$$
$$q_{13} = \overline{\Omega}(t_1) \cdot \Omega(t_2)$$
$$q_{14} = \Omega(t_1) \cdot \Omega(t_2)$$
$$q_{21} = \overline{\Omega}(t_2) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4))$$
$$q^{\emptyset}_{22} = \overline{\Omega}(t_2) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4)$$
$$q^{b}_{22} = \Omega(t_2) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3))$$
$$q_{23} = \Omega(t_2) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4))$$
$$q_{24} = \Omega(t_2) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4)$$
$$q_{25} = \overline{\Omega}(t_2) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3))$$
$$q_{31} = \overline{\Omega}(t_1) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3))$$
$$q_{32} = \Omega(t_1) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3))$$
$$q^{\emptyset}_{33} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4)$$
$$q^{a}_{33} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4))$$
$$q_{34} = \Omega(t_1) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4)$$
$$q_{36} = \overline{\Omega}(t_1) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4))$$
$$q_{41} = \Omega(t_3) \cdot \Omega(t_4)$$
$$q_{42} = \overline{\Omega}(t_3) \cdot \Omega(t_4)$$
$$q_{43} = \Omega(t_3) \cdot \overline{\Omega}(t_4)$$
$$q_{42} = \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4)$$
$$q_{52} = \Omega(t_2)$$
$$q_{55} = \overline{\Omega}(t_2)$$
$$q_{63} = \Omega(t_1)$$
$$q_{66} = \overline{\Omega}(t_1)$$

$$r_{12} = q_{12}/(1 - q_{11}) \qquad r_{13} = q_{13}/(1 - q_{11})$$
$$r_{14} = q_{14}/(1 - q_{11}) \qquad r^{a}_{22} = q_{21} \cdot r_{12}/(1 - q^{\emptyset}_{22})$$
$$r^{b}_{22} = (q^{b}_{22} + q_{25})/(1 - q^{\emptyset}_{22}) \qquad r_{23} = (q_{23} + q_{21} \cdot r_{13})/(1 - q^{\emptyset}_{22})$$
$$r^{b}_{24} = q_{24}/(1 - q^{\emptyset}_{22}) \qquad r^{\{a,b\}}_{24} = q_{21} \cdot r_{14}/(1 - q^{\emptyset}_{22})$$
$$r_{32} = (q_{32} + q_{31} \cdot r_{12})/(1 - q^{\emptyset}_{33}) \qquad r^{a}_{33} = (q^{a}_{33} + q_{36})/(1 - q^{\emptyset}_{33})$$
$$r^{b}_{33} = q_{31} \cdot r_{13}/(1 - q^{\emptyset}_{33}) \qquad r^{a}_{34} = q_{34}/(1 - q^{\emptyset}_{33})$$
$$r^{\{a,b\}}_{34} = q_{31} \cdot r_{14}/(1 - q^{\emptyset}_{33}) \qquad r^{a}_{42} = (q_{41} \cdot r_{12} + q_{43} \cdot r_{32})/(1-q_{44})$$
$$r^{b}_{42} = q_{42} \cdot r^{b}_{22}/(1 - q_{44}) \qquad r^{a}_{43} = q_{43} \cdot r^{a}_{33}/(1 - q_{44})$$
$$r^{b}_{43} = (q_{41} \cdot r_{13} + q_{42} \cdot r_{23})/(1 - q_{44}) \qquad r^{a}_{44} = q_{43} \cdot r^{a}_{34}/(1 - q_{44})$$
$$r^{b}_{44} = q_{42} \cdot r^{b}_{24}/(1 - q_{44}) \qquad r^{\{a,b\}}_{44} = q_{41} \cdot r^{\{a,b\}}_{14}/(1 - q_{44})$$

Trace equivalences

Definition 3 *An* interleaving trace *of a DTSPN N is a pair $(\sigma, \mathcal{P})$, where $\sigma = a_1 \cdots a_n \in Act^*$ and:*

$$\mathcal{P} = \sum_{\{M_1,\ldots,M_n \mid M_{in} \xrightarrow{a_1}\!\!\!\twoheadrightarrow_{\mathcal{P}_1} M_1 \xrightarrow{a_2}\!\!\!\twoheadrightarrow_{\mathcal{P}_2} \cdots \xrightarrow{a_n}\!\!\!\twoheadrightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

We denote a set of all interleaving traces *of a DTSPN N by $IntTraces(N)$. Two DTSPNs N and N' are* interleaving trace equivalent, *denoted by $N \equiv_i N'$, if:*

$$IntTraces(N) = IntTraces(N').$$

Definition 4 *A* step trace *of a DTSPN N is a pair $(\Sigma, \mathcal{P})$, where $\Sigma = A_1 \cdots A_n \in \mathcal{M}(Act)^*$ and:*

$$\mathcal{P} = \sum_{\{M_1,\ldots,M_n \mid M_{in} \xrightarrow{A_1}\!\!\!\twoheadrightarrow_{\mathcal{P}_1} M_1 \xrightarrow{A_2}\!\!\!\twoheadrightarrow_{\mathcal{P}_2} \cdots \xrightarrow{A_n}\!\!\!\twoheadrightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

We denote a set of all step traces *of a DTSPN N by $StepTraces(N)$. Two DTSPNs N and N' are* step trace equivalent, *denoted by $N \equiv_s N'$, if:*

$$StepTraces(N) = StepTraces(N').$$

Bisimulation equivalences

Definition 5 *Let N be a DTSPN. An* equivalence *relation $\mathcal{R} \subseteq RS^*(N)^2$ is an* interleaving bisimulation *between two markings M_1 and M_2 of N (i.e., $(M_1, M_2) \in \mathcal{R}$), denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_i M_2$, if $\forall a \in Act\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$:*

$$M_1 \xrightarrow{a}\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L} \ \Leftrightarrow\ M_2 \xrightarrow{a}\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L}.$$

Two markings M_1 and M_2 are interleaving bisimulation equivalent*, denoted by $M_1 \underline{\leftrightarrow}_i M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_i M_2$.*

Definition 6 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is an* interleaving bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_i N'$, if $\mathcal{R} : M_{in} \underline{\leftrightarrow}_i M'_{in}$.*

Two DTSPNs N and N' are interleaving bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_i N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_i N'$.*

Definition 7 *Let N be a DTSPN. An* equivalence *relation $\mathcal{R} \subseteq RS^*(N)^2$ is a* step bisimulation *between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_s M_2$, if $\forall A \in \mathcal{M}(Act)\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$:*

$$M_1 \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L} \ \Leftrightarrow\ M_2 \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} \mathcal{L}.$$

Two markings M_1 and M_2 are step bisimulation equivalent*, denoted by $M_1 \underline{\leftrightarrow}_s M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_s M_2$.*

Definition 8 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is a* step bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_s N'$, if $\mathcal{R} : M_{in} \underline{\leftrightarrow}_s M'_{in}$.*

Two DTSPNs N and N' are step bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_s N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_s N'$.*

Backward bisimulation equivalences

Definition 9 *Let N be a DTSPN. An* equivalence *relation $\mathcal{R} \subseteq RS^*(N)^2$ is an* interleaving backward bisimulation *between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_{ib} M_2$, if $\forall a \in Act\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$:*

$$M_1 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N) \ \Leftrightarrow\ M_2 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N),$$

$$\mathcal{L} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} M_1 \ \Leftrightarrow\ \mathcal{L} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} M_2 \ \text{ and } [M_{in}]_{\mathcal{R}} = \{M_{in}\}.$$

Two markings M_1 and M_2 are interleaving backward bisimulation equivalent*, denoted by $M_1 \underline{\leftrightarrow}_{ib} M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_{ib} M_2$.*

Definition 10 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is an* interleaving backward bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{ib} N'$, if $\forall a \in Act\ \forall \mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}\ \forall M_1, M_2 \in \mathcal{L}$:*

$$M_1 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_1)) \ \Leftrightarrow\ M_2 \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_2)),$$

$$[M_{in}]_{\mathcal{R}} = \{M_{in}, M'_{in}\} \text{ and}$$

$$\mathcal{K} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_1))|}{|\mathcal{K}\cap RS^*(\Gamma(M_1))|}} M_1 \ \Leftrightarrow\ \mathcal{K} \xrightarrow{a}\!\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_2))|}{|\mathcal{K}\cap RS^*(\Gamma(M_2))|}} M_2.$$

Two DTSPNs N and N' are interleaving backward bisimulation equivalent*, denoted by $N \underline{\leftrightarrow}_{ib} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{ib} N'$.*

Definition 11 *Let N be a DTSPN. An* equivalence *relation $\mathcal{R} \subseteq RS^*(N)^2$ is a* step backward bisimulation *between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_{sb} M_2$, if $\forall A \in \mathcal{M}(Act)\ \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$:*

$$M_1 \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N) \ \Leftrightarrow\ M_2 \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} RS^*(N),$$

$$\mathcal{L} \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} M_1 \ \Leftrightarrow\ \mathcal{L} \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} M_2 \text{ and } [M_{in}]_{\mathcal{R}} = \{M_{in}\}.$$

Two markings M_1 and M_2 are step backward bisimulation equivalent, *denoted by $M_1 \underline{\leftrightarrow}_{sb} M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_{sb} M_2$.*

Definition 12 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is a* step backward bisimulation *between N and N', denoted by $\mathcal{R} : N \underline{\leftrightarrow}_{sb} N'$, if $\forall A \in \mathcal{M}(Act)\ \forall \mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}\ \forall M_1, M_2 \in \mathcal{L}$:*

$$M_1 \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_1)) \ \Leftrightarrow\ M_2 \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}} RS^*(\Gamma(M_2)),$$

$$[M_{in}]_{\mathcal{R}} = \{M_{in}, M'_{in}\} \text{ and}$$

$$\mathcal{K} \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_1))|}{|\mathcal{K}\cap RS^*(\Gamma(M_1))|}} M_1 \ \Leftrightarrow\ \mathcal{K} \xrightarrow{A}\!\!\!\rightarrow_{\mathcal{Q}\cdot\frac{|\mathcal{L}\cap RS^*(\Gamma(M_2))|}{|\mathcal{K}\cap RS^*(\Gamma(M_2))|}} M_2.$$

Two DTSPNs N and N' are step backward bisimulation equivalent, *denoted by $N \underline{\leftrightarrow}_{sb} N'$, if $\exists \mathcal{R} : N \underline{\leftrightarrow}_{sb} N'$.*

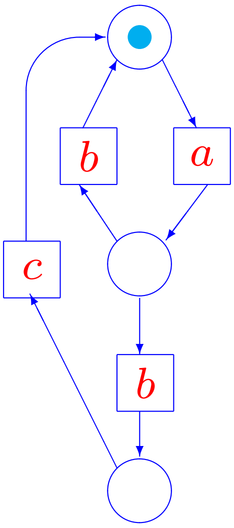
Back and forth bisimulation equivalences

Definition 13 *Two DTSPNs N and N' are* interleaving back and forth bisimulation equivalent, *denoted by* $N \underline{\leftrightarrow}_{ibf} N'$, *if* $N \underline{\leftrightarrow}_{i} N'$ *and* $N \underline{\leftrightarrow}_{ib} N'$.

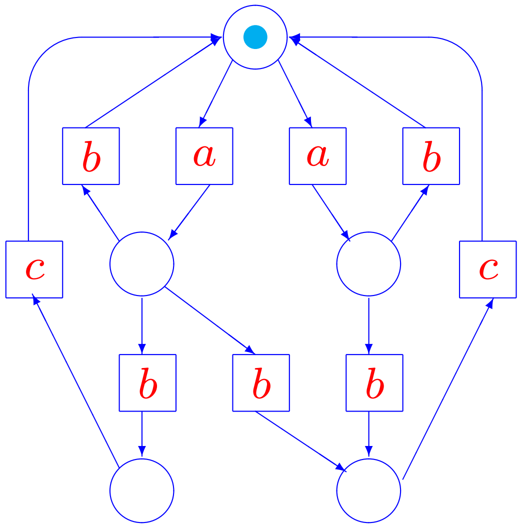
Definition 14 *Two DTSPNs N and N' are* step back and forth bisimulation equivalent, *denoted by* $N \underline{\leftrightarrow}_{sbf} N'$, *if* $N \underline{\leftrightarrow}_{s} N'$ *and* $N \underline{\leftrightarrow}_{sb} N'$.

Examples of the equivalences

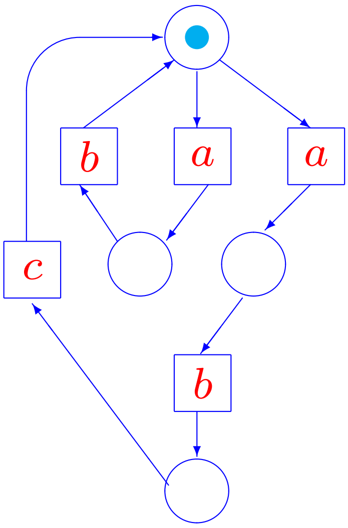
N_1



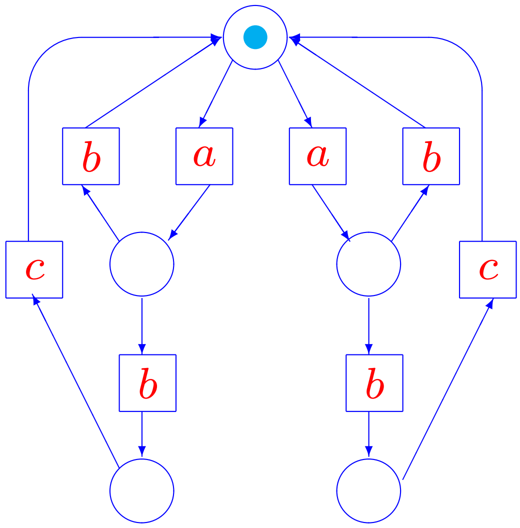
N_2



N_3



N_4



Nets related via different equivalences

$N_1 \equiv_s N_2 \equiv_s N_3 \equiv_s N_4$ $\quad$ $N_1 \underline{\leftrightarrow}_s N_2 \underline{\leftrightarrow}_s N_4$ $\quad$ $N_1 \underline{\leftrightarrow}_{sb} N_3 \underline{\leftrightarrow}_{sb} N_4$

$N_1 \underline{\leftrightarrow}_{sbf} N_4$ $\quad$ $N_2 \underline{\not\leftrightarrow}_{i} N_3$ $\quad$ $N_2 \underline{\not\leftrightarrow}_{ib} N_3$

Comparing the equivalences

$$\begin{tikzcd}
 & \underline{\leftrightarrow}_i \arrow[d] & \underline{\leftrightarrow}_s \arrow[l] \arrow[d] & \\
\underline{\leftrightarrow}_{ibf} \arrow[ur] \arrow[dr] & \equiv_i & \equiv_s \arrow[l] & \underline{\leftrightarrow}_{sbf} \arrow[ul] \arrow[dl] \arrow[lll, bend right] \\
 & \underline{\leftrightarrow}_{ib} \arrow[u] & \underline{\leftrightarrow}_{sb} \arrow[l] \arrow[u] &
\end{tikzcd}$$

Interrelations of the equivalences

Proposition 1 *Let* $\star \in \{i, s\}$. *For DTSPNs* N *and* N' *the following holds:*

1. $N \underline{\leftrightarrow}_\star N' \;\Rightarrow\; N \equiv_\star N'$;

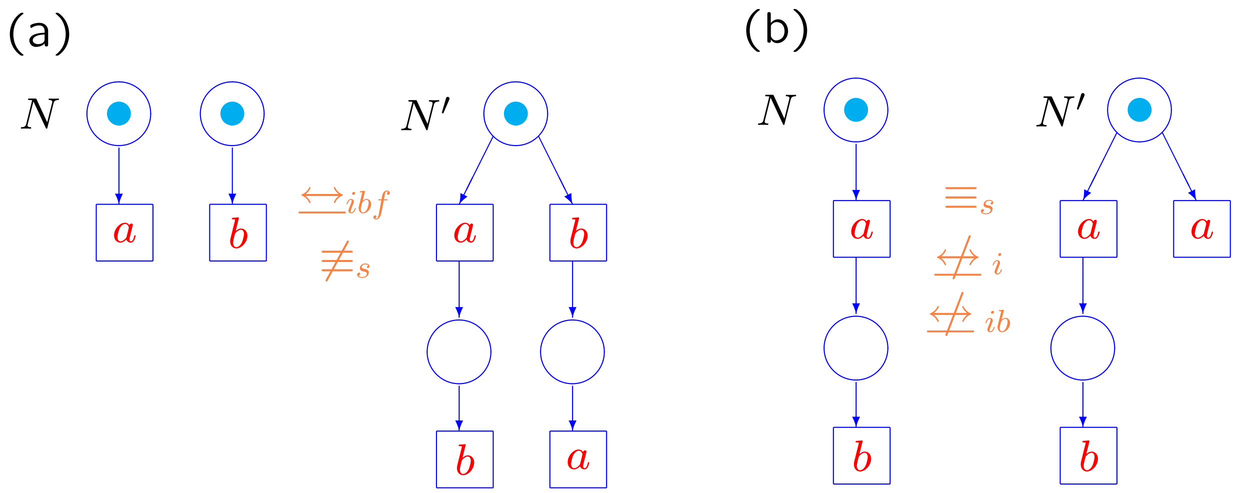
2. $N \underline{\leftrightarrow}_{\star b} N' \;\Rightarrow\; N \equiv_\star N'$;

3. $N \underline{\leftrightarrow}_{\star bf} N' \;\Rightarrow\; N \underline{\leftrightarrow}_\star N'$ *and* $N \underline{\leftrightarrow}_{\star b} N'$.

Theorem 1 *Let* $\leftrightarrow, \twoheadleftarrow\!\!\!\twoheadrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$ *and* $\star, \star\star \in \{i, s, ib, sb, ibf, sbf\}$. *For DTSPNs* N *and* N' *the following holds:*

$$N \leftrightarrow_\star N' \;\Rightarrow\; N \twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star} N'$$

iff in the graph in figure above there exists a directed path from $\leftrightarrow_\star$ *to* $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\star\star}$.



Examples of the equivalences

Stationary behavior

The *embedded steady state distribution after the observation of a visible event* is the unique solution of the set linear equation:

$$ps^*(M) = \sum_{\widetilde{M} \in RS^*(N)} ps^*(\widetilde{M}) \cdot PS^*[\widetilde{M}, M]$$

subject to $\sum_{M \in RS^*(N)} ps^*(M) = 1$.

A *step trace starting in marking* $M \in RS^*(N)$ is defined as $(\Sigma, \mathcal{P})$, where $\Sigma = A_1 \cdots A_n \in Act^*$ and:

$$\mathcal{P} = \sum_{\{M_1, \ldots, M_n \mid M \xrightarrow{A_1}\!\!\!\!\rightarrow_{\mathcal{P}_1} M_1 \xrightarrow{A_2}\!\!\!\!\rightarrow_{\mathcal{P}_2} \cdots \xrightarrow{A_n}\!\!\!\!\rightarrow_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

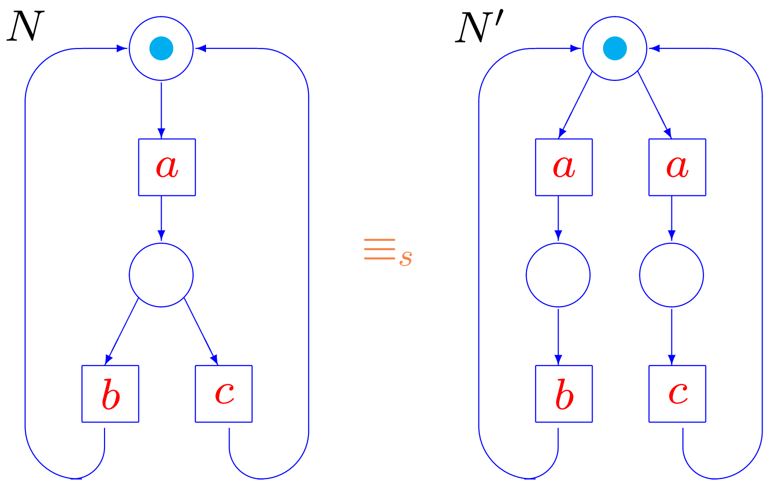
$StepTraces(N, M)$ be the set of *all step traces of starting in marking* M.

Definition 15 *A* step trace in steady state *is a triple* $(M, \Sigma, ps^*(M) \cdot \mathcal{P})$ *s.t* $M \in RS^*(N)$ *and* $(\Sigma, \mathcal{P}) \in StepTraces(N, M)$*.*

The set of all step traces in steady state *is denoted by* $StepTracesSS(N)$*.*

Theorem 2 *Let N and N' be backward or forward bisimulation equivalent DTSPNs, then:*

$$StepTracesSS(N) = StepTracesSS(N').$$



Two step trace equivalent nets with
$StepTracesSS(N) \neq StepTracesSS(N')$

The results obtained

- A new class of Stochastic Petri Nets with labeled transitions and a step semantics for transition firing (DTSPNs).

- Equivalences for DTSPNs which preserve interesting aspects of behavior and thus can be used to compare systems and to compute for a given one a minimal equivalent representation [Buch95].

- A diagram of interrelations for the equivalences.

- An application of the equivalences for comparing stationary behavior of DTSPNs.

Further research

- Other equivalences in interleaving and step semantics:

  interleaving branching bisimulation [PRS92] (respecting conflicts with invisible transitions),

  back-forth bisimulations [NMV90,Pin93] (moving backward along history of computation).

- True concurrent equivalences:

  partial word and *pomset bisimulations* [PRS92,Vog92] (partial order models of computation).

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